Business-to-Consumer Single Period Fulfillment Strategies

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Order fulfillment is vital for successful business-to-consumer e-commerce firms. Customers place orders from various geographically disperse locations or, in the presence of multiple distribution channels, from various channels. The firm must make a tactical decision from which fulfillment centers to serve all of the forecasted demand and how much to procure for the forthcoming season. We present a single period distribution model that captures different customer locations or channels, each one with its own stochastic demand. The distribution cost for shipping goods from fulfillment centers to customers and the procurement costs are captured together with the associated revenue. We develop and analyze two approximate models, of which one can be analytically solved. In addition, we give a lower bound on the exact model that is used in the computational study. The solutions from the approximate models are reasonably close to the lower bound and they give an improvement over the strategy of assigning a unique fulfillment center to each demand location.

1. Introduction

In late nineties we witnessed a boom in e-commerce. The proliferation of internet led to new business concepts such as business-to-business exchanges, portals, e-procurement, online auctions, and business-to-consumer strategies. Especially the latest paradigm of business-to-consumer had the most significant influence on everyday life of consumers since it involves them directly. In addition, online retailing is rising quickly; from $8 billion in revenues generated by the US retailers in 1998 to $90 billion in 2004, Grosso et. al. (2005). Many firms use internet to sell directly to consumers, i.e., they use the direct channel. While direct sales are not confined to internet (catalog and mail sales exist for a long time), they definitely became the most widespread direct channel. The pioneers in this direction are the book retailer Amazon.com and the computer manufacturer Dell who originally was selling only through the direct online channel before cooperating with Best Buy.

On the other hand, there are several firms that use various distribution channels. Barnes & Noble, another book retailer, uses its stores as a distribution channel and it offers a direct channel through its own web site. Several firms, e.g., Barnes & Noble, Gap, Best Buy, Levi Strauss & Co., used or are using both channels: the direct channel and the reseller channel. It is expected, due to lower distribution costs, that the price in a direct channel to be lower than the one in the reseller channel. However, not to undercut and interfere with retail sales, several firms price their products in the direct channel equal to or above their reseller price. Estee Lauder offers the same price on their Clinique.com web site (Machlis (1998b)) and Nike sells online for the retail price (King (1999)). Mattel Toys (Bannon (2000)) and Intuit (Machlis (1998a)) even offer higher prices on their web sites.

To boost profitability, order fulfillment processes are critical. Amazon.com and Dell are two examples of efficiency machines when it comes to fulfillment. When an online order is placed, the firm faces the decision from which fulfillment center to ship. Fulfillment centers can be stores and warehouses owned by the company or outsourced facilities to carriers such as FedEx Express and UPS. For example, Amazon.com has 19 fulfillment centers and the fulfillment decision factors in real-time order data and ship dates in order to develop optimal pick, pack, and ship processes. Which fulfillment center to use is an important decision especially when the shipping cost is covered by the company. This is usually the case when the customer selects the 'standard' shipping option.

For seasonal items such as apparel, the firm must make the procurement or manufacturing decision before the season starts. During the season, the demand realizes in several sales locations, which are geographically dispersed. In the case of multiple channel operations, each channel can have several sales locations. Pure e-tailers have only a single direct channel and therefore sales locations can be identified with geographic markets. In addition to the procurement or manufacturing cost, the firm incurs the shipping cost, which depends on the sales location. Clearly the firm wants to minimize the distribution cost. In addition, the revenue stream comes from selling the product at the regular price and the potential income from selling at the markdown or salvage value in case the procurement or production quantity exceeds the demand.

In this paper we lay down a modeling framework for such tactical planning of seasonal items. The model has two intriguing components. The first one is that the demand at sales locations can be fulfilled from any fulfillment center. Thus, we allow, for example, that only a fraction of the demand be fulfilled from one fulfillment center and a different fraction of the demand from another fulfillment center. The second component is in modeling the optimal shipping cost, which depends on the realized demand and the procurement or production quantity. Unfortunately, the underlying model is very hard to analyze analytically and therefore we develop two approximate models. One of them can be analytically analyzed, i.e., the desired quantities can be explicitly computed and used as a heuristic. In addition, we develop a lower bounding procedure on the value of the optimal model. Numerical experiments performed on randomly generated instances show that the heuristic on average performs reasonably well.

The main novelties of this work are the two components described above. They yield an intriguing version of the standard newsvendor problem, which is analyzed by employing an approximation and then decomposition. The numerical experiments show that the proposed heuristic produces better results than the standard heuristic of assigning a sales location to a unique fulfillment center (thus not allowing the demand from a sales location to be split among several centers). The improvement increases with the increased variability in the demand.

In Section 2 we first present the underlying model. Next the two approximate models are provided and we show that both of them yield an upper bound. Section 3 focuses on one of the two approximate models by giving an explicit solution to the model. In Section 4 we give a lower bound on the exact model, which is then used in computational experiments in Section 5. Conclusions are wrapped up in Section 6. We finish the introduction with a literature review.

Literature Review

Most of the related research concerns the multi-channel studies of firms. Many authors study decentralized systems with multi-channels. The main question is under what conditions is beneficial to establish a direct channel. Another important question is the pricing strategies for the direct and the reseller channel, i.e., should the price be the same and if not, how to set them up. Tsay and Agrawal (2000), Chiang et. al. (2003), Tsay and Agrawal (2004a), and Jain et. al. (2005) study cons and pros of using both channels in a decentralized system. They also compare three possible scenarios: the firm has only a direct channel, the firm has only the reseller channel, and the firm is using both channels. Cattani et. al. (2003) consider also different price setting strategies between the two channels. Similarly, Boyaci and Gallego (2002) study pricing and channel profits in a single warehouse multiple store setting. Boyaci (2005) assumes that channels are differentiated based on the location and the channel related demand is substitutable. The distribution cost is not a factor. Bernstein et. al. (2005) study the impact of setting up a channel to 'taste' the product, which then hopefully induces additional reseller demand. Cattani et. al. (2004) and Tsay and Agrawal (2004b) provide recent surveys related to this line of research.

There is also limited inventory management literature in multi-channel settings. Unfortunately in a multi-channel setting soon the underlying models become very hard to analyze. Chiang and Monahan (2005) study the two echelon continuous review model with a single direct channel. The reseller channel consists of a warehouse that supplies a single retailer. The retailer has exogenous demand where customers shop at the store. In addition, the direct demand is fulfilled from the warehouse. The authors study base stock policies. A similar system is studied by Allgor et. al. (2004) where several heuristics are proposed for the multi-item version of the problem. The closest work to ours is the publication by Alptekinoglu and Tang (2005). Their model captures several cross-docking depots not carrying inventory and multiple sales locations. Like our work, the demand from a sales location can be fulfilled from several depots. They study the periodic review model in which the distribution cost, the holding cost at sales locations, and backlogging costs are captured. They propose a decomposition heuristic where in the first step a fraction of each demand is assigned to a depot. In the next step they approximately solve the problem with a single depot and several sales locations. Our model is a single period model where we also capture revenue. In addition, our fulfillment centers carry inventory. Another important difference is in the allocation of the distribution costs. Alptekinoglu and Tang (2005) model explicitly the flow from a depot to the sales location. On the other hand, our model assumes an optimal shipping strategy once the demand is observed. Such a choice leads to a much more complicated shipping cost function. Our heuristic is complementary to the one presented by Alptekinoglu and Tang (2005) since we decompose the problem with respect to a single sales location.

The business-to-consumer setting is also considered in Bagga et. al. (2005). In their work a single warehouse supplies several stores, which fulfill the direct demand. They assume that a fixed order up-to-level replenishment policy is followed and they study day-to-day operations, i.e., execution planning. They do not allow demand from a location to be split among several stores, i.e., a single store must serve the entire demand from a location. They present an integer program to perform the assignment.

Since we study the single period problem, the standard newsvendor problem and its variants relate to our work as well (see e.g. Khouja (1999) for a recent survey). Of particular interest are the newsvendor networks, where several items and several locations with capacity restrictions are considered (see e.g. Van Mieghem and Rudi (2002) and the references therein). Their model is a more general model than ours from a modeling standpoint. The key difference between our paper and theirs lies in the solution methodology. Their focus is not on a practical algorithm but insights. Due to their generality with respect to locations and items, analytical solutions cannot be obtained. Our paper does provide a practical approximation algorithm with an analytical solution.

Our model generalizes the work by Swaminathan and Tayur (1998) who study how to delay product differentiation by using semi-finished products while managing broader product lines. They refer to these semi-finished products as vanilla boxes. Given a set of vanilla box configurations, the problem is to decide the inventory level of each vanilla box before demand is realized, and the allocation of vanilla boxes to assemble a variety of finished products after demand is realized. Their inventory level of each vanilla box corresponds to the procurement quantity at each facility in our model, while their allocation of vanilla boxes corresponds to the shipment decisions in our model. They use a subgradient based method to solve the underlying two stage stochastic program with recourse. The subgradient algorithm is not suited for a large number of demand realizations (Birge and Louveaux 1997). The algorithms developed in this paper apply to their model since it is a special case of our model. Furthermore, one of the proposed algorithms can be used to solve practical problems with a large number of demand realizations.

2. Models

We consider n fulfillment centers or distribution facilities operated by a single corporation, i.e., a centralized system, and m sales locations or markets in a single time period and a single item. Before the time period starts, each facility must procure or produce a certain amount of the item for which it incurs a certain cost. The procurement lead time is long such that a second procurement is not possible during the time period after observing some demand. The procurement or production cost of facility i is denoted by c_i. During the time period the demand at each market realizes. If the procurement quantity exceeds the total demand, then the surplus is salvaged with s being the salvage value. If the total demand exceeds the procurement quantity, there is no penalty for unfulfilled demand. The selling price of the item is r and it does not depend on the market. While this does not need to hold in general for multi-channel firms, we gave in the introduction several examples of firms using multiple channels, which price the product equally in all of these channels. For a pure e-retailer such an assumption holds in practice. We assume $r > c_i \geq s$ for every $i \in N$, which is a standard assumption for newsvendor type problems. The product that is sold at a market incurs additional distribution cost to ship it from the facility to the market. Let d_{ij} be the per unit distribution cost between facility i and market j. Surplus procurement quantity is salvaged at the fulfillment center and incurs no distribution cost. Stochastic demand $\tilde{D}_j$ of a particular market j can be fulfilled from several facilities, see Figure 1. A realization of the stochastic demand of market j is denoted by D_j. Once the demand realizes, we assume that an optimal shipping decision is made. The demands can be correlated; however, we do not allow substitutions. Substitutions happen in multi-channel settings, but they are very hard to quantify. The no substation assumption is easily justified in the pure e-retailer setting. We assume that the customer demand is fulfilled as long as there is sufficient supply at the facilities. Let $N = \{1,...n\}$ be the set of all distribution facilities and let $M = \{1,...,m\}$ be the set of all markets.

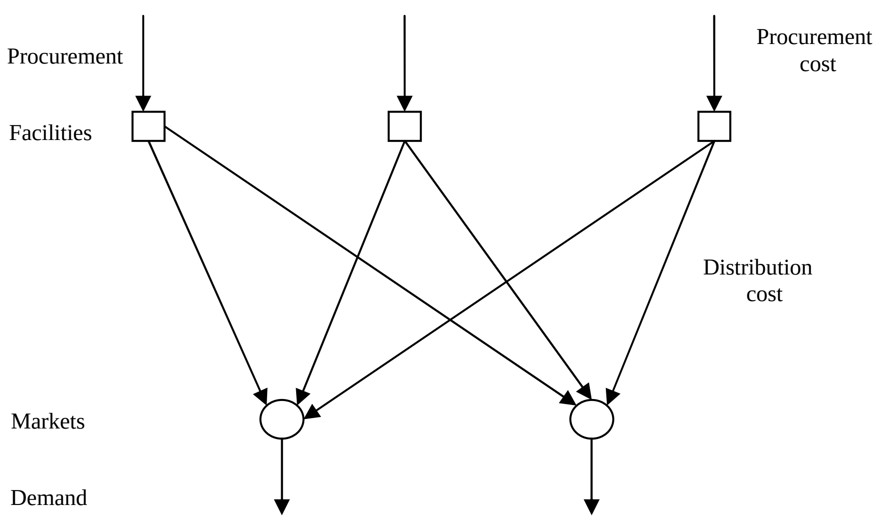

Figure 1: The material flow with n=3, m=2

A two stage stochastic linear model is formulated to find the optimal procurement quantity $\boldsymbol{y}^* = (y_1^*,..., y_n^*)$ and the optimal distribution decision $v_{ij}^*(\boldsymbol{y}^*, \boldsymbol{D})$ under each realization of demand $\boldsymbol{D} = (D_1,..., D_m)$. Here $v_{ij}(\boldsymbol{y}, \boldsymbol{D})$ is the amount shipped from distribution facility i to market j . (Vectors are denoted in bold.)

The single period profit maximization problem (we actually minimize the negative of the profit) is

$$Z = \min_{\boldsymbol{y} \geq \boldsymbol{0}} E_{\tilde{\boldsymbol{D}}} \{ \sum_{i \in N} c_i y_i + g(\boldsymbol{y}, \tilde{\boldsymbol{D}}) - r \min(\sum_{i \in N} y_i, \sum_{j \in M} \tilde{D}_j) - s(\sum_{i \in N} y_i - \sum_{j \in M} \tilde{D}_j)^+ \} . \quad (1)$$

Here we denote $\tilde{\boldsymbol{D}} = (\tilde{D}_1, ..., \tilde{D}_m)$. The first term $\sum_{i \in N} c_i y_i$ is the procurement cost. The third term $[r \min(\sum_{i \in n} y_i, \sum_{j \in M} D_j)]$ is the selling income and the forth term $[s(\sum_{i \in N} y_i - \sum_{j \in M} D_j)^+]$ equals to the salvage value. The second term $g(\boldsymbol{y}, \tilde{\boldsymbol{D}})$ is the distribution cost, which is the optimal objective value of the second stage problem derived next.

Given demand realization $\boldsymbol{D} = (D_1, ..., D_m)$ and fix procurement quantities $\boldsymbol{y} = (y_1, ..., y_n)$, the distribution cost is given by

$$g(\boldsymbol{y}, \boldsymbol{D}) = \min \sum_{i \in N, j \in M} d_{ij} v_{ij}$$

$$\sum_{j \in M} v_{ij} \leq y_i \quad i = 1, ..., n \quad (2)$$

$$\sum_{i \in N} v_{ij} \leq D_j \quad j = 1, ..., m \quad (3)$$

$$\sum_{i \in N, j \in M} v_{ij} = \min(\sum_{i \in N} y_i, \sum_{j \in M} D_j) \quad (4)$$

$$v_{ij} \geq 0 \quad i = 1, ..., n \;\; j = 1, ..., m$$

Constraints (2) impose that we do not ship more than we have available, while (3) guarantee that the shipping quantity does not exceed the demand. Constraint (4) specifies that we ship as much as possible.

Remark: Under the assumption that $r \geq d_{ij}$ for every $i \in N$ and $j \in M$, the previous model is equivalent to the following model:

$$Z = \min_{\boldsymbol{y} \geq \boldsymbol{0}} E_{\tilde{\boldsymbol{D}}} \{ \sum_{i \in N} c_i y_i - g(\boldsymbol{y}, \tilde{\boldsymbol{D}}) - s(\sum_{i \in N} y_i - \sum_{j \in M} \tilde{D}_j)^+ \},$$

where

$$g(\boldsymbol{y}, \boldsymbol{D}) = \max \sum_{i \in N, j \in M} (r - d_{ij}) v_{ij}$$

$$\sum_{j \in M} v_{ij} \leq y_i \;\; i = 1, ..., n$$

$$\sum_{i\in N} v_{ij} \le D_j \quad j=1,\dots,m$$

$$v_{ij} \ge 0 \quad i=1,\dots,n \quad j=1,\dots,m.$$

To see the equivalence between this model and model (1), note that under the assumption $r \ge d_{ij}$, the optimal shipping quantities v_{ij} of this equivalent model must satisfy constraint (4) for any given demand realization $\boldsymbol{D}$ and procurement quantities $\boldsymbol{y}$. In practice, the assumption may be violated especially for e-tailers. Hence, in what follows, we will focus on model (1), which is more general.

The exact two stage stochastic linear program (1) is hard to analyze. Analytical functional form of $g(\boldsymbol{y},\boldsymbol{D})$ cannot be derived in general. Furthermore, $g(\boldsymbol{y},\boldsymbol{D})$ is neither convex nor concave. Standard algorithms like the L-shaped method do not apply here. Therefore next we present two approximation models, namely the *production based shipping* (PBS) and *link based shipping* (LBS) models. Both of them are based on the concept of greedily approximating the distribution cost $g(\boldsymbol{y},\boldsymbol{D})$. To this end, we select and fix an order of shipping links (i, j), which we denote by $l_1, l_2, \dots, l_{m\cdot n}$. For ease of notation, let $l_k = (i_k, j_k)$ and $d_k = d_{i_k j_k}$ for $k = 1,\dots,mn$. Each shipping order can in essence gives a different PBS and LBS model. One example of a shipping order is the one corresponding to the non-increasing order of the shipping cost d, which is shown to be the optimal shipping order under some condition on d, for LBS later in the paper.

The first approximate model called the *production based shipping* model reads

$$Z_1 = \min_{\boldsymbol{y}\ge \boldsymbol{0}} E_{\tilde{\boldsymbol{D}}}\{\sum_{i\in N} c_i y_i - r\min(\sum_{i\in N} y_i, \sum_{j\in M}\tilde{D}_j) - s(\sum_{i\in N} y_i - \sum_{j\in M}\tilde{D}_j)^+ + \sum_{k=1}^{m\cdot n} d_k w_k(\boldsymbol{y},\tilde{\boldsymbol{D}})\}.$$

The only difference between the exact model and the PBS model is in replacing the term $g(\boldsymbol{y},\boldsymbol{D})$ by $\sum_{k=1}^{m\cdot n} d_k w_k(\boldsymbol{y},\boldsymbol{D})$. Here $w_k(\boldsymbol{y},\boldsymbol{D})$ is the shipping quantity on link l_k under demand $\boldsymbol{D}$ and procurement quantity $\boldsymbol{y}$. For fixed procurement vector $\boldsymbol{y}$ and demand vector $\boldsymbol{D}$, we employ the shipping strategy of first shipping as much as possible along l_1, then along l_2, and so forth. Under this shipping strategy, $w_k(\boldsymbol{y},\boldsymbol{D})$ is calculated as shown in Algorithm 1. In the algorithm, $\boldsymbol{y}^k$ and $\boldsymbol{D}^k$ are the remaining procurement quantity and demand before assigning shipment for link l_k, respectively. For $u \le v$, $\boldsymbol{e}_u^v$ is the v dimensional u'th unit vector.

Initialization: $\boldsymbol{y}^1 = \boldsymbol{y}, \boldsymbol{D}^1 = \boldsymbol{D}$
For $k = 1,..,m \cdot n$
$\quad w_k = \min\{y_{i_k}^k, D_{j_k}^k\}$
$\quad \boldsymbol{y}^{k+1} = \boldsymbol{y}^k - w_k \boldsymbol{e}_{i_k}^n$
$\quad \boldsymbol{D}^{k+1} = \boldsymbol{D}^k - w_k \boldsymbol{e}_{j_k}^m$
End For

Algorithm 1: Production based shipping

The second approximate model called the *link based shipping* model reads

$$Z_2 = \min_{\boldsymbol{x} \ge \boldsymbol{0}} E_{\tilde{\boldsymbol{D}}}\{\sum_{i \in N} c_i \sum_{j \in M} x_{ij} - r \sum_{j \in M} \min(\sum_{i \in N} x_{ij}, \tilde{D}_j) - s \sum_{j \in M} (\sum_{i \in N} x_{ij} - \tilde{D}_j)^+ + \sum_{k=1}^{m \cdot n} d_k w_k(\boldsymbol{x}, \tilde{\boldsymbol{D}})\}.$$

The second approximate model uses different decision variables. Namely, for each shipping link (i, j) we have a decision variable x_{ij}, which specifies the reserved procurement quantity of fulfillment center i for market j. Actual shipping quantity v_{ij} may be smaller than x_{ij}. In this case, the surplus amount $x_{ij} - v_{ij}$ is salvaged at fulfillment center i and cannot be used to satisfy demand at markets other than j. The procurement quantity y_i of distribution facility i is now $\sum_{j \in M} x_{ij}$. In this model the second stage function $g(\boldsymbol{y}, \boldsymbol{D})$ is approximated by $\sum_{k=1}^{m \cdot n} d_k w_k(\boldsymbol{x}, \boldsymbol{D})$. Here $w_k(\boldsymbol{x}, \boldsymbol{D})$ is the shipping quantity on link l_k under demand $\boldsymbol{D}$ and available link shipping quantity $\boldsymbol{x}$. In this setting, the total shipment $\sum_{i \in N} v_{ij}$ to market j is independent from the shipments to other markets; so is the salvage value for market j, which is now $s \min(\sum_{i \in N} x_{ij} - D_j)^+$. The distribution quantities $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{D})$ are again computed greedily based on Algorithm 2. Here $\boldsymbol{D}^k$ is the remaining demand before assigning shipment to link l_k.

Initialization: $\boldsymbol{D}^1 = \boldsymbol{D}$
For $k = 1,..,m \cdot n$
$\quad w_k = \min\{x_{i_k j_k}, D_{j_k}^k\}$
$\quad \boldsymbol{D}^{k+1} = \boldsymbol{D}^k - w_k \boldsymbol{e}_{j_k}^m$.
End For

Algorithm 2: Link based shipping

Example 1: Consider the following example with $n = 2, m = 3$, Figure 2.

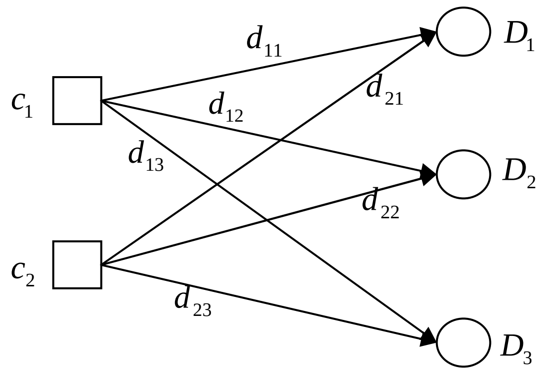

Figure 2: Data for Example 1

Let us assume that the shipping order is given by the following sequence $(1,3) \to (2,2) \to (2,3) \to (1,1) \to (2,1) \to (1,2)$, where the shipping cost is denoted as $d_1 = d_{(1,3)}, d_2 = d_{(2,2)}, d_3 = d_{(2,3)}, d_4 = d_{(1,1)}, d_5 = d_{(2,1)}, d_6 = d_{(1,2)}$. The distribution cost for Z_1 based on Algorithm 1 is as follows:

$$\sum_{k=1}^{2\cdot 3} d_k w_k(\boldsymbol{y}, \boldsymbol{D}) = d_1 \min(y_1, D_3) + d_2 \min(y_2, D_2) + d_3 \min((y_2 - D_2)^+, (D_3 - y_1)^+)$$
$$+ d_4 \min((y_1 - D_3)^+, D_1) + d_5 \min((y_2 - D_2)^+ - \min((y_2 - D_2)^+, (D_3 - y_1)^+),$$
$$D_1 - \min((y_1 - D_3)^+, D_1)) + d_6 \min((y_1 - D_3)^+ - \min((y_1 - D_3)^+, D_1), (D_2 - y_2)^+).$$

Similarly, the distribution cost for Z_2 based on Algorithm 2 is as follows:

$$\sum_{k=1}^{2\cdot 3} d_k w_k(\boldsymbol{x}, \boldsymbol{D}) = d_1 \min(x_{13}, D_3) + d_2 \min(x_{22}, D_2) + d_3 \min(x_{23}, (D_3 - x_{13})^+)$$
$$+ d_4 \min(x_{11}, D_1) + d_5 \min(x_{21}, (D_1 - x_{11})^+) + d_6 \min(x_{12}, (D_2 - x_{22})^+). \qquad \square$$

Our first result states that Z_1 and Z_2 overestimate the optimal value Z.

Theorem 1: For Z, Z_1 and Z_2 defined earlier, we have $Z \le Z_1$. In addition, if $r \ge s + d_{\max}$, where $d_{\max} = \max_{i \in N, j \in M} d_{ij}$, then we have $Z_1 \le Z_2$ under the same shipping order.

The following lemma is used in the proof.

Lemma 1: If $r-d_{\max} \geq s$, then for any n and any $x_1,...,x_n, y_1,...,y_n$, we have

$$(r-d_{max})min(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i) + s(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i)^+ \geq (r-d_{max})\sum_{i=1}^{n} min(x_i, y_i) + s\sum_{i=1}^{n}(x_i - y_i)^+ .$$

Proof: A simple case analysis proves the statement for $n=2$. The general case then follows by induction. □

Proof of Theorem 1: The only difference between the PBS model and the exact model is in obtaining the distribution quantities for fixed purchasing quantities $\boldsymbol{y}$ and demand realization $\boldsymbol{D}$. The exact model requires the optimal shipping quantities while the PBS model assumes only feasible shipping quantities. Hence we have $Z \leq Z_1$.

Given any solution $\boldsymbol{x}$ for the LBS model, we can always construct a feasible solution $y_i = \sum_{j \in M} x_{ij}$ for the PBS model under the same shipping order. Let $w_k^{PBS}(\boldsymbol{y}, \boldsymbol{D})$, $w_k^{LBS}(\boldsymbol{x}, \boldsymbol{D})$ be the shipping quantities obtained from Algorithm 1 and 2, respectively. Then

$$\sum_{k=1}^{m \cdot n} d_k (w_k^{PBS}(\boldsymbol{y}, \boldsymbol{D}) - w_k^{LBS}(\boldsymbol{x}, \boldsymbol{D})) \leq$$

$$d_{\max}(\sum_{k=1}^{m \cdot n}(w_k^{PBS}(\boldsymbol{y}, \boldsymbol{D}) - w_k^{LBS}(\boldsymbol{x}, \boldsymbol{D}))) =$$

$$d_{\max}(\min(\sum_{j \in M}\sum_{i \in N} x_{ij}, \sum_{j \in M} D_j) - \sum_{j \in M} \min(\sum_{i \in N} x_{ij}, D_j))$$

provides an upper bound on the transportation cost difference. The other cost difference is the selling income plus the salvage value (i.e., the revenue). The revenue is

$$r \min(\sum_{j \in M}\sum_{i \in N} x_{ij}, \sum_{j \in M} D_j) + s(\sum_{j \in M}\sum_{i \in N} x_{ij} - \sum_{j \in M} D_j)^+$$

for the PBS model and $r\sum_{j \in M} min(\sum_{i \in N} x_{ij}, D_j) + s\sum_{j \in M}(\sum_{i \in N} x_{ij} - D_j)^+$ for the LBS model. By Lemma 1, if $r \geq s + d_{\max}$, we know that Z_2 is no larger than the upper bound of corresponding Z_1. Hence we have $Z_1 \leq Z_2$. □

The following example shows that when $r < s + d_{\max}$, the reverse inequality $Z > Z_2$ can happen.

Example 2: Consider the example with $n=1, m=2$. Demand D_1 can be either 2 or 4 with equal probability, while demand D_2 is fixed at 1. The selling price is $r = 100$, the procurement cost is $c = 41$, and the salvage value is $s = 40$. The shipping costs are $d_{11} = 1$ and $d_{12} = 71$. Note that $r < s + d_{12}$ in this example. The optimal Z is obtained by $y = 4$, which gives

$Z = -(0.5 \times 4 \times (100 - 41 - 1) + 0.5 \times (2 \times (100 - 41 - 1) + 1 \times (100 - 41 - 71) + 1 \times (40 - 41))) = -167.5$.

The optimal Z_2 under the shipping order $(1,1) \to (1,2)$ is obtained when $x_{11} = 4$ and $x_{12} = 0$, which gives

$$Z_2 = -(0.5 \times 4 \times (100 - 41 - 1) + 0.5 \times (2 \times (100 - 41 - 1) + 2 \times (40 - 41))) = -173 < Z. \qquad \square$$

Next we give conditions under which $Z_1 = Z$. These conditions are closely related to the Monge property of the transportation problem, see e.g. Burkard (1996).

Theorem 2: Let us assume that the shipping cost has the following property:

$$d_{ij} \le d_{i,j+1} \quad \text{for all } i \in N \text{ and } j \in M, \tag{5}$$

$$d_{ij} \le d_{i+1,j} \quad \text{for all } i \in N \text{ and } j \in M, \text{ and} \tag{6}$$

$$d_{ij} + d_{i+1,j+1} \le d_{i,j+1} + d_{i+1,j} \quad \text{for all } i \text{ and } j. \tag{7}$$

Then $Z = Z_1$ for the shipping orders given either by the sequence

$$(1,1),(1,2),\ldots,(1,m),(2,1),\ldots,(2,m),\ldots,(n,1),\ldots,(n,m) \tag{8}$$

or

$$(1,1),(2,1),\ldots,(n,1),(1,2),\ldots,(n,2),\ldots,(1,m),\ldots,(n,m) \tag{9}$$

Proof: Consider $g(\boldsymbol{y}, \boldsymbol{D})$ for any $\boldsymbol{y}$ and a demand realization $\boldsymbol{D}$. Assume first that $\sum_{i \in N} y_i > \sum_{j \in M} D_j$.

Thus $g(\boldsymbol{y}, \boldsymbol{D})$ equals to the transportation problem

$$\min \sum d_{ij} v_{ij}$$

$$\sum_{j \in m} v_{ij} + s_i = y_i \qquad i = 1,\ldots,n$$

$$\sum_{i \in n} v_{ij} = D_j \qquad j = 1,\ldots,m$$

$$v_{ij} \ge 0 \qquad i = 1,\ldots,n \quad j = 1,\ldots,m$$

$$s_i \ge 0 \qquad i = 1,\ldots,n$$

Properties (5) and (7) imply that this problem has the Monge property and by the result of Hoffman (1963), the northwest corner rule is optimal. This rule corresponds to the sequence given either by (8) or (9) in the statement of the theorem. We conclude that for any $(\boldsymbol{y}, \boldsymbol{D})$ with $\sum y_i > \sum D_j$, $\boldsymbol{w}(\boldsymbol{y}, \boldsymbol{D})$ from Algorithm 1 using either the shipping order (8) or (9) is optimal to this transportation problem. By using (6) and (7), we get the same conclusion for the case $\sum y_i \le \sum D_j$ with either the shipping order (8) or (9). Hence clearly we have $Z = Z_1$. □

3. Single Market Analysis

In this section we study the problem with a single market. We first show that the LBS model decomposes with respect to markets. Then we analyze the optimal solutions of the LBS model with a single market. We first make an observation regarding the shipping order.

Note that each shipping order can in essence give a different PBS and LBS models. The optimal shipping order for Algorithm 1 depends on $\boldsymbol{D}$ and $\boldsymbol{y}$ and therefore we are not able to determine an optimal shipping order for the PBS model. On the other hand, as shown next, we can explicitly give an optimal order giving the lowest Z_2. We start by showing that the LBS model decomposes with respect to the markets.

Proposition 1: Let $\boldsymbol{x}^j = (x_{1j}, ..., x_{nj})$. Then $Z_2 = \sum_{j \in M} \min E_{\tilde{D}_j}(\hat{Z}_j(\boldsymbol{x}^j, \tilde{D}_j))$ for some functions $\hat{Z}_j : R^{n+1} \to R$.

Proof: Recall that from Algorithm 2 $D_j^{\,k}(\boldsymbol{x}, \boldsymbol{D})$ is the remaining demand of market j before assigning shipment for link l_k. We first show that $D_j^{\,k}(\boldsymbol{x}, \boldsymbol{D}) = \hat{D}_j^k(\boldsymbol{x}^j, D_j)$ for every $j \in M$, $k = 1, ..., n \cdot m$, and functions $\hat{D}_j^k : R^{n+1} \to R$. This means that D_j^k is a function of only $\boldsymbol{x}^j$ and D_j.

We prove this by induction. The statement holds for $k = 1$. Let now $k > 1$. By Algorithm 2 and induction hypothesis we have $w_k(\boldsymbol{x}, \boldsymbol{D}) = \min(x_{i_k j_k}, \hat{D}_{j_k}^k(\boldsymbol{x}^{j_k}, D_{j_k}))$, which is a function of only $\boldsymbol{x}^{j_k}$ and D_{j_k}. For $j \ne j_k$ we have $D_j^{\,k+1}(\boldsymbol{x}, \boldsymbol{D}) = D_j^{\,k}(\boldsymbol{x}, \boldsymbol{D}) = \hat{D}_j^k(\boldsymbol{x}^j, D_j)$ by the induction hypothesis. On the other hand, for $j = j_k$ we have $D_{j_k}^{\,k+1}(\boldsymbol{x}, \boldsymbol{D}) = D_{j_k}^{\,k}(\boldsymbol{x}, \boldsymbol{D}) - w_k$ $= \hat{D}_{j_k}^{\,k}(\boldsymbol{x}^{j_k}, D_{j_k}) - w_k$, which is again a function of $\boldsymbol{x}^{j_k}$ and D_{j_k}.

Above we have also proved that w_k is a function of only $\boldsymbol{x}^{j_k}$ and D_{j_k}. The statement now easily follows from the definition of Z_2. □

As opposed to the PBS model, the greedy order does give the lowest Z_2 as shown next.

Theorem 3: Among all shipping orders, the order corresponding to the non-increasing order of the shipping cost $\boldsymbol{d}$ gives the lowest value of Z_2.

Proof: By Proposition 1 it suffices to consider each market separately. Let j be fixed and let $\boldsymbol{x}^j$ be the available shipping quantities for market j. Then the optimal shipping quantity $\tilde{\boldsymbol{w}}^*$ given any demand realization D_j and procurement quantities $\boldsymbol{x}^j$ is given by

$$\min \sum_{k \in N} d_{kj} \tilde{w}_k$$

$$\sum_{k \in N} \tilde{w}_k \le D_j$$

$$\sum_{k \in N} \tilde{w}_k = \min\{D_j, \sum_{k \in N} x_{kj}\}$$

$$0 \le \tilde{w}_k \le x_{kj} \quad k \in N .$$

The optimal solution to this linear program is to ship as much as possible based on the non-increasing order of $\boldsymbol{d}$. This strategy corresponds to Algorithm 2 when the fixed shipping order corresponds to the non-increasing order of $\boldsymbol{d}$. □

From Proportion 1 and Theorem 3, the LBS model decomposes and for each model the optimal shipping order is the non-increasing order of $\boldsymbol{d}$. We conclude that it suffices to consider only a single market. For the remainder of this section, we consider the case with n distribution facilities and only a single market $j = 1$ with demand distribution $\tilde{D}$. The facilities are ordered with respect to the non-increasing order of d. Given demand realization D and fixed procurement quantity $\boldsymbol{x} = (x_1,..., x_m)$ for this market, the objective function (the negative of the profit) is

$$\hat{Z}(\boldsymbol{x}, D) = \sum_{i \in N} c_i x_i - r \min(\sum_{i \in N} x_i, D) - s(\sum_{i \in N} x_i - D)^+ + \sum_{i=1}^{n} d_i \min(x_i, (D - \sum_{j=1}^{i-1} x_j)^+) .$$

Our goal is to solve $\bar{Z}_2 = \min_{\boldsymbol{x}>\boldsymbol{0}} E_{\tilde{D}}(\hat{Z}(\boldsymbol{x}, \tilde{D}))$. By the imposed order we have $d_1 \le d_2 \le ... \le d_n$. By the standard scaling argument, we can without loss of generality assume that $E(\tilde{D}) = 1$. We also assume that $d_n + c_n \le r$. If $d_n + c_n > r$ (and from $c_n \ge s$), it follows that we can fix $x_n = 0$. This can be seen by a long but elementary argument combined with a case analysis.

Now we characterize the solutions corresponding to $\overline{Z}_2$. First we define a partial order $i \propto j$ if $c_i + d_i \le c_j + d_j$, $d_i < d_j$, and $c_i \ge c_j$. Let $i_1 \propto i_2 \propto ... \propto i_k$ be a chain. We define $\overline{\sigma}_i = \dfrac{r - d_i - c_i}{r - d_i - s}$ and $\xi(i, j) = \dfrac{c_j + d_j - c_i - d_i}{d_j - d_i}$. We say that the chain is compatible if $0 \le \xi(i_1, i_2) \le \xi(i_2, i_3) \le ... \le \xi(i_{k-1}, i_k) \le \overline{\sigma}_{i_k} \le 1$. An example of a compatible chain is as follows.

Example 3: Consider $n = 6$, a single market, and in addition let us assume that $d_1 \le d_2 \le d_3 \le ... \le d_6$, and $c_1 \ge c_2 \ge c_3 \ge ... \ge c_6$. We also have the condition $c_1 + d_1 \le c_2 + d_2$ $\le c_5 + d_5$. Consider the chain $1 \propto 2 \propto 5$. Furthermore, we assume that $\xi(1,2) \le \xi(2,5) \le \overline{\sigma}_5$. Thus the chain $1 \propto 2 \propto 5$ is compatible. □

If $S = \{i_1, i_2, i_3, ..., i_k\}$ is a compatible chain, it has a solution $\boldsymbol{x}(S)$ for $\overline{Z}_2$ with the following property. For $i \in N \setminus S$ we have $x_i = 0$. The corresponding solution for $i \in S$ is computed from

$$\int_0^{x_{i_1}} dF_{\tilde{D}} = \xi(i_1, i_2), \int_0^{x_{i_1}+x_{i_2}} dF_{\tilde{D}} = \xi(i_2, i_3), ..., \int_0^{x_{i_1}+...+x_{i_{k-1}}} dF_{\tilde{D}} = \xi(i_{k-1}, i_k), \text{ and } \int_0^{x_{i_1}+...+x_{i_k}} dF_{\tilde{D}} = \overline{\sigma}_{i_k}.$$

Here $F_{\tilde{D}}$ is the cumulative distribution function of demand $\tilde{D}$. By definition of the partial order we have $0 \le \xi(i_k, i_{k+1}) \le 1$, $0 \le \overline{\sigma}_{i_k} \le 1$. By the compatibility property, it easily follows that $x_i \ge 0$ for $i \in N$. Hence the solution defined on each compatible chain S is feasible for the LBS model with objective function $\overline{Z}_2$. For each compatible chain S, let $\alpha(S)$ be the corresponding objective value, i.e., $E_{\tilde{D}}(\hat{Z}(\boldsymbol{x}(S), \tilde{D}))$. Hence $\overline{Z}_2 \le \alpha(S)$ for any S.

The compatible chains can be chains of a single element $S = \{i_1\}$. Such a chain is compatible if and only if $0 \le \overline{\sigma}_{i_1} \le 1$. The corresponding solution is $\int_0^{x_{i_1}} dF_{\tilde{D}} = \overline{\sigma}_{i_1}, x_i = 0$ for every $i \in N \setminus \{i_1\}$. Note that this corresponds to the solution of the standard newsvendor problem with production cost c_i, revenue $r - d_i$ and salvage value s. Due to our assumptions, any single element chain is a compatible chain.

We now characterize the optimal solutions. We first need to prove convexity of the objective function.

Lemma 2: $\hat{Z}(\boldsymbol{x},D)$ is a convex function of $\boldsymbol{x}$ and therefore $E_{\tilde{D}}(\hat{Z}(\boldsymbol{x},\tilde{D}))$ is convex in $\boldsymbol{x}$.

Proof: It is easy to see from definition that $\hat{Z}(\boldsymbol{x},D)=\sum_{i\in N}c_i x_i+\sum_{i=1}^{n}h_i\, min(\sum_{j=1}^{i}x_j,D)$, where $h_i(\sum_{j=1}^{i}x_j,D)=(d_i-d_{i+1})\min(\sum_{j=1}^{i}x_j,D)$ for $i=1,...,n-1$. For $i=n$, the definition reads $h_n(\sum_{j=1}^{i}x_j,D)=-(r-d_n)\min(\sum_{j=1}^{i}x_j,D_1)-s(\sum_{j=1}^{i}x_j-D)^+$. Since $d_1\le d_2\le....\le d_n$, all h_i for $i=1,..,n-1$ are convex in *u*. The function $-h_n$ is concave since $r\ge d_n$ and $s\ge 0$. From all these observations it is clear that $\hat{Z}$ is convex. □

Theorem 3: We have $\bar{Z}_2=\min_{S\text{ compatible chain}}\alpha(S)$.

Proof: Let $\alpha=\min_{S\text{ compatible chain}}\alpha(S)$. In view of the above discussion we have $\bar{Z}_2\le\alpha$.

In order to show that $\alpha\le\bar{Z}_2$, let $\boldsymbol{x}^*$ be an optimal solution corresponding to $\bar{Z}_2$ and $S=\{i\in N\mid x_i^*>0\}$. We consider only variables in S and discard those not in S since they are fixed at zero. For ease of notation, let $S=N$. Let $U=\{\boldsymbol{x}\mid x_i>0\}$ and $u(x)=E_{\tilde{D}}(\hat{Z}(\boldsymbol{x},\tilde{D}))$. Since U is an open set and $\boldsymbol{x}^*$ is an optimal solution for $\bar{Z}_2$, we must have $\frac{\partial u}{\partial x_i}|_{x^*}=0$ for every $i\in S$. A long but straight forward calculation shows that this is equivalent to $\int_0^{\sum_{i=1}^{n}x_i^*}dF_{\tilde{D}}=\bar{\sigma}_n$ and $\int_0^{\sum_{i=1}^{k}x_i^*}dF_{\tilde{D}}=\frac{c_{k+1}+d_{k+1}-c_k-d_k}{d_{k+1}-d_k}$ for all $k=1,...,n-1$. It follows now by definition that S is a chain. In addition, since $\boldsymbol{x}^*>0$, the chain must be compatible. By Lemma 2, every local minimum is a global minimum. This shows that $\alpha(S)\le\bar{Z}_2$, which completes the proof. □

For the simple case $n=2$, i.e., when there are only two fulfillment facilities, a close formula can be derived for the optimal solution.

Corollary 1: If n = 2, then

- if $c_1+d_1\le c_2+d_2$, $d_1\le d_2$, $c_1\ge c_2$ and $\xi(1,2)\le\bar{\sigma}_2$, the optimal solution is given by

$$\int_0^{x_1+x_2} dF_{\tilde{D}} = \overline{\sigma}_2\,,\ \int_0^{x_1} dF_{\tilde{D}} = \xi(1,2),$$

- otherwise $x_1 = 0$, $\int_0^{x_2} dF_{\tilde{D}} = \frac{r-d_2-c_2}{r-d_2-s}$, or $x_2 = 0$, $\int_0^{x_1} dF_{\tilde{D}} = \frac{r-d_1-c_1}{r-d_1-s}$.

4. A Lower Bound

In this section we provide a lower bound on *Z*, i.e. on the optimal cost. The bound is based on linear programming duality.

We first provide a lower bound on the distribution cost $g(\boldsymbol{y},\boldsymbol{D})$. By linear programming duality $-\sum_{j\in M} D_j\beta_j - \sum_{i\in N} y_i\alpha_i + \lambda\min(\sum_{j\in M} D_j, \sum_{i\in N} y_i) \le g(\boldsymbol{y},\boldsymbol{D})$ for any $\boldsymbol{y},\boldsymbol{D}$ and $\boldsymbol{\alpha}\ge 0, \boldsymbol{\beta}\ge 0$, λ unrestricted with $-\alpha_i - \beta_j + \lambda \le d_{ij}$ for all $i\in N$, $j\in M$. Therefore, a lower bound of the objective function in (4) is provided by

$$E_{\tilde{D}}[-\sum_{j\in M}\tilde{D}_j\beta_j + \sum_{i\in N}(c_i-\alpha_i)y_i - (r-\lambda)\min(\sum_{j\in M}\tilde{D}_j,\sum_{i\in N} y_i) - s(\sum_{i\in N} y_i - \sum_{j\in M}\tilde{D}_j)^+] = -\sum_{j\in M} E_{\tilde{D}}(\tilde{D}_j)\beta_j$$
$$+ E_{\tilde{D}}[\sum_{i\in N}(c_i-\alpha_i)y_i - (r-\lambda)\min(\sum_{j\in M}\tilde{D}_j,\sum_{i\in N} y_i) - s(\sum_{i\in N} y_i - \sum_{j\in M}\tilde{D}_j)^+].$$

In turn, for any $\boldsymbol{\alpha}\ge 0, \boldsymbol{\beta}\ge 0$, λ unrestricted, we have

$$-\sum_{j\in M} E_{\tilde{D}}(\tilde{D}_j)\beta_j + \min_{y\ge 0} E_{\tilde{D}}[\sum_{i\in N}(c_i-\alpha_i)y_i - (r-\lambda)\min(\sum_{j\in M}\tilde{D}_j,\sum_{i\in N} y_i) - s(\sum_{i\in N} y_i - \sum_{j\in M}\tilde{D}_j)^+] \le Z.$$

It is obvious that there is an optimal solution to the second term on the left-hand side with $y_i^* = 0$ for every $i \in N/\{k\}$, where $k = \arg\min_{i\in N}(c_i - \alpha_i)$. The corresponding y_k^* is the solution to $\int_0^{y_k^*} dF_{\tilde{D}} = \frac{r-\lambda-(c_k-\alpha_k)}{r-\lambda-s}$, where $\tilde{D} = \sum_{j\in M}\tilde{D}_j$. The resulting objective value of this term is

$$(\lambda+s-r)\int_0^{y_k^*} D\,dF_{\tilde{D}}(D).$$

Since we want to find the largest lower bound, we need to solve $\Lambda = \max_{\beta,\alpha,\lambda}\{-\sum_{j\in M} E_{\tilde{D}_j}(\tilde{D}_j)\beta_j + (\lambda+s-r)\int_0^{y_k^*} D\,dF_{\tilde{D}}(D)\}$. From $-\alpha_i - \beta_j + \lambda \le d_{ij}$, $\boldsymbol{\beta}\ge 0$, and the fact that $\boldsymbol{\beta}$ should be as small as possible, we get $\beta_j = \max_{i \in N}(\lambda - \alpha_i - d_{ij})^+$ for every $j \in M$. We also need to impose $0 < \dfrac{r - \lambda - \min_{i \in N}(c_i - \alpha_i)}{r - \lambda - s} < 1$. Let

$$\Omega = \{(\boldsymbol{\alpha}, \lambda) \mid \boldsymbol{\alpha} \geq 0, \lambda \leq r - s, \lambda + \min_{i \in N}(c_i - \alpha_i) \leq r, s \leq \min_{i \in N}(c_i - \alpha_i)\}.$$

We conclude that

$$\Lambda = \max_{(\boldsymbol{\alpha},\lambda) \in \Omega} \{ -\sum_{j \in M} E(D_j) \max_{i \in N}(\lambda - \alpha_i - d_{ij})^+ + (\lambda + s - r) \int_0^{y_k^*} D dF_{\tilde{D}}(D) \}.$$

is a lower bound on Z . Since for linear programs Lagrangian duality is equivalent to standard linear programming duality, we call this bound the *Lagrangian bound*. We note that in general it is a nonlinear optimization problem to solve for the largest lower bound.

5. Numerical Experiments

In this section we present numerical experiments that evaluate the strength of the two approximations and the Lagrangian lower bound. All computational experiments were performed on a Toshiba Portégé M200 tablet PC running an Intel Pentium 1.7GHz central processing unit and equipped with a 512 Mbytes of random access memory. The operating system is Windows XP SP2. The development has been done in Microsoft Excel 2003 by using VBA. For solving optimization problems we used What’s Best 7.0 (www.lindo.com).

The exact optimization problem and the PBS model do not have an analytical solution. We have approximated their values by Monte-Carlo sampling. We generated 1,000 demand scenarios denoted by D^1, D^2, …, D^{1000} and next we solved the nonlinear optimization problem $\dfrac{1}{1000} \min_{y \geq 0} \sum_{j=1}^{1000} \hat{z}(y, D^j)$, where $\hat{z}$ is the corresponding cost function. It takes on average 60 minutes to compute this value. On the other hand, the lower bounding optimization problem and the computation of the link based shipping heuristic requires less than a second of computing time. For this reason most of the computational experiments focus on the gap of the LBS value and the lower bound. Computing these two values is computationally feasible while computing PBS by Monte-Carlo sampling is in many cases excessive.

We consider only the case with 2 distribution facilities and 2 markets. The demand is always normal and independent across the two markets. Unless otherwise stated, the mean of the demand for the two locations is equal. The two facilities and markets are selected as geographical locations and the shipping cost is proportional to the travel distance between the two locations. The baseline case has the following parameters:

$$c_1 = 44, c_2 = 42, r = 60, s = 40, d_{(1,1)} = 6.5, d_{(1,2)} = 9.5, d_{(2,1)} = 9, d_{(2,2)} = 12.$$

We investigate only the shipping order $(1,1) \to (2,1) \to (1,2) \to (2,2)$, which corresponds to the greedy order. In all of the presented instances that are different from the baseline case we always maintain the following conditions on the shipping costs:

$$d_{(1,1)} \le d_{(1,2)}, d_{(2,1)} \le d_{(2,2)}, d_{(1,1)} \le d_{(2,1)}, d_{(1,2)} \le d_{(2,2)}, d_{(1,1)} \le d_{(2,1)} \le d_{(1,2)} \le d_{(2,2)}.$$

The first four conditions are equivalent to (5) and (6) in Theorem 2. The last order imposing condition justifies the use of the greedy order.

In view of Theorem 2 the following quantity defined as $l = (d_{(1,1)} + d_{(2,2)}) - (d_{(2,1)} + d_{(1,2)})$ plays a role. We call it the *shipping factor*. If $l \le 0$, then Theorem 2 states that the PBS model is optimal. In the baseline case $l = 0$ and therefore PBS gives an optimal value.

In our first experiment we study the performance of the Lagrangian bound, the LBS model, and the PBS model. We selected the demand mean in each market to be 4000, 5000, and 6000 and we vary the standard deviation of the demand from 100 to 1,500 in increments of 100, Figure 3. In what follows, “Lag” denotes the Lagrangian lower bound. The gap is defined either as (PBS-Lag)/PBS or (LBS-Lag)/LBS. The value of LBS is obtained by letting $y_i^* = \sum_{j \in M} x_{ij}^*$ in the exact model and calculating the expected transportation cost by simulation. Here x_{ij}^* is the optimal solution from LBS model. Since $l = 0$, the first ratio corresponds to the gap between the Lagrangian lower bound and the optimal solution while the second ratio corresponds to the gap between the Lagrangian lower bound and the link based shipping heuristic. In other words, the series denoted by “Lag/PBS” denote the effectiveness of the Lagrangian lower bound and those denoted by “Lag/LBS” the quality of the link based heuristic with respect to the Lagrangian lower bound.

The series using production based shipping fluctuate due to the Monte-Carlo sampling computation. Nevertheless we believe the sample size of 1,000 is indicative. Two trends can be immediately observed. The Lagrangian lower bound gives a solid lower bound whenever the standard deviation is small. The quality of the bound deteriorates with the increasing standard deviation. A similar trend is observed with the gap of the heuristic; it keeps increasing as the standard deviation rises. Both gaps keep decreasing as the mean of the demand increases. For even larger mean values the heuristic is only within a few percent from the lower bound. We also observe that the gap between the lower bound and PBS represents a significant portion and therefore it seems that the LBS heuristic performs well. We further elaborate on this later.

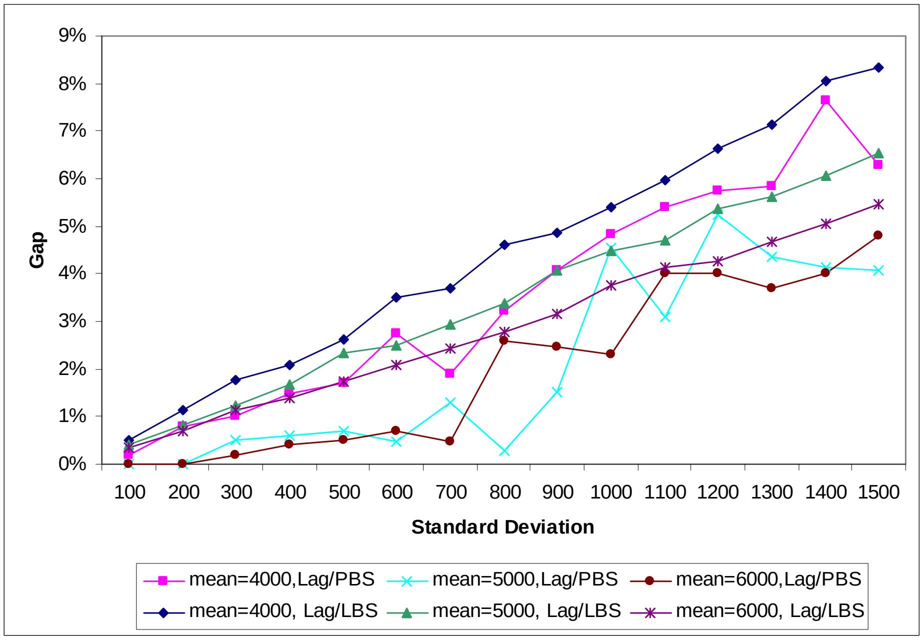

Figure 3: Performance with respect to standard deviation

Unfortunately the gaps keep growing with even larger values of standard deviation, Figure 4. In this experiment we consider only the mean of 6,000, which actually yields the smallest gaps. We observe that the gap becomes fairly large for excessive values of standard deviation. Again it seems that most of the gap in "Lag/LBS" comes from the Lagrangian lower bound.

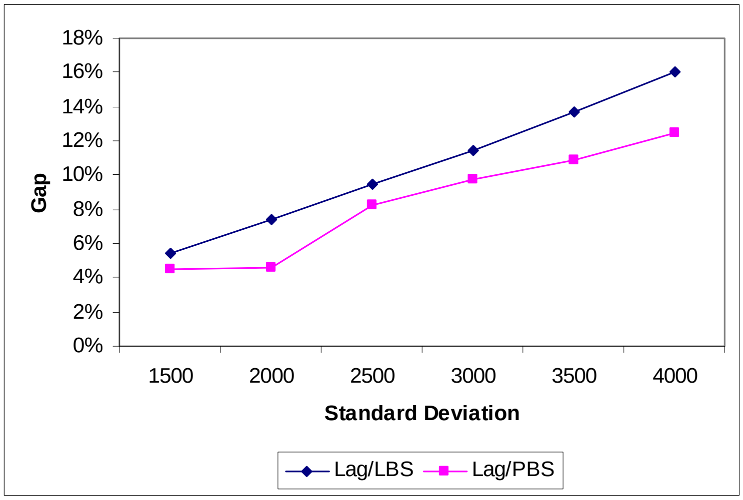

Figure 4: Performance for larger standard deviation

Next we explore the impact of shipping factor *l*, Figure 5. We consider several discrete values of *l*, which were all obtained by appropriately varying the geographical locations of markets. The demand mean is fixed at 6,000. In view of Theorem 2, we expected that the negative shipping factor would create easier problems. However this is not the case. The gap is substantial for very negative shipping factors while for positive shipping factors the gap is usually within 5%. It is interesting that even for a very small standard deviation the gap is still large. There is a slight upwards trend for l=-1.95. For this case PBS represents an optimal solution while for l=6.5 PBS might be suboptimal. In the latter case nevertheless the gap between LBS and the Lagrangian lower bound is much smaller. The natural question to ask is if the lower bound is weak or the LBS heuristic does not yield good solutions.

A further analysis of the case l=-1.95 is given in Figure 6. Here "PBS/LBS" shows the quantity (LBS-PBS)/LBS. The gap between the LBS heuristic and the optimal solution represented by PBS (recall that Theorem 2 holds here) is always lower than 10%. On the other hand the gap between the Lagrangian solution and the optimal solution (Lag/PBS) is substantial. We conclude that in this case the Lagrangian lower bound is very weak. The quality of the LBS heuristic is satisfactory.

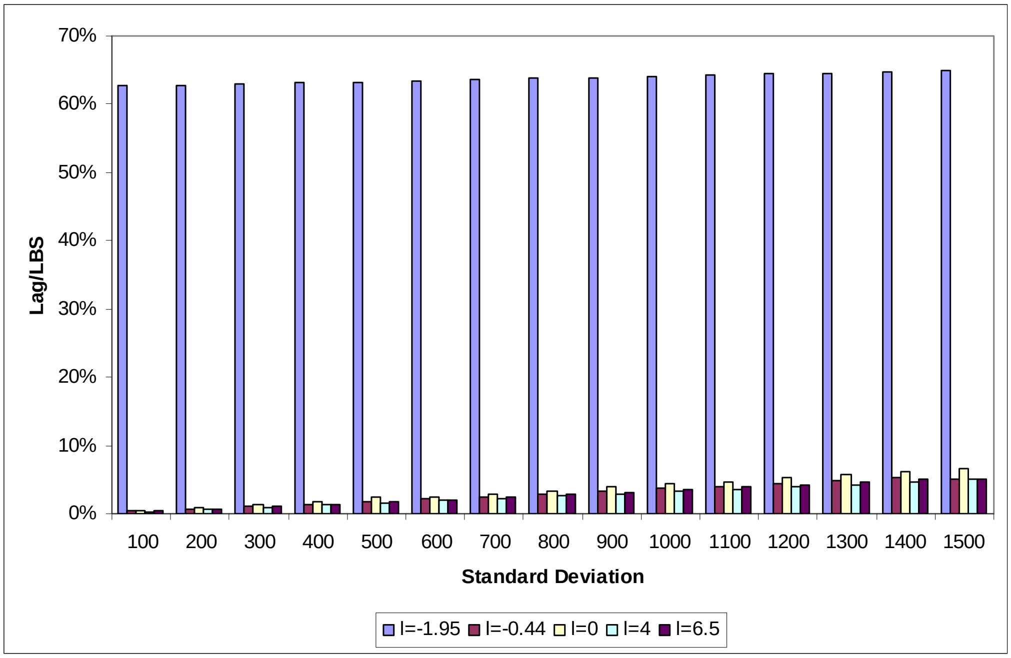

Figure 5: Dependency with respect to the shipping factor

Now we compare the LBS model with a less sophisticated method. A naïve strategy is to assign every market j to a unique distribution facility i^*, where $i^* = \arg\min_{i \in N}(c_i + d_{ij})$. Then we solve the newsvendor problem for each facility. The demand for a facility is the sum of demands of all the markets assigned to it. A distribution facility can serve more than a single market. For the 2x2 case the optimal naïve heuristic can be computed by solving 2 newsvendor problems. In Figure 7 we compare the naïve heuristic against the LBS heuristic. We consider the baseline case with the demand mean of 4,000 and we vary the shipping factor. Except for the case l=-0.44 there is an upwards trend in the gap. Thus the benefits of using the LBS heuristic increase as the standard deviation increases. The most significant benefits are obtained for low shipping factors. On average the gains are between 1% and 2%.

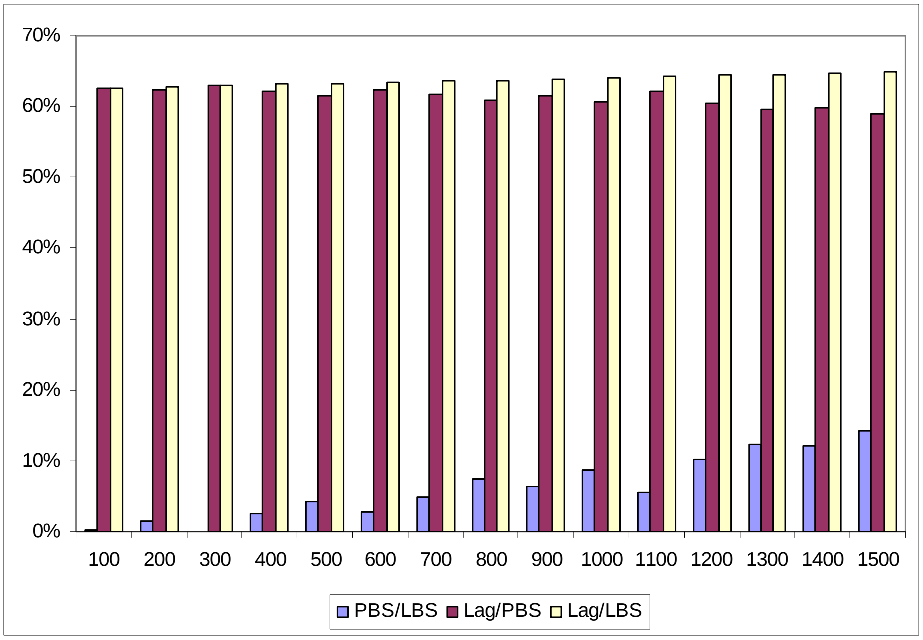

Figure 6: Detailed analysis for the shipping factor *l* = -1.95

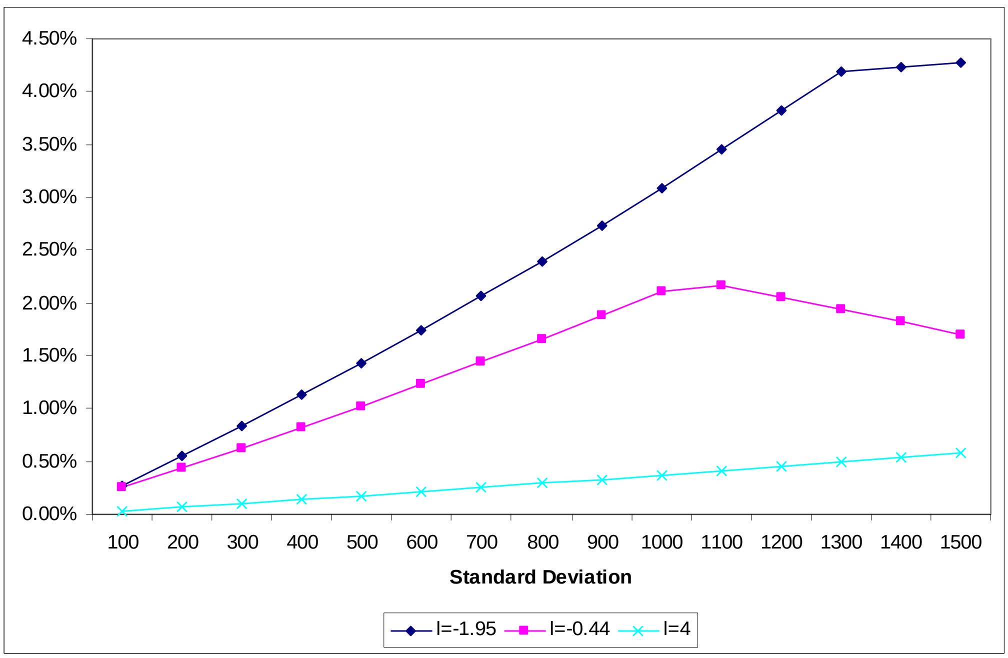

Figure 7: Naive heuristic vs. link based shipping

We have also performed a sensitivity analysis with respect to the operating margin. We define the margin to be $\max_{i\in N, j\in M}\left(r-(c_i+d_{ij})\right)$, i.e., the largest margin among all links considering the procurement and the distribution costs. Figure 8 shows that the gap between the LBS heuristic and the lower bound decreases with the increased margin. In this experiment the mean of the demand is 4,000 and the shipping factor is 0. The distribution cost and the salvage value do not vary, however the procurement cost and the selling price vary in order to achieve the desired margin. As in all cases so far, the gap increases with the increased standard deviation.

In all of these experiments, the demand distribution in both markets was equal. The last set of experiments considers different demand mean values for the two markets. All the remaining values are the same as in the baseline case. Figure 9 considers three cases: (4000,4000), (3000,5000), (2000,6000), where the first number is the mean of the first market and the second number corresponds to the mean of the second market. Note that the total demand is always 8000. The gaps do not differ substantially, nevertheless with the increased difference between the two mean values the gap increases.

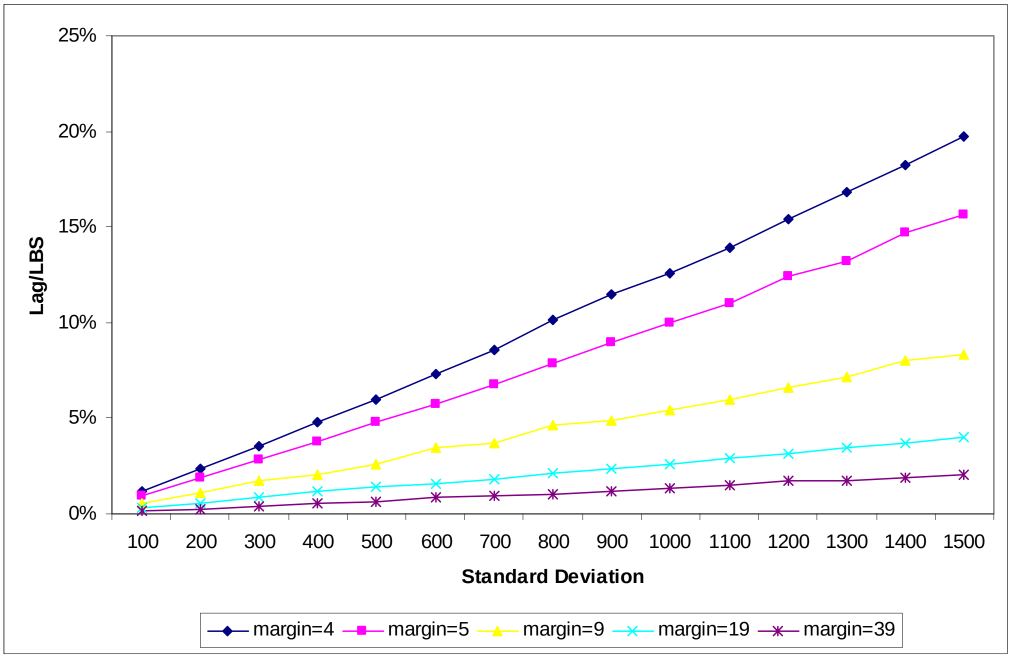

Figure 8: Gaps with respect to the margin

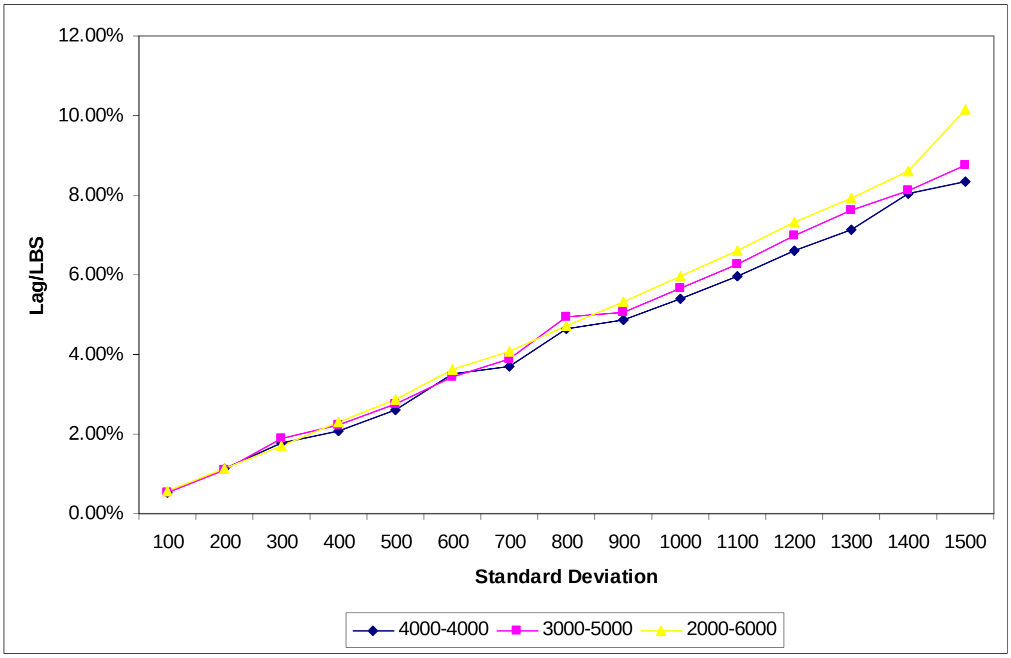

Figure 9: Different demand means for the two markets

6. Conclusions

In this work we present a heuristic using the LBS model and non-increasing shipping order for deciding procurement quantities in a multi-market environment. In addition, we provide a lower bound on the optimal value. Both of these two values can be quickly computed by today's optimization software packages.

We perform a computational study with the case consisting of 2 distribution facilities and 2 markets under demand being normally distributed. There are several important conclusions of these experiments.

1. The heuristic performs well for small standard deviations. As the standard deviation increases, the performance deteriorates.
2. The heuristic improves on the naïve heuristic by several percent. The improvements increase as the standard deviation increases. These several percent can correspond to significant savings in a season.
3. The performance of the heuristics improves with increased margins.
4. There are cases where the gap is large. Fortunately in these cases the heuristic is close to optimality and the lower bound is weak.

We conclude that there are benefits of using our heuristic instead of the naïve heuristic. On the other hand, there is still room for development of a better heuristic or solution.

The most important extension to our work is the multi-period periodic review case. It is very unlikely that analytical solutions for such cases exist. Indeed, we believe that even the multi-period extension of LBS does not have an analytical solution since the state space in the underlying dynamic program is multi dimensional.

There are also other assumptions in our model that are limiting in many real word cases.

- We do not consider a fixed cost associated with procurement.
- In many cases there is a set up cost of using a link (or in other words, of establishing a channel). These costs are neglected in our model.
- We have already argued that many companies to avoid channel conflicts offer the same selling price among all channels. There are cases where companies set up different prices for different channels. Such an extension of our model is yet to be studied.
- We do not allow for demand substitution. This is a reasonable assumption in the context of geographically dispersed multi-markets. In the multi channel setting, substitution happens. Indeed, some companies deliberately induce substitutions among channels.

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