Approximations to Auctions of Digital Goods with Share-averse Bidders

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Abstract

We consider the case of a digital product for share-averse bidders, where the product can be sold to multiple buyers who experience some disutility from other firms or consumers owning the same product. We model the problem of selling a digital product to share-averse bidders as an auction and apply a Bayesian optimal mechanism design. We also design constant-approximation algorithms in the prior-free setting including both average- and worst-case analyses.

1 Introduction

Consider the problem of selling a valuable piece of information or dataset. In principle, the information or dataset can be sold to all bidders at no marginal cost to the seller. In general, many digital goods share the same property. In other words, digital goods are expensive to produce but cheap to reproduce, since the unit cost of reproduction is negligible and virtually zero. They can be consumed by more than one user at the same time. However, in reality, the value of the information or dataset to a bidder decreases as increasing numbers of bidders obtain the information because the competitive advantage of possessing the information becomes weaker with more receivers of the information. The seller needs to know:

1. What is the optimal number of copies, k that they should sell to?
2. What is the profit maximizing price to charge the k buyers?

Digital product is available in unlimited supply. The firm can sell as many copies as there are buyers. On the other side of this tug-of-war, each of the buyers that obtains a copy incurs some disutility from others obtaining the same product. This indicates that the buyers are willing to pay less if more copies are sold. This represents the basic trade-off to the seller. Since the firms can sell to multiple parties who obtain some disutility from sharing, we call this a digital product with share-averse bidders.

We model this example as an auction. The possible setup is selling either to an individual buyer (standard single item auction) or to multiple buyers depending upon the valuations of all buyers. We introduce a deterministic function to capture the decreasing valuations when sharing the product. To make the problem tractable, we assume that this deterministic function is known to all bidders and hence it is not part of bidder's private information. This assumption implies a single-parameter auction. A distinct attribute of our problem is the task of modeling the number of winners. This can be computed by applying the VCG auction on the virtual values in the Bayesian setting. We apply the well-known Myerson mechanism for maximizing the expected revenue in the Bayesian setting where the values are drawn from a prior distribution. Unlike an auction for standard physical goods, the number of items sold or the number of winners can not be determined in advance in the auction for digital goods with share-averse bidders. We need to use the prior distribution to compute the number of winners in this auction. In prior-free settings, it is challenging to model the number of winners. We study prior-free auctions and establish the approximation ratio in both average- and worst-cases of the appropriately designed algorithms. We design an algorithm in the prior-free average case where the number of winners can be obtained by the VCG auction on bids. The proposed algorithm approximately maximizes the revenue against a certain benchmark. The algorithm provides good techniques to handle the prior-free approximations and to resolve the issue of coping with the number of winners. We design another prior-free auction/algorithm in the worst case where the bidders are divided into sample and market groups and the number of winners is determined by computing the winning price from the sample group.

We first show a single-sample approximation algorithm in the prior-free average case where the revenue benchmark is the expected optimal revenue as shown in the Bayesian setting. By single-sample, we mean that the algorithm is developed over only one sample of bids from all bidders. The average case analysis can be used to compare the revenue performance in this prior-free setting with the Bayesian optimal revenue. The second prior-free approximation algorithm designed is a random sampling algorithm in the worst case. The benchmark in the second approximation algorithm is the revenue of the optimal single price auction. The optimal single price auction is optimal among all auctions where the price by which the winners need pay is unique. In other words, in the optimal single price auction the winners pay the same price. The worst case approximation is more challenging since the approximation ratio is applied to every possible realization of valuations.

An important contribution of our work is the design of prior-free algorithms for auctioning digital goods with share-averse bidders. The single-sample algorithm is a constant-approximation algorithm to the revenue benchmark. In the single-sample auction, the number of winners is a decision variable. Another major contribution of this work is the technique to deal with the number of winners in a complicated single-parameter auction setting. We need to sacrifice the revenue performance for determining the number of winners in the approximations. For example, in the prior-free worst-case analysis, determining the number of winners makes the approximation even less "optimal" in comparison with the standard single-parameter problem without shareability. This is not a problem in the standard single-parameter auction because of the indivisible nature of standard products.

2 Literature Review

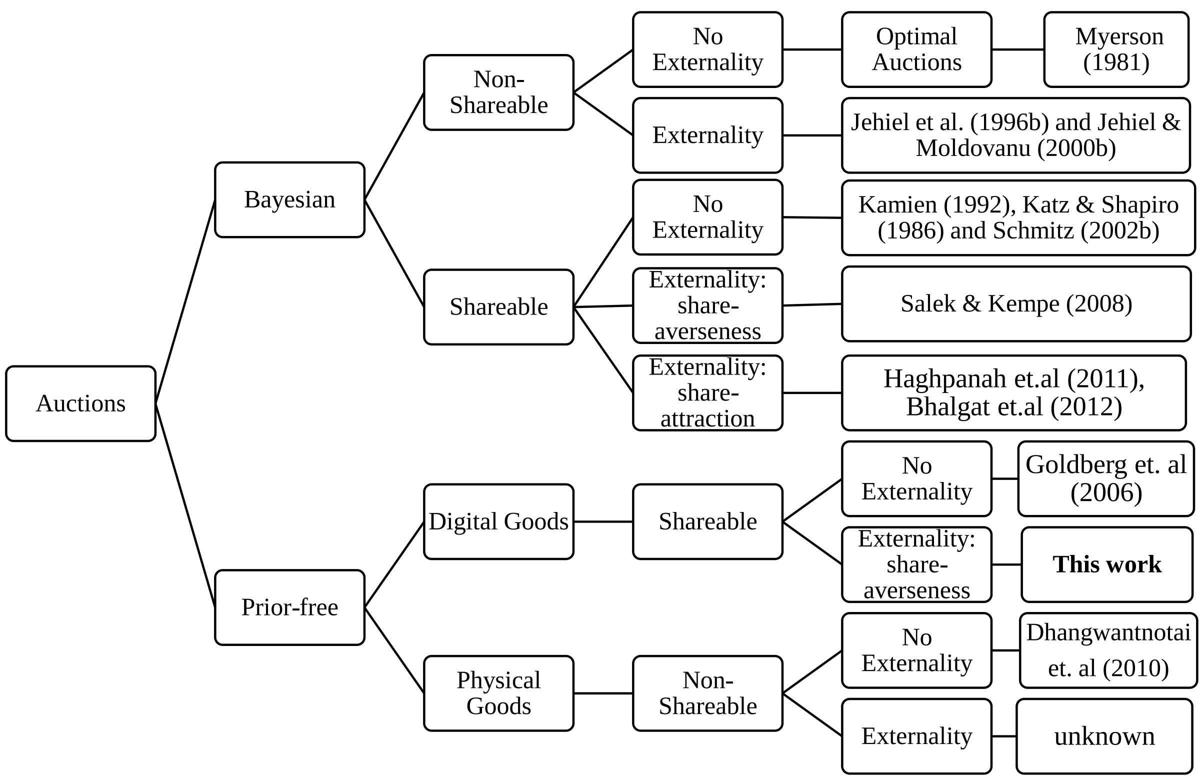

Figure 1: Summary of Literatures

We mainly differentiate auctions by two features: externality and shareability. By externality, we mean the allocation externality where a party obtaining the product influences the remaining parties. Shareability is the property of one unit of the product consumed by more than one party at the same time. At the top level, we divide auctions by prior distributions: Bayesian auctions with known prior distributions and prior-free auctions without any prior distributions. The traditional approach to auction design is to study optimal auctions (i.e., revenue-maximizing auctions) in the Bayesian setting, Myerson (1981). There are cases in the Bayesian setting where shareability has been taken into account, e.g., the patent licensing. The problem of licensing an innovation to firms that are competitors in a downstream market has been well studied. Kamien (1992) provides an excellent survey of patent licensing. Katz & Shapiro (1986) show a licensing game in which the bidders are identical and their signals are publicly observable. In our work, bidder's signal of willingness to pay is private. Schmitz (2002) analyzes a revenue-maximizing auction for a sale of multiple licenses where each bidder's signal is private. All these papers assume no allocation externality, i.e., a firm who gets a license does not affect other firms obtaining no licenses. Our study adds allocation externality into the setting. Our work is also related to the literature on sales with externality. Both Jehiel *et al.* (1996) and Jehiel & Moldovanu (2000) discuss auctions with externality. However, neither considers the shareability of a product at the same time, i.e., it is impossible to share the product in the auction. We allow multiple bidders to share a product. Salek & Kempe (2008) is closest to our work. They study auctions in which items being auctioned can be shared among multiple winners, and the valuation of winners decreases in the number of winners. They exhibit an optimal truthful auction for a single item in the sense of Myerson. We advance this by studying the prior-free auctions. In addition, in the Myerson's setting our model is different because we do not allow fractional allocations. On the flip side of externality, i.e., positive externality or share-attraction, Haghpanah *et al.* (2011) and Bhalgat *et al.* (2012) show how to model bidder's preference regarding positive externality in a social network setting.

Literature on prior-free auction design, the focus of our work, is rare. In practice, the Bayesian approach is restrictive since the prior distribution is usually unknown. A prior-free mechanism design improves understanding of the auction without the assumption of prior distributions. Goldberg *et al.* (2006) investigate such a prior-free mechanism design problem where the monopolist has a constant marginal cost of supplying units. They completely eliminate the prior distribution assumption in their analysis. Dhangwantnotai *et al.* (2010) propose single-sample approximations for a prior-free mechanism design. We apply the single-sample techniques to the auctions for digital goods with share-averse bidders and analyze their performance.

The paper is structured as follows. In Section 3 we state the model for auctioning digital goods with share-averse bidders and show the Bayesian optimal mechanism design in Section 4. We then focus on the approximation algorithms and their analysis in Section 5. We conclude the introduction with a literature review.

3 Model

We model the sales of a digital good with share-averse bidders by an auction. In the models that follow we use the following notation. We use bold letters to denote vectors. Let $\mathcal{N}$ be the set of all bidders and n be the total number of bidders, i.e., $|\mathcal{N}| = n$. Let $\mathbf{v} = (v_1, ..., v_n)$ be the vector of strict valuations of all bidders for the single digital good with share-averse bidders. Namely, v_i is the valuation of bidder i as the individual winner and v_i is drawn from continuous distribution G_i, which we assume are i.i.d. Let $g(\cdot)$ be the corresponding probability density function of $G = G_i$. We denote by $\mathbf{v}_{-i} = (v_1, ..., v_{i-1}, v_{i+1}..., v_n)$ the mask vector after removing bidder i's value and $\mathbf{v}_{-i-j} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_{j-1}, v_{j+1}, \cdots, v_n)$ the mask vector after removing both bidder i and j's values. Similarly, we denote the joint distribution function without bidder i by mask vector G_{-i}. We assume that the distribution has the monotone hazard rate property, i.e., $h(\cdot) = \frac{g(\cdot)}{1-G(\cdot)}$ is increasing. We denote $\phi(v_i) = v_i - \frac{1-G(v_i)}{g(v_i)}$ as the virtual value. Let $f(\cdot) : \mathbb{Z} \to \mathbb{R}$ be a decreasing function such that $0 \leq f(\cdot) \leq 1$. This function models the disutility a bidder obtains from the product being shared with other bidders. For example, if the auction awards 2 copies of the product, then the value to agent i is $v_i \cdot f(2)$. Since we have a finite number of bidders, there exists a (possibly non-unique) optimal number of bidders k such that the revenue obtained by selling to those k bidders is greater than any other number. In case of a tie, we always pick the smaller k. In this setting, we can model bidder's utility using utility function $u_i(\mathbf{v}, k) = x_i(\mathbf{v}) \cdot v_i \cdot f(k) - p_i(\mathbf{v})$, where $x_i \in \{0, 1\}$ is the allocation of an item to agent i or not and $p_i \in \mathbb{R}_+$ is the payment of bidder i to the seller. Finally, for simplicity we assume that all feasible allocations are single units, i.e., there is no additional utility in a bidder receiving multiple copies or a fraction of a copy. This is a reasonable assumption for digital goods with share-averse bidders, given that they are allocated as discrete units.

We denote by

$$Q_i(v_i) = \int_{\mathbf{v}_{-i}} x_i(\mathbf{v}) f(\sum_j x_j(\mathbf{v})) dG_{-i}(\mathbf{v}_{-i})$$

the conditional allocation to bidder i by which we model the incentive constraint in the Bayesian setting. The seller's problem is to determine a subset $S \subseteq \mathcal{N}$ of bidders to allocate to while maintaining the Bayesian incentive constraint and individual rationality. We use the following lemma to simplify the seller's objective, i.e., maximizing the total payment from all bidders.

Lemma 1 (Myerson's Lemma)**.** *For every truthful mechanism* $(\mathbf{x}, \mathbf{p})$*, the expected payment of bidder* i *with valuation distribution* G *satisfies* $E_{\mathbf{v}}[p_i(\mathbf{v})] = E_{\mathbf{v}}[\phi(v_i) x_i(\mathbf{v}) f(\sum_j x_j(\mathbf{v}))]$.

The seller's problem is

$$\max_x \quad E_{\mathbf{v}}[\sum_{i=1}^{n} \phi(v_i) x_i(\mathbf{v}) f(\sum_j x_j(\mathbf{v}))] \tag{1}$$

$$Q_i(v_i) \quad \text{is monotone in } v_i \text{ for every } i \tag{2}$$

$$u_i(\mathbf{v}, k) \geq 0 \quad \text{for every } i, \mathbf{v} \text{ and } k \tag{3}$$

$$Q_i(v_i) \geq 0 \quad \text{for every } i,\, v_i. \tag{4}$$

Note that objective (1) is the total payment from all bidders by Lemma 1. In a single-parameter Bayesian setting it is known that for a monotone allocation there exists a payment scheme that maintains the incentive constraint. Hence incentive constraint (2) requires monotonicity of Q. Constraints (3) and (4) are standard non-negativity requirements.

We denote by $\phi_{(i)}$ the ordered virtual values, i.e., $\phi_{(i)} \geq \phi_{(i+1)}$ for $i = 1, 2, \cdots, n-1$. The seller's objective (1) can be further reduced to

$$\max_{S \in 2^{\mathcal{N}}} \mathbb{E}\left[f(|S|) \sum_{i \in S} \phi_{(i)} \right]$$

while maintaining a monotone allocation where the seller charges each winner the critical payment.

The VCG auction allocates items in a socially optimal manner, while ensuring each bidder receives at most one item. This system charges each individual the externality they cause to other bidders and ensures that the optimal strategy for a bidder is to bid the true valuation. Mathematically, the VCG auction is a pair $(\mathbf{x}, \mathbf{p})$ such that $\mathbf{x}$ maximizes $\sum_i x_i(\mathbf{v}) v_i f(k)$ where $k = \sum_i x_i(\mathbf{v})$, and $p_i(\mathbf{v}) = \max_{x'} \left[f(\sum_{j \neq i} x'_j(\mathbf{v})) \sum_{j \neq i} x'_j(\mathbf{v}) v_j \right] - \sum_{j \neq i} x_j(\mathbf{v}) v_j f(k)$. It can be seen that the optimal auction in the Bayesian setting is a VCG auction on virtual valuations. We next study the optimal auction in Section 4.1 and use the VCG auction to design our prior-free algorithms for digital goods with share-averse bidders in Section 5.

4 Bayesian Optimal Mechanism

Bayesian optimal mechanism design is well studied for the standard single-parameter setting where the number of units of goods is determined. The auction for digital goods with share-averse bidders introduces a new variable, i.e., the number of goods to be allocated (or the number of winners), which does not exist in the standard setting. We explicitly model this variable in Bayesian optimal mechanism design. In this section, we study an optimal mechanism that mainly involves computing k^*, the optimal number of winners and the "critical-type" payments. We further show that the optimal allocation rule is monotone.

Consequently, the critical type payment rule exists by the Myerson's lemma and we show that the VCG virtual payment is not as simple as the Vickrey payment in the standard k-unit auction.

4.1 Computing allocation and payments

To derive an optimal mechanism, we first relax the incentive constraint and solve the non-game theoretic optimization problem. We then show that the resulting mechanism is also incentive compatible. The optimal mechanism is as follows.

1. Solicit and accept sealed bids $\mathbf{b} = (b_1, ..., b_n)$.
2. Compute $\boldsymbol{\phi} \leftarrow$ the virtual values over $\mathbf{b}$.
3. $(x, p') \leftarrow VCG(\phi, k^*)$, i.e., apply the VCG auction on virtual values, where k^* is the optimal number of bidders to allocate 1 unit of the good.
4. For all i, set $p_i \leftarrow \phi_i^{-1}(p'_i)$.

Each ϕ_i is computed at b_i. In step 3, we obtain the optimal k^* by solving the following problem (where $\phi_{(i)}$ are the sorted virtual values, i.e., $\phi_{(i)} \geq \phi_{(i+1)}$ for every i)

$$k^* = \arg\max_k \{f(k) \sum_{i=1}^{k} \phi_{(i)}\}.$$

The algorithm calculates the profit maximizing set of allocations x that are awarded to a set of bidders $S(b)$ corresponding to the top k^* virtual values. The virtual payments p'_i are obtained as

$$p'_i(\phi) \quad = \quad f(k^-) \sum_{\substack{j \in S(\mathbf{b_{-i}}) \\ j \neq i}} \phi_j - f(k^*) \sum_{\substack{j \in S(\mathbf{b}) \\ j \neq i}} \phi_j,$$

where $S(\mathbf{b_{-i}})$ is the winner set of re-running the VCG auction after removing bidder i and $k^- = |S(\mathbf{b_{-i}})|$. If $k^* = 1$, then we can reduce the problem to a regular single-item auction, where the second price with reserve value $\phi^{-1}(0)$ is an optimal revenue maximizing auction. It is straightforward to see that the algorithm runs in $O(n^3)$ time.

4.2 Analysis of the optimal auction

We now analyze the optimal auction mechanism (allocation and payment) computed in the previous section. We first show incentive compatibility and then discuss the payment rule.

Recall the interim allocation $Q_i(v_i) = \int_{\mathbf{v}_{-i}} x_i(\mathbf{v}) f(\sum_j x_j(\mathbf{v})) dG_{-i}(\mathbf{v}_{-i})$. In order to show that mechanism (x, p) is incentive compatible it suffices to show that the interim allocations Q_i are monotone. Without loss of generality, we assume that the values are decreasingly ordered, i.e., $v_1 \geq v_2 \geq \cdots \geq v_n$. Let $v_i < v_i^{'}$. If $x_i(\mathbf{v}_{-i}, v_i) = 0$ for some $\mathbf{v}_{-i}$, then $x_i(\mathbf{v}_{-i}, v_i^{'})$ can be either 0 or 1. It remains to show that when $x_i(\mathbf{v}_{-i}, v_i) = 1$ for some $\mathbf{v}_{-i}$, then $x_i(\mathbf{v}_{-i}, v_i^{'}) = 1$ and the total number of winners corresponding to $v_i^{'}$ is no bigger than the number with v_i. Suppose with valuation v_i bidder i is one of the winners and the total number of served bidders is k^*. Then clearly the virtual welfare function is maximized at k^*, i.e., $f(k^*) \sum_{j=1}^{k^*} \phi_{(j)}$ is maximal, see Figure 2. Note that the virtual welfare function in Figure 2 by assumption is discrete, but for the purpose of illustration, we smooth the curves in the next three figures.

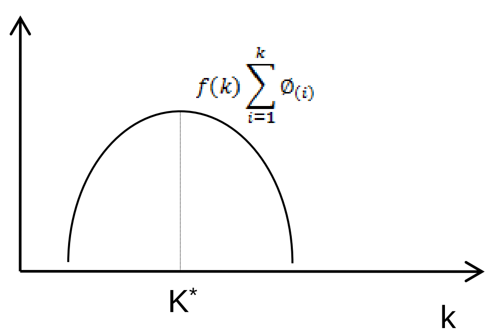

Figure 2: The optimal allocation of k^* winners

We distinguish two cases. In the first case $v_{i-1} \geq v_i^{'} \geq v_i \geq v_{i+1}$. In other words, changing value for bidder i from v_i to $v_i^{'}$ does not change the ranking of the values. The difference of the virtual welfare functions before and after changing the value from v_i to $v_i^{'}$ is decreasing

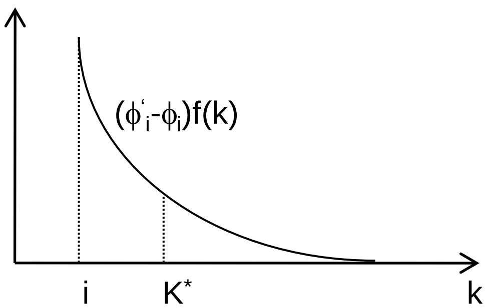

Figure 3: The difference between virtual welfare functions

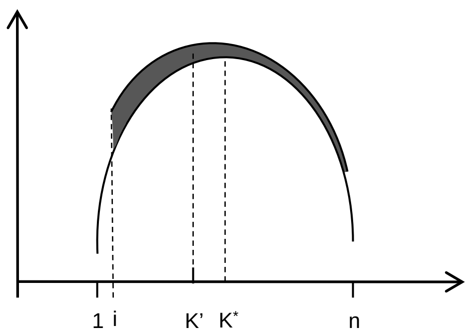

Figure 4: Monotonicity of $Q_i(v_i)$: Case 1

in k for $k \geq i$, see Figure 3. It shows that the optimal number of winners k' corresponding to $(v_1, v_2, \cdots, v_{i-1}, v_i^{'}, v_{i+1}, \cdots, v_n)$ is no bigger than k^*. Further, it is between i and k^*, see Figure 4. The shaded area in Figure 4 represents the increase of virtual welfare in the number of winners. The upper boundary curve is the virtual welfare function corresponding to $v_i^{'}$ and the lower boundary the virtual welfare function for v_i. Bidder i is also one of the winners after changing the value to $v_i^{'}$. Hence, the interim allocation is non-decreasing. On the other hand, suppose increasing the value for bidder i from v_i to $v_i^{'}$ does improve his ranking in the second case. Following the similar argument as in the first case, it is not hard to show that the total number of winners k' is still no larger than the number of winners k^*. Now it is possible that $k' < i$ depending on the new ranking based on $v_i^{'}$. However, bidder i is in the winner set. This completes the statement on monotonicity.

We next show that computing the VCG virtual payment p' is not the same (i.e., as simple) as the Vickrey payment in the standard k-unit auction. The differences between the VCG and Vickerey payments will also be illustrated later in the prior-free setting. The optimal allocation rule x combined with the standard Vickrey payment that is restricted to k units may be infeasible. Next we exhibit an instance with the Vickrey payment in which a bidder can be better off by reporting a non-truthful valuation. Suppose for bidder i with true valuation v_i his allocation associated with reporting v_i is winning together with total k bidders. It implies that his payoff $u_i(v_i, v_i)$ of reporting the true valuation is

$$u_i(v_i, v_i) = f(k)[v_i - max\{r, v_{(k+1)}\}].$$

Recall the reserve price $r = \phi^{-1}(0)$. Suppose instead bidder i reports a higher value $v_i^{'}$, i.e., $v_i^{'} > v_i$. We have

$$u_i(v_i, v_i^{'}) = f(k^{'})[v_i - max\{r, v_{(k^{'}+1)}\}],$$

where $u_i(v_i, v_i^{'})$ corresponds to a new allocation with total $k^{'}$ winners. We already discussed the allocation of the auction mechanism corresponding to increasing bids. Suppose that the new allocation has the number of winner $k^{'} < k$. We might have $u_i(v_i, v_i^{'}) > u_i(v_i, v_i)$. In other words, when bidder i increases his reported valuation he improves his interim allocation (leading to a potentially larger payment). At the same time, he also lifts up the bar, namely winning with fewer winners. This might eventually improve his payoff.

Consider the following numerical example. There are 3 bidders where values are independently drawn from uniform distribution on $[0, 1]$. We assume $f(k) = \frac{1}{\sqrt{k}}$ for $1 \leq k \leq 3$, which is decreasing and convex. The reserve price is $r = 0.5$ and the virtual value is $\phi(v_i) = 2v_i - 1$ for $i = 1, 2, 3$. Let the input value profile be $\mathbf{v} = (0.8, 0.625, 0.6)$. We correspondingly have the virtual value $\phi(\mathbf{v}) = (0.6, 0.25, 0.2)$. It is easy to see that the optimal number of winners is $k = 3$. Bidder 1's payoff of reporting true valuation is $u_1(v_1, v_1) = f(k)[v_1 - r_1]$, i.e., $u_1(0.8, 0.8) = f(3)[0.8 - 0.5] = 0.173$. On the other hand, suppose now bidder 1 reports $v_1^{'} = 1$. We can similarly compute that $k^{'} = 1$ and bidder 1's corresponding payoff is $u_1(v_1, v_1^{'}) = f(k^{'})[v_1 - v_{(k^{'}+1)}]$, i.e., $u_1(0.8, 1) = 0.8 - 0.625 = 0.175$. We conclude that

$u_1(0.8, 1) > u_1(0.8, 0.8)$.

5 Prior-free Approximations

The optimality of the Bayesian algorithm in Section 4 depends on the prior distribution of bidder's valuations. The algorithm is based on the VCG auction on virtual values. There are a variety of appealing reasons to consider prior-free auctions, starting with the fact that determining the prior distributions is very costly and unreliable. For this reason in this section we focus on the design of prior-free algorithms. We show constant approximations in both average- and worst-case scenarios.

5.1 An average-case constant-approximation

Let us consider an algorithm in which a single reserve price is randomly picked and VCG with respect to values is conducted on the sub-economy of the remaining bidders excluding the reserve bidder. The single sample algorithm includes steps:

1. Solicit and accept bids $\mathbf{b} = (b_1, ..., b_n)$
2. Randomly pick a reserve bidder i (i.e., bid b_i is to be used as the reserve price for remaining bidders in the next step)
3. Run the VCG auction with the lazy random reserve price b_i on $\mathbf{b_{-i}}$, i.e., $(x_{-i}, t_{-i}) \leftarrow VCG_{b_i}(\mathbf{b_{-i}})$. In other words, we first run the VCG allocation on b_{-i} and then apply the random reserve price on the allocation x_{-i}. This yields the winner set $W^{VCG(\mathbf{b_{-i}})}$.
4. Charge each winner j in the set $W^{VCG(b_{-i})}$ the maximum between the VCG payment t_j and $f(k')b_i$, where $k' = |W^{VCG(b_{-i})}|$.

In step 3, the allocation x_{-i} is determined by choosing the top m bidders by their bids in the set of all bidders excluding bidder i, where

$$m = \arg\max_k \{f(k) \sum_{j=1}^{k} b_{(j)} : b_j \in \mathbf{b_{-i}} \quad and \quad b_{(j)} \geq b_{(j+1)}\}.$$

If bidder j is one of the top m bidders and also has a value $v_j > v_i$, which is the selected reserve price, then bidder j belongs to the winner set $W^{VCG(\mathbf{b_{-i}})}$. The direct VCG payment rule in step 4 assures that our approximation algorithm is incentive compatible. Hence, we can use b_i and v_i interchangeably. The VCG payment rule calculates the payments based on

$$t_j(\mathbf{v}_{-i}) = f(|S_{-j-i}|) \sum_{\substack{l \in S_{-j-i} \\ l \neq i \\ l \neq j}} v_l - f(|S_{-i}|) \sum_{\substack{l \in S_{-i} \\ l \neq i}} v_l,$$

where S_{-j-i} is the winner set induced by running the VCG auction on $\mathbf{v_{-j-i}}$ and likewise S_{-i} is the winner set induced by running the VCG auction on $\mathbf{v_{-i}}$.

We show below that the expected revenue from the single sample algorithm is approximately optimal compared to an optimal auction with respect to the original environment with n bidders. We denote by $E_v[OPT(\mathbf{v})]$ the revenue benchmark which is the expected revenue of an optimal mechanism, i.e., the objective (1) in Section 3. We denote by $E_v[VCG_{v_i}(\mathbf{v_{-i}})]$ the expected revenue of the single sample algorithm with respect to the induced values $\mathbf{v_{-i}}$. The result is stated in the following theorem.

Theorem 1. *Suppose the strict valuations v_i for any bidder i are independently and identically drawn from distribution G that satisfies the monotone hazard rate (MHR) condition. In addition, we assume function $f(\cdot)$ is linearly decreasing. Then the expected revenue of the single sample algorithm is a constant approximation to the expected revenue of an optimal auction in the original environment $\mathbf{v}$, precisely, $E_v[VCG_{v_i}(\mathbf{v_{-i}})] \geq \frac{1}{2e}\frac{n-1}{n}E_v[OPT(\mathbf{v})] \geq \frac{1}{4e}E_v[OPT(\mathbf{v})]$.*

The proof of the theorem is fairly technical. The sketch of the proof is as follows. Intuitively, we have

$$
\begin{aligned}
E_v[OPT(\mathbf{v})] &\approx E_v[OPT(\mathbf{v_{-i}})] && (5)\\
&= E_v[VCG(\phi(\mathbf{v_{-i}}))] && (6)\\
&\approx E_v[VCG_{r^*}(\mathbf{v_{-i}})] && (7)\\
&\approx E_v[VCG_{v_i}(\mathbf{v_{-i}})], && (8)
\end{aligned}
$$

where

- $E_v[OPT(\mathbf{v_{-i}})]$ is the expected revenue from an optimal mechanism with respect to environment $\mathbf{v_{-i}}$ induced by random reserve bidder i;
- $E_v[VCG(\phi(\mathbf{v_{-i}}))]$ is the expected revenue of the VCG mechanism with respect to induced virtual values $\phi(\mathbf{v_{-i}})$;
- $E_v[VCG_{r^*}(\mathbf{v_{-i}})]$ is the expected revenue of the VCG mechanism with monopoly reserve price r^* with respect to induced values $\mathbf{v_{-i}}$.

Next we provide a complete proof including several partial results and observations. We first make a few observations.

Observation 1. *The virtual welfare function and the welfare function are sub-modular in the number of winners.*

Let us assume that the virtual values are decreasingly ordered, i.e, $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_n$. We define the virtual welfare function by $L(k) = f(k)\sum_{i=1}^{k}\phi_i$. It is straightforward to verify that $L(k)$ is submodular in k. Further, $L(k+1) - L(k)$ is strictly decreasing since $f(k) - f(k+1)$ is non-decreasing. The case of welfare function $LV(k) = f(k)\sum_{i=1}^{k} v_i$ is similar.

Observation 2. *Based upon the same economy* $\mathbf{v}$ *the total number of winners in the VCG auction with monopoly reserve price with respect to bid values is no smaller than the number of winners with respect to virtual values.*

To see this, it suffices to show that for every k at which the virtual welfare $L(k)$ is increasing, so is $LV(k)$. Mathematically, we want to show that $L(k+1) - L(k) \geq 0$ implies $LV(k+1) - LV(k) \geq 0$ for every k. In other words, we want to show for every such k that

$$\frac{\sum_{i=1}^{k} \phi_i}{\phi_{k+1} + \sum_{i=1}^{k} \phi_i} \geq \frac{\sum_{i=1}^{k} v_i}{v_{k+1} + \sum_{i=1}^{k} v_i}. \tag{9}$$

Since $L(k+1) - L(k) \geq 0$ is equivalent to $\frac{f(k+1)}{f(k)} \geq \frac{\sum_{i=1}^{k} \phi_i}{\phi_{k+1} + \sum_{i=1}^{k} \phi_i}$, which from (9) implies that $LV(k+1) - LV(k) \geq 0$. As a conclusion, it suffices to show (9).

Let us consider $v_1 \geq v_2 \geq \cdots \geq v_{k+1}$. By the monotone hazard rate condition, i.e., the hazard rate $h(v)$ is monotone increasing in v, we have

$$\frac{1}{h(v_1)} \leq \frac{1}{h(v_2)} \leq \cdots \leq \frac{1}{h(v_{k+1})}.$$

This implies $-v_{k+1}[\frac{1}{h(v_1)} + \cdots + \frac{1}{h(v_k)}] \geq -[v_1 + \cdots + v_k]\frac{1}{h(v_{k+1})}$ and in turn

$$\begin{aligned} &[v_1 + \cdots + v_k][v_1 + \cdots + v_{k+1}] - [v_1 + \cdots + v_{k+1}][\frac{1}{h(v_1)} + \cdots + \frac{1}{h(v_k)}] \\ &\geq [v_1 + \cdots + v_k][v_1 + \cdots + v_{k+1}] - [v_1 + \cdots + v_k][\frac{1}{h(v_1)} + \cdots + \frac{1}{h(v_{k+1})}]. \end{aligned}$$

This further implies that

$$\frac{[v_1 + \cdots + v_k] - [\frac{1}{h(v_1)} + \cdots + \frac{1}{h(v_k)}]}{[v_1 + \cdots + v_{k+1}] - [\frac{1}{h(v_1)} + \cdots + \frac{1}{h(v_{k+1})}]} \geq \frac{v_1 + \cdots + v_k}{v_1 + \cdots + v_{k+1}}$$

and

$$\frac{\phi_1 + \cdots + \phi_k}{\phi_1 + \cdots + \phi_{k+1}} \geq \frac{v_1 + \cdots + v_k}{v_1 + \cdots + v_{k+1}}.$$

This shows the second observation.

Observation 3. *The allocation $x(\mathbf{v})$ in the optimal auction is monotone. Namely, for any i and fixed $\mathbf{v}_{-i}$, if bidder i increases the bid from v_i to $v_i^{'}$, i.e., $v_i^{'} > v_i$, then we have two cases: (1) $x_i(\mathbf{v}_{-i}, v_i) = 0$ and $x_i(\mathbf{v}_{-i}, v_i^{'}) \in \{0, 1\}$, or (2) $x_i(\mathbf{v}_{-i}, v_i) = 1$ and $x_i(\mathbf{v}_{-i}, v_i^{'}) = 1$, but the total number of winners is non-increasing.*

It is straightforward to show monotonicity when $x_i(\mathbf{v}_{-i}, v_i) = 0$ and bidder i increases the bid from v_i to $v_i^{'}$. For bidder i with strict value v_i, if $x_i(\mathbf{v}) = 1$ and bidder i increases the bid from v_i to $v_i^{'}$, it is easy to argue that $x_i(v_i^{'}, \mathbf{v}_{-i}) = 1$ and the total number of winners $\sum_j x_j(v_i^{'}, \mathbf{v}_{-i})$ is not larger than the number with v_i. On the other hand, if $x_i(\mathbf{v}) = 1$ and bidder i reduces the bid from v_i to $v_i^{''}$, it can easily be seen that either $x_i(v_i^{''}, \mathbf{v}_{-i}) = 0$ (i.e., bidder i drops out) or $x_i(v_i^{''}, \mathbf{v}_{-i}) = 1$ and the total number of winners $\sum_j x_j(v_i^{''}, \mathbf{v}_{-i})$ is not lower than the number corresponding to v_i. This shows the third observation.

We continue the overall proof with the following proposition that demonstrates the expected revenue of an optimal auction after randomly throwing out a bidder from the original economy $\mathbf{v} = (v_1, \cdots, v_n)$.

Proposition 1. *The expected revenue of the optimal auction on sub-economy v_{-i} after randomly removing a bidder, for example i, is at least $\frac{n-1}{n}$ fraction of the expected revenue of the optimal auction on $\mathbf{v}$, i.e., $E_v[OPT(\mathbf{v_{-i}})] \geq \frac{n-1}{n} E_v[OPT(\mathbf{v})]$.*

Proof. Given an input valuation profile $\mathbf{v} = (v_1, \cdots, v_n)$, let us suppose the winner set is $W^{OPT(\mathbf{v})}$ in the original economy with $|W^{OPT(\mathbf{v})}| = k$. Since the reserve bidder i is selected independently of all valuations, each bidder in the winner set $W^{OPT(\mathbf{v})}$ is a non-reserve bidder with probability $\frac{n-1}{n}$. Conditioning on the valuations, the expected (over the choice of reserve bid v_i) virtual welfare of an optimal auction on the sub-economy $\mathbf{v_{-i}}$ induced by reserve bidder i is at least a $\frac{n-1}{n}$ fraction of the expected virtual welfare of an optimal auction on the original economy $\mathbf{v}$. To be specific, if the chosen reserve bidder is outside the winner set (i.e., this occurs with probability $\frac{n-k}{n}$) the expected virtual welfare of the optimal auction on the sub-economy is identical to that on the original economy. On the other hand, if the chosen reserve bidder i is one of the winners (each with probability $\frac{1}{n}$), the optimal auction can be modeled by replacing bidder i's bid with 0. By applying Observation 3, the virtual welfare of the optimal auction on the sub-economy can be computed as

$$\begin{aligned}
\text{Virtual Welfare}(\mathbf{v_{-i}}) &\geq \frac{n-k}{n}\left[f(k)\sum_{t=1}^{k}\phi_{(t)}\right] + \frac{1}{n}\left[f(k-1)\sum_{t=2}^{k}\phi_{(t)}\right] \\
&\quad + \frac{1}{n}\left[f(k-1)\sum_{\substack{t\neq 2\\ t=1,\ldots,k}}\phi_{(t)}\right] + \cdots + \frac{1}{n}\left[f(k-1)\sum_{t=1}^{k-1}\phi_{(t)}\right] \\
&\geq \frac{n-k-1}{n}\left[f(k)\sum_{t=1}^{k}\phi_{(t)}\right] + \frac{1}{n}\left[f(k)\sum_{t=1}^{k}\phi_{(t)}\right] \\
&\quad + \frac{1}{n}\left[f(k)\sum_{t=1}^{k}\phi_{(t)}\right] + \cdots + \frac{1}{n}\left[f(k)\sum_{t=1}^{k}\phi_{(t)}\right] \\
&\geq \frac{n-1}{n} f(k)\sum_{t=1}^{k}\phi_{(t)} = \frac{n-1}{n}\cdot\ \text{Virtual Welfare}(\mathbf{v}).
\end{aligned}$$

From Lemma 1, it can be seen that the optimal expected revenue is the expected virtual welfare. Hence, we apply a linear transformation and conclude that the expected revenue satisfies $E_v[OPT(\mathbf{v_{-i}})] \geq \frac{n-1}{n}E_v[OPT(\mathbf{v})]$. □

The relationship between the expected revenue and welfare, summarized below, in a given auction has already been established before.

Lemma 2 (Dhangwantnotai *et al.* (2010)). *For any monotone-hazard-rate distribution G the expected welfare from any individual bidder i in a truthful auction is at most e times more than the expected monopoly revenue from the same bidder, i.e.,* $E[\phi_i(v_i)] \geq \frac{1}{e}E[v_i]$.

Lemma 2 implies that the expected revenue of the VCG mechanism with lazy reserve prices is competitive with the expected optimal welfare in the single bidder case. We need to extend the above result from an individual bidder to a VCG auction. Suppose the distribution G with respect to bidder valuations $\mathbf{v}$ has monotone hazard rate. We apply a

VCG auction with the monopoly reserve price on the values. In particular, k' is the total number of winners of the VCG_{r^*} auction under a valuation profile $\mathbf{v}$ and total n bidders, i.e., $k^{'} = \arg\max_{k=1}^{n}\{f(k)\sum_{i}^{k} v_{(i)}\}$. We next show that the expected revenue of the above auction is at least a constant fraction of the expected welfare related to this auction. Recall that $v_{(i)}$ and $\phi_{(i)}$ are the order statistics of v_i and ϕ_i respectively, i.e., $v_{(i)} \geq v_{(i+1)}$ and $\phi_{(i)} \geq \phi_{(i+1)}$ for every i.

Lemma 3. *For any monotone hazard rate distribution G the expected revenue in a VCG auction with lazy monopoly reserve price is at least $\frac{1}{e}$ fraction of the expected social welfare of the same auction, i.e., $E_v[f(k^{'}(v,n))\sum_{i=1}^{k'} \phi_{(i)}] \geq \frac{1}{e} E_v[f(k^{'}(v,n))\sum_{i=1}^{k'} v_{(i)}]$.*

Proof. See Appendix for a detailed proof. □

As shown in the previous section, the Bayesian optimal auction could be interpreted as the VCG auction on the virtual values. Formally, we have $E_v[OPT(\mathbf{v_{-i}})] = E_v[VCG(\phi(\mathbf{v_{-i}}))$ for the sub-economy $\mathbf{v_{-i}}$ induced by reserve bidder i. In other words, the steps in Section 4 give us an optimal auction. We next show that the VCG auction with the monopoly reserve price on values is approximately optimal.

Proposition 2. *The expected revenue of the VCG auction with lazy monopoly reserve price r^* on values of $\mathbf{v_{-i}}$ is a constant $\frac{1}{e}$ fraction of the expected revenue of the VCG auction with monopoly reserve price r^* on induced virtual values of $\phi(\mathbf{v_{-i}})$.*

Proof. The detailed proof is deferred to Appendix. □

We are now prepared to start the analysis of the performance of the single-sample approximation algorithm. Since a winning bidder incurs disutility while sharing with other potential winners, we use $G_k(p)$ to represent the distribution function in the winning scenario with a total k winners. Note that $G_1(\cdot) = G(\cdot)$ is the regular distribution function of the strict valuation. It is straightforward to verify that $G_k(p) = G(\frac{p}{f(k)})$ and $g_k(p) = \frac{1}{f(k)} g(\frac{p}{f(k)})$. Similarly, the monopoly reserve price in the case of total k winners is $r^* f(k)$. We denote by

$R_k(p) = p[1 - G_k(p)]$ the revenue function in the case of k winners as a function of price p. For a single-bidder and single-item case, the expected revenue of the approximation algorithm with a random reserve price is at least one half of that of an optimal auction, see Lemma 3.6 in Dhangwantnotai *et al.* (2010). We have a similar result for the auction of digital goods with n bidders.

Lemma 4. *Let v_i denote a random valuation from distribution G. For any nonnegative number $t \geq 0$ and $k \geq 1$, we have*

$$E_{v_i}[R_k(max\{t, v_i f(k)\})] \geq \frac{1}{2} R_k(max\{t, r^* f(k)\}).$$

As shown in step 3 of the single sample algorithm, we use a random reserve price rather than the monopoly reserve price. The performance loss of using a random reserve price is shown by the following proposition.

Proposition 3. *The expected revenue of the VCG auction with a random reserve price $v_i f(k'(v_{-i}))$ on induced values $\mathbf{v_{-i}}$ is a 2-approximation to the expected revenue of the VCG auction with the monopoly reserve price $r^* f(k'(v_{-i}))$ on $\mathbf{v_{-i}}$. Namely, we have*

$$E_v[VCG_{v_i}(\mathbf{v_{-i}})] \geq \frac{1}{2} E_v[VCG_{r^*}(\mathbf{v_{-i}})].$$

Proof. We compare the expected revenue of an individual bidder in two different scenarios. The first one is the scenario of a single-item with a single-bidder. The expected revenue of an individual bidder j in a single-item single-bidder auction is

$$R_k(max\{t_j, r^* f(k'(v_{-i}))\}) = max\{t_j, r^* f(k'(v_{-i}))\}[1 - G_k(max\{t_j, r^* f(k'(v_{-i}))\})]$$

for any $t_j \geq 0$. The other scenario is the revenue contribution of the same individual bidder in a VCG auction on induced values v_{-i}. We next show that the revenue contribution of the same bidder j in the VCG auction with the random reserve price $v_i f(k'(v_{-i}))$ on v_{-i} is the same as in the single-item auction.

We consider a reserve bidder i. For fixed reserve price $v_i f(k'(v_{-i}))$ we condition on all valuations $\mathbf{v_{-i-j}}$ other than bidder i and j in order to compute the VCG payment t_j for any bidder $j \neq i$. For fixed $\mathbf{v}_{-i}$, every non-reserve bidder is isolated and hence evaluated by using two prices $v_i f(k'(v_{-i}))$ and $t_j(\mathbf{v_{-i-j}})$. This is equivalent to the single-item auction with random reserve price $v_i f(k'(v_{-i}))$ for the same bidder j. Hence, the expected revenue is $E_{v_i}[R_{k'}(max\{t_j, v_i f(k'(v_{-i}))\})]$. In order to show the revenue contribution of individual bidder $j \neq i$, we need to ensure that the number of the new set of winners after removing bidder j is at least the number of winners $k'(\mathbf{v_{-i}})$ on the originally induced values $\mathbf{v_{-i}}$. This point is illustrated by Observation 2. By Lemma 4, we further have

$$E_{v_i}[R_{k'}(max\{t_j, v_i f(k'(v_{-i}))\})] \geq \frac{1}{2} R_{k'}(max\{t, r^* f(k'(v_{-i}))\}).$$

Taking expectations over previously fixed valuations $\mathbf{v_{-i-j}}$, summing over all non-reserve bidders j, applying the linearity of expectation, and eventually taking expectations over the reserve bids v_i, we obtain $E_v[VCG_{v_i}(\mathbf{v_{-i}})] \geq \frac{1}{2} E_v[VCG_{r^*}(\mathbf{v_{-i}})]$. □

Now, we are ready to state the entire proof to the theorem.

Proof of Theorem 1: First, from Proposition 3 we have $E_v[VCG_{v_i}(\mathbf{v_{-i}})] \geq \frac{1}{2} E_v[VCG_{r^*}(\mathbf{v_{-i}})]$, which is inequality (8). Further, from Proposition 2 we have $E_v[VCG_{r^*}(\mathbf{v_{-i}})] \geq \frac{1}{e} E_v[OPT(\mathbf{v_{-i}})]$ that is inequality (7). Finally, Proposition 1 yields $E_v[OPT(\mathbf{v_{-i}})] \geq \frac{n-1}{n} E_v[OPT(\mathbf{v})]$, which is inequality (5). All these results yield $E_v[VCG_{v_i}(\mathbf{v_{-i}})] \geq \frac{1}{2e} \frac{n-1}{n} E_v[OPT(\mathbf{v})] \geq \frac{1}{4e} E_v[OPT(\mathbf{v})]$.

This completes the proof to Theorem 1 of the constant approximation ratio of the single-sample algorithm to the optimal mechanism in terms of the expected revenue.

5.2 Numerical analysis of average-case approximation

In Section 5.1 we presented a single sample constant approximation algorithm to the optimal expected revenue. Next by means of a numerical study we validate the performance of the single sample approximation algorithm in various settings. Each setting consists of a prior distribution of the valuation distribution G, size of the population N and decreasing function f. Let $f(k) = 1 - \frac{k-1}{N}$, where N is selected from $\{5, 10, 100\}$. We consider 3 distributions with increasing hazard rate functions. The first distribution is the continuous uniform distribution between 0 and 1. The second one is the gamma distribution with parameters $(\lambda, \alpha) = (1, 2)$ and the density function

$$g(t, \lambda, \alpha) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\tau(\alpha)}.$$

The third distribution is the truncated normal distribution between 0 and 1 with parameters of mean and standard deviation $(\mu, \sigma) = (1/2, 1/2)$, which we denote by T-normal. The three distributions are chosen in the numerical analysis because they are commonly used in practice and parameters are further configured to make the hazard rate functions monotone increasing. Notice that the expected values for the 3 distributions are the same.

The numerical analysis is summarized in Figure 5 below. The optimum column is the

Uniform	benchmark	single sample algorithm	Optimum	Approx. Ratio	95% C.I.	
N=5	0.16	0.60	1.06	0.57	0.58	0.62
N=10	0.34	1.22	2.07	0.59	1.19	1.26
N=100	3.71	12.77	20.38	0.63	12.56	12.98

Gamma	benchmark	single sample algorithm	Optimum	Approx. Ratio	95% C.I.	
N=5	0.12	0.52	0.82	0.63	0.49	0.54
N=10	0.27	1.09	1.60	0.68	1.05	1.12
N=100	2.92	11.68	16.03	0.73	11.52	11.84

T-Normal	benchmark	single sample algorithm	Optimum	Approx. Ratio	95% C.I.	
N=5	0.15	0.61	1.03	0.60	0.59	0.64
N=10	0.33	1.23	2.01	0.61	1.20	1.27
N=100	3.59	12.54	19.70	0.64	12.33	12.75

Figure 5: Numerical results in average-case analysis

optimal expected revenue shown in Section 4.1. The benchmark is set to be $\frac{1}{2e}\frac{N-1}{N}$ fraction of the optimum. The approximation ratio is defined as the ratio of the revenue from the single sample algorithm to the optimum. We also report the 95% confidence interval (C.I.) after $1,000$ replications.

In Figures 6, 7 and 8 we show that the revenue from the single sample algorithm lies between the benchmark and optimum. Further we show the approximation ratio is increasing in N. Not surprisingly, more revenue can be gained from the single sample algorithm with more bidders.

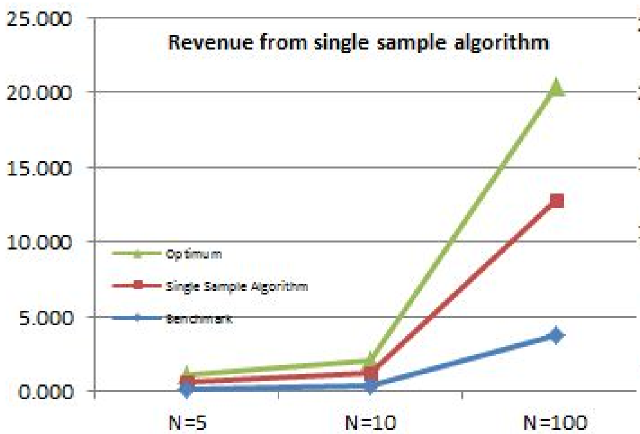

Figure 6: Uniform Distribution

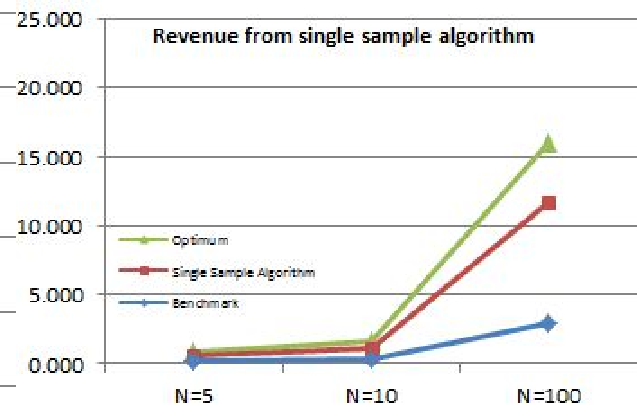

Figure 7: Gamma Distribution

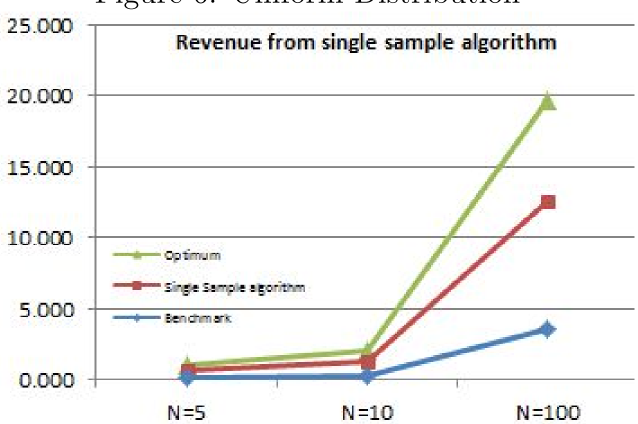

Figure 8: T-Normal Distribution

This numerical analysis shows that the single sample algorithm yields solutions that are between 55% and 80% from the optimum. It also asserts that the approximation ratio is increasing in the size of the population.

5.3 A worst-case constant-approximation

In this section, we exhibit a constant approximation algorithm for auctioning the digital goods with share-averse bidders in the worst case scenario. We conduct a sequence of two auctions. The first auction is used to learn bidders' valuations and thus no allocation of goods is actually assigned. The optimal price learned from the first auction over a subset of bidders is then applied to the remaining bidders in the second auction. In both auctions, the values are given and we choose the number of winners that maximizes the revenue in the auction. We use the random sampling technique to learn bidders' valuations from a subset of bidders and then apply the learning to the remaining bidders. The random sampling algorithm A is as follows.

1. Solicit and accept bids $\mathbf{b} = (b_1, ..., b_n)$
2. Partition the bidders into two groups: market group $s^{'}$ and sample group $s^{''}$.
   - First sort all bids in decreasing order.
   - Assign the bidder with the largest value to the market group.
   - Assign the remaining bidders to groups and an arbitrary bidder is placed in the market group with probability $\frac{1}{2}$.
3. Compute the optimal price of the auction from the sample group $s^{''}$.
   - Return the optimal price v_k where k represents the index of the winner with the smallest value in the auction.
4. Apply the optimal price v_k to the auction conducted in the market group and collect the revenue.

The random sampling algorithm is incentive compatible because the price by which the revenue is collected is determined in the sample group and does not depend on the bidder's value in the market group. For this reason value v and bid b can be used interchangeably.

We next show that the random sampling algorithm gives a constant approximation to the optimal single price auction. We partition the bidders by randomly assigning bidder i into two groups according to random variable x_i with binary values. Namely, bidder i is in the sample group if $x_i = 1$ and the market group otherwise. Recall that the bidders are sorted. Let m_i be the number of bidders in the sample group after the first i bidders have been assigned, i.e., the number of bidders whose values are greater than v_i and have been placed in the sample group. It is easy to see that the optimal revenue in the auction from the sample group is $R(s^{''}) = m_k \cdot v_k \cdot f(m_k)$, where v_k is the winning price and m_k represents the number of winners in the auction conducted in the sample group. Hence, the revenue from the market group is $R(s^{'}) = (k - m_k) \cdot v_k \cdot f(k - m_k)$. We first show a lemma which will be used later.

Lemma 5. *For any $i = 1, 2, \cdots, n$ we have $P(i - m_i \geq \frac{m_i}{3}) \geq 0.9$.*

Proof. Let $z_i = 3(i - m_i) - m_i + 1$. The boundary conditions are $x_1 = 0$ and $z_1 = 4$ since the bidder with the largest value is assigned to the market group. Now for $i \geq 2$ we define if $z_{i-1} = t$, then

$$z_i = \begin{cases} t - 1 & \text{if } x_i = 1, \\ t + 3 & \text{if } x_i = 0. \end{cases}$$

For a fixed t, we define $r_t = P(z_i \leq 0 \quad \text{for some} \quad j \leq i \leq n \quad \text{with} \quad z_j = t)$. This leads to the recursion $r_t = \frac{1}{2}(r_{t-1} + r_{t+3})$. It is easy to check that $r_t = r_1^t$ with boundary condition $r_0 = 1$ is a solution to the recursion if $r_1^4 - 2r_1 + 1 = 0$. Applying the Ferrari's method, it can be seen that $r_4 = r_1^4 \leq 0.1$. Subsequently, by the boundary condition $z_1 = 4$, for $1 \leq i \leq n$ we have

$$\begin{aligned} 1 - r_4 &= P(z_i > 0) \\ &= P(3(i - m_i) - m_i + 1 > 0) \\ &= P(i - m_i \geq \frac{m_i}{3}) \geq 0.9. \end{aligned}$$

$\square$

We denote by $\mathcal{G}_{\mathbf{v}}$ the revenue of an optimal single price auction, i.e., $\mathcal{G}_{\mathbf{v}} = \max_{1\leq i\leq n}\{i\cdot v_i\cdot f(i)\}$ and R($\mathbf{v}$) the revenue of the random sampling algorithm with input $\mathbf{v}$. The following theorem shows the constant approximation performance of the random sampling algorithm to the optimal single price auction.

Theorem 2. *The random sampling algorithm is a constant approximation to the optimal single price auction, i.e.,* $R(\mathbf{v}) \geq \frac{1}{15}\mathcal{G}_{\mathbf{v}}$ *for every* $\mathbf{v}$.

Proof. We prove the theorem in several steps. We first show that the revenue from the sample group is close to the revenue of the optimal single price auction, i.e., $R(s^{''}) \geq \frac{1}{2}\mathcal{G}_{\mathbf{v}}$. It can be seen that m_i follows the Binomial distribution with parameters $(i, \frac{1}{2})$. This implies that $P(m_i \geq \frac{i}{2}) \geq \frac{1}{2}$ for any $1 \leq i \leq n$. Under event $E_1 = \{m_k \geq \frac{k}{2}\}$ we have

$$\begin{aligned} R(s^{''}) &= m_k \cdot v_k \cdot f(m_k) \\ &\geq m_{k^*} \cdot v_{k^*} \cdot f(m_{k^*}) \\ &\geq \frac{k^*}{2} \cdot v_{k^*} \cdot f(k^*) \\ &= \frac{1}{2}\mathcal{G}_{\mathbf{v}}, \end{aligned}$$

where k^* is the number of winners in an optimal single price auction with input $\mathbf{v}$. The first inequality follows by the definition of m_k and in the second inequality we use $f(m_{k^*}) \geq f(k^*)$ since f is decreasing and $m_{k^*} \leq k^*$.

Next, we show that the revenue from the market group is also close to the revenue of the sample group. Under event E_1 and $E_2 = \{k - m_k \geq \frac{m_k}{3}\}$, the revenue from the market group satisfies

$$
\begin{aligned}
R(s^{'}) &= (k-m_k)\cdot v_k\cdot f(k-m_k) \\
&\geq \frac{m_k}{3}\cdot v_k\cdot f(k-m_k) \\
&\geq \frac{1}{3}\cdot m_k\cdot v_k\cdot f(m_k) \\
&= \frac{1}{3}R(s^{''}).
\end{aligned}
$$

The first inequality follows because of event E_2 and the second inequality is due to event E_1. Specifically, event E_1 implies that $k-m_k \leq m_k$. We have $f(k-m_k) \geq f(m_k)$ because function $f(\cdot)$ is decreasing. By Lemma 5 we have $P(E_1E_2) \geq 1-P(E_1^c)-P(E_2^c) \geq 1-0.5-0.1 = 0.4$. Finally, we have

$$
\begin{aligned}
R(s^{'}) &\geq P(E_1E_2)\frac{1}{3}R(s^{''}) \\
&\geq 0.4\cdot\frac{1}{3}\cdot\frac{1}{2}\mathcal{G}_{\mathbf{v}} \\
&= \frac{1}{15}\mathcal{G}_{\mathbf{v}}.
\end{aligned}
$$

This shows that the revenue from the market group is at least an $\frac{1}{15}$ fraction of the optimal benchmark revenue, which completes the proof. □

5.4 Numerical analysis of worst-case approximation

The worst-case approximation study is to show how far is the performance of the random sampling algorithm from $\frac{1}{15}$ fraction of the revenue of the optimal single-price auction. We use the same three distributions from Section 5.2. The total population in the worst-case numerical analysis is chosen as $N = 100$ and we again use $f(k) = 1 - \frac{k-1}{N}$.

We ran 1,000 replications and in Figures 9, 10, and 11 we report the performance of the worst-case approximation algorithm against the revenue benchmark of the optimal single price auction which is the upper bound. The lower bound is a constant proportion (i.e., 1/15 in this case) of the upper bound. The revenue from the random sampling algorithm clearly lies between the lower and upper bound.

In order to show the tightness of the bound, consider the following case. In Figure 12 we show the result under the setting of $N = 10$ and discrete distribution g defined by

$$T = \begin{cases} 0.01 & \text{with probability } 1/3, \\ 0.5 & \text{with probability } 1/3, \\ 0.99 & \text{with probability } 1/3. \end{cases}$$

It can be seen that the revenue of the random sampling algorithm in this case is close to the upper bound. The implication is that in this case, we designed a better worst-case approximation algorithm.

6 Conclusions and Future Research

Auctioning digital goods for share-averse bidders is challenging because digital goods can be auctioned in any quantity and disutility of a particular bidder of sharing the same product with others makes the problem complicated. We study auctions of digital goods for share-averse bidders that are optimal in the Bayesian setting or approximately optimal in the prior-free setting. Specifically, we first apply the single-parameter optimal auction design to digital goods for share-averse bidders and show an optimal auction algorithm. We concluded that the standard single-parameter auction design is not applicable to digital goods with share-averse bidders since the new decision variable about the final number of winners needs to be integrated in the algorithm in an optimized way. In other words, an optimal number of final winners can not be prior determined.

We next proposed a single sample algorithm in the prior-free average-case setting that is a constant approximation to the optimal auction in terms of the expected revenue. In other words, the revenue from the single sample algorithm is not worse than a constant proportion

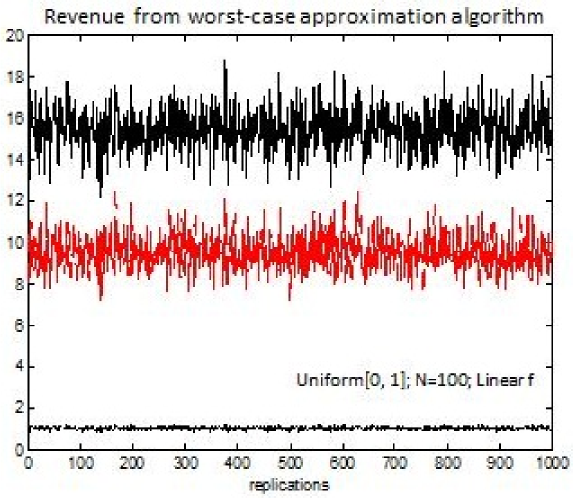

Figure 9: Uniform Distribution

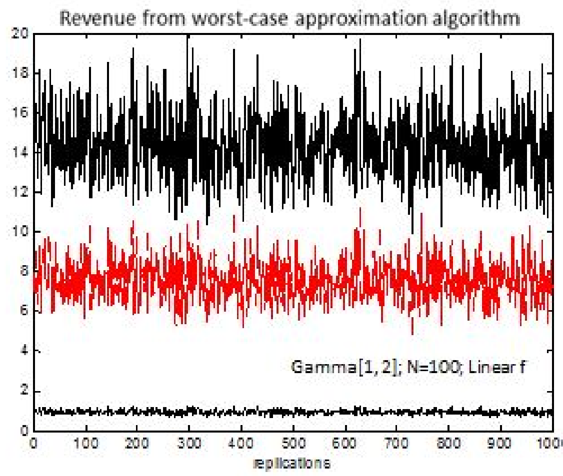

Figure 10: Gamma Distribution

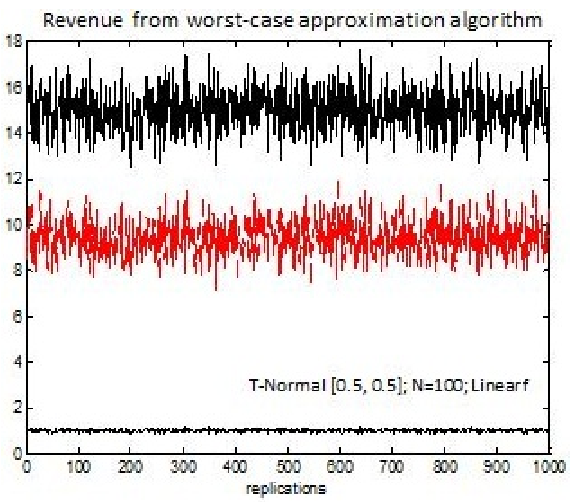

Figure 11: T-Normal Distribution

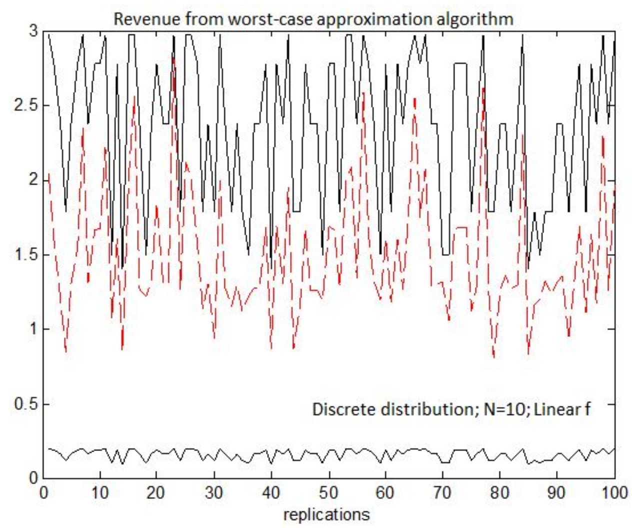

Figure 12: Discrete Distribution

of the optimal revenue in expectation. It can be much better than $\frac{1}{4e}$ of the optimal revenue as the numerical study establishes when $N = 100$ in Section 5.2. We concluded that in prior-free setting, we can design a good auction algorithm of selling digital goods for share-averse bidders by using a single sample to determine the winner set and selling prices.

We further exhibited a constant approximation algorithm in the prior-free worst-case setting. The worst-case bound is distribution independent, thus practically useful in selling digital goods with share-averse bidders. In most cases, the worst-case bound is not tight as shown in our numerical analysis. It remains an interesting open question to design an algorithm with a tighter bound, even for such distributions as those shown in Figure 12.

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Appendix

Proof to Lemma 3:

Proof. We want to show that $E_v[f(k^{'}(v,n))\sum_{i=1}^{k'}\phi_{(i)}] \geq \frac{1}{e}E_v[f(k^{'}(v,n))\sum_{i=1}^{k'}v_{(i)}]$. It suffices to show $E_v[\sum_{i=1}^{k'}\phi_{(i)}] \geq \frac{1}{e}E_v[\sum_{i=1}^{k'}v_{(i)}]$.

Consider the distribution of $\sum_{i=1}^{k'}\phi_{(i)}$ and $\sum_{i=1}^{k'}v_{(i)}$, which is a convolution of distributions of $\{\phi_{(i)}\}_i$ and $\{v_{(i)}\}_i$, respectively. We know that the convolution of distributions with monotone hazard rate functions is still of monotone hazard rate, e.g., Theorem 3.2 in Barlow *et al.* (1963). Since the valuation distribution G has a monotone hazard rate, the bidder with the ith largest virtual value $\phi_{(i)}$ among n bidders is the same person as the one who has the ith largest value $v_{(i)}$. Thus it suffices to show $E[\phi_{(i)}] \geq \frac{1}{e}E[v_{(i)}]$ for every i regardless of the number of winners. Let us denote $g_{(i)}(t)$ as the density and $G_{(i)}(t)$ as the cumulative distribution function of the ith largest value in $\mathbf{v}$. By applying known facts about standard order statistics of $\mathbf{v}$, we obtain

$$g_{(i)}(t) = n\binom{n-1}{n-i}G(t)^{n-i}[1-G(t)]^{i-1}g(t),$$

and

$$G_{(i)}(t) = \sum_{k=0}^{i-1}\binom{n}{k}G(t)^{n-k}[1-G(t)]^{k}.$$

These equations can be derived from Result 2.3 in Gungor *et al.* (2009). It remains to show that $v_{(i)}$ has a distribution function with a monotone hazard rate. Then by invoking Lemma 2 it follows that $E[\phi_{(i)}] \geq \frac{1}{e}E[v_{(i)}]$ for every i. The hazard rate function of $v_{(i)}$ is

$$\begin{aligned} h_{(i)}(t) &= \frac{g_{(i)}(t)}{1-G_{(i)}(t)} \\ &= \frac{n\binom{n-1}{n-i}G(t)^{n-i}[1-G(t)]^{i-1}g(t)}{1-\sum_{k=0}^{i-1}\binom{n}{k}G(t)^{n-k}[1-G(t)]^{k}}. \end{aligned}$$

It suffices to show that $h_{(i)}(t)$ is increasing in t. Let $p = 1 - G(t)$ and consider

$$\begin{aligned} \frac{1}{h_{(i)}(t)} &= \frac{\sum_{k=i}^{n} \binom{n}{k} p^k [1-p]^{n-k}}{n\binom{n-1}{n-i} p^{i-1}[1-p]^{n-i} g(t)} && (10) \\ &= \frac{1}{g(t)} \int_0^p \left(\frac{x}{p}\right)^{(i-1)} \left(\frac{1-x}{1-p}\right)^{(n-i)} dx, && (11) \end{aligned}$$

which further implies

$$\frac{1}{h_{(i)}(t)} = \frac{p}{g(t)} \int_0^1 y^{(i-1)} \left(\frac{1-yp}{1-p}\right)^{(n-i)} dy.$$

Equality (11) follows because $P(Beta(i, n+1-i) \leq p) = P(Binomial(n,p) \geq i)$. We note that $h(t) = \frac{g(t)}{p}$ is the hazard rate function of the original distribution of v which is increasing. It is also easy to see that $\frac{1-yp}{1-p}$ is decreasing in $[0, 1]$. Thus, we obtain $E[\phi_{(i)}] \geq \frac{1}{e} E[v_{(i)}]$ for every i and hence $E_v[f(k^{'}(v,n)) \sum_{i=1}^{k'} \phi_{(i)}] \geq \frac{1}{e} E_v[f(k^{'}(v,n)) \sum_{i=1}^{k'} v_{(i)}]$. □

Proof to Proposition 2:

Proof. We denote by $W^{OPT(\mathbf{v_{-i}})}$ the winner set in an optimal auction and by $W^{VCG(\mathbf{v_{-i}})}$ the winner set in the VCG auction with monopoly reserve price r^*. By the Myerson's lemma, we have that the expected revenue from an optimal auction is

$$E_v[OPT(\mathbf{v_{-i}})] = E_v\left[f(k^*(v,n)) \sum_{i \in W^{OPT(\mathbf{v_{-i}})}} \phi_i\right],$$

where $k^*(v,n)$ is the total number of winners in an optimal auction for a given valuation profile $\mathbf{v_{-i}}$ and total number of n bidders. Likewise, the expected revenue from the VCG auction is

$$E_v[VCG_{r^*}(\mathbf{v_{-i}})] = E_v\left[f(k^{'}(v,n)) \sum_{i \in W^{VCG(\mathbf{v_{-i}})}} \phi_i\right],$$

where $k^{'}(v,n)$ is the total number of winners in the VCG auction for a given valuation profile $\mathbf{v_{-i}}$ and total number of bidders n. This yields

$$\begin{aligned}
\frac{E_v[VCG_{r^*}(\mathbf{v_{-i}})]}{E_v[OPT(\mathbf{v_{-i}})]} &= \frac{E_v\left[f(k^{'}(v,n))\sum_{i\in W^{VCG(\mathbf{v_{-i}})}}\phi_i\right]}{E_v\left[f(k^*(v,n))\sum_{i\in W^{OPT(\mathbf{v_{-i}})}}\phi_i\right]} \\
&= \frac{E_v\left[f(k^{'}(v,n))\left(\sum_{i\in W^{OPT(\mathbf{v_{-i}})}}\phi_i+\sum_{i\in W^{VCG(\mathbf{v_{-i}})}\setminus W^{OPT(\mathbf{v_{-i}})}}\phi_i\right)\right]}{E_v\left[f(k^*(v,n))\sum_{i\in W^{OPT(\mathbf{v_{-i}})}}\phi_i\right]} \\
&= \frac{E_v[f(k^{'}(v,n))\sum_{i=1}^{k^{'}}\phi_{(i)}]}{E_v[f(k^*(v,n))\sum_{i=1}^{k^*}\phi_{(i)}]},
\end{aligned}$$

which further implies that

$$\frac{E_v[VCG_{r^*}(\mathbf{v_{-i}})]}{E_v[OPT(\mathbf{v_{-i}})]} \geq \frac{1}{e}\frac{E_v[f(k^{'}(v,n))\sum_{i=1}^{k^{'}}v_{(i)}]}{E_v[f(k^*(v,n))\sum_{i=1}^{k^*}\phi_{(i)}]} \tag{12}$$

$$\geq \frac{1}{e}\frac{E_v[f(k^{'}(v,n))\sum_{i=1}^{k^{'}}v_{(i)}]}{E_v[f(k^*(v,n))\sum_{i=1}^{k^*}v_{(i)}]} \tag{13}$$

$$\geq \frac{1}{e}. \tag{14}$$

Inequality (12) follows from Lemma 3, and inequality (13) is satisfied due to $v_{(i)} \geq \phi_{(i)}$ for every i. Inequality (14) follows by the definition of the VCG auction, i.e., $f(k^{'}(v,n))\sum_{i=1}^{k^{'}}v_{(i)}$ is socially optimal. This completes the proof. $\square$