Submitted to *Operations Research*
manuscript 1

Optimal Recharging Policies for Electric Vehicles

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Recharging decisions for electric vehicles require many special considerations due to battery dynamics. Battery longevity is prolonged by recharging less frequently and at slower rates, and also by not charging the battery too close to its maximum capacity. In this paper, we address the problem of finding an optimal recharging policy for an electric vehicle along a given path. The path consists of a sequence of nodes, each with a charging station, and the driver must decide where to stop and how much to recharge at each stop. We present efficient algorithms for finding an optimal policy in general instances and also for two specialized cases. In addition, we develop two heuristic procedures that we characterize analytically and explore empirically.

Key words: electric vehicles; optimal recharging policies; lot sizing; convex ordering cost
History: This paper was first submitted on March 5, 2013.

1. Introduction

For drivers seeking to reduce their dependence on fossil fuels, battery electric vehicles (EVs) have become a practical and affordable alternative in recent years to conventional gasoline-powered vehicles. EVs are powered by electricity only, requiring no gasoline, and connect to the electrical grid to recharge. While the ability to plug in and recharge offers the potential for significant savings in fuel costs as well as other benefits, such as fewer greenhouse gas emissions and reduced dependence on foreign oil, there are still a number of obstacles to mass EV adoption.

One reason why many buyers are reluctant to purchase an EV is range anxiety (Klabjan and Sweda 2011). The maximum range of an EV is less than that of a comparable gasoline-powered vehicle, and charging stations are scarcer than gasoline stations. Furthermore, if an EV runs out of charge along its route, there is no convenient method of recharging it from the side of the road. Spare batteries are prohibitively costly and bulky, in addition to being difficult to swap with the vehicle's depleted battery, and roadside charging services are either extremely limited or unavailable. As a result, EV drivers must be cautious when planning their routes to ensure that their vehicles do not run out of charge.

Unfamiliarity with battery dynamics also deters many potential EV purchasers. Recharging an EV's battery requires much more time than refueling a conventional vehicle, and unlike gasoline refueling, where the time required to refuel is roughly linearly related to the amount refueled (i.e., the refueling rate is constant), battery recharging occurs at a varying rate that depends on the charge level. In addition, charging too close to its maximum capacity (known as *overcharging*) can adversely affect the battery's lifespan, which is a major concern to EV owners since batteries are one of the most expensive and critical components of EVs. It is therefore important to understand the impacts of recharging decisions in order to minimize the costs of owning and operating an EV.

To overcome the aforementioned issues, this paper addresses the problem of finding an optimal recharging policy for an EV along a given path. The path consists of a sequence of nodes, each with a charging station, and the vehicle must decide where to stop and how much to recharge at each stop. Whenever the vehicle stops to recharge, it incurs a fixed stopping cost, a charging cost based on the total amount it recharges, and an additional cost when the battery becomes overcharged. The goal is to minimize the total cost of recharging along the path, including all stopping, charging, and overcharging costs. Thus, optimal recharging policies provide the most favorable tradeoff between the number of times that the vehicle stops to recharge and the amount it recharges whenever it stops.

The model presented in this work represents the first effort in the literature to optimize recharging behavior specifically for EVs. We begin by identifying several properties of optimal recharging policies along a fixed path. Using these properties, we develop efficient algorithms for finding an optimal recharging policy in the general case and in two specialized cases: when the vehicle can stop to recharge anywhere along the path (not just at prespecified nodes), and when the nodes with charging stations along the path are equidistant. We also describe two heuristic methods based on the properties of optimal paths that we use to obtain reasonable policies quickly, and we derive bounds on the quality of their solutions. To demonstrate the performance of these heuristics in practice, we implement them using actual highway data and conduct a numerical study to compare their solutions with those of optimal recharging policies.

The main contributions of this paper are: (*i*) an efficient algorithm for obtaining an optimal recharging policy for the vehicle recharging problem; (*ii*) closed-form optimal policies for instances in which either charging capability is available continuously along the path or charging stations are equidistantly spaced; (*iii*) two heuristic methods that are easy to implement and yield reasonable recharging policies with little computational effort, along with bounds on their solution quality; and (*iv*) a numerical study that demonstrates the actual performance of the heuristics using data from U.S. Interstate 90.

The remainder of the paper is organized as follows. Section 2 provides an overview of the existing literature on topics related to EV recharging. Section 3 describes the model studied in this work along with some properties of optimal recharging policies, and Section 4 details algorithms that can be used to obtain recharging policies both optimally and heuristically under different scenarios. A case study implementation of the algorithms is demonstrated in Section 5. Lastly, Section 6 summarizes the conclusions and future directions of this research.

2. Literature review

The refueling problem for gasoline-powered vehicles, where drivers must decide at which nodes to refuel as well as how much to refuel in order to minimize the total cost of fuel, has been well studied. Khuller et al. (2007) and Lin et al. (2007) show that the optimal refueling policy along a fixed path can be solved easily with dynamic programming when fuel prices at each node are static and deterministic. For such a problem, the optimal decision at each node is always one of the following: do not refuel, refuel completely, or refuel just enough to reach the next node where refueling occurs. An algorithm for simultaneously finding the optimal path and refueling policy in a network is detailed by Lin (2008a), and some combinatorial properties of the optimal policies are explored by Lin (2008b). Specifically, it is proven that the problem of finding all-pairs optimal refueling policies reduces to an all-pairs shortest path problem that can be solved in polynomial time. However, all of the aforementioned analyses only consider fuel costs and not stopping or other costs. We include these additional costs in our analysis because they can comprise a significant portion of the total cost of traveling along a path and therefore can influence optimal recharging policies.

Several models have expanded on the vehicle refueling problem by introducing costs for stopping to refuel and traveling to refueling stations. A generic model for vehicle refueling is presented in Suzuki (2008) that attempts to capture such aspects, penalizing longer routes and routes with more refueling stops. Like other papers that study the vehicle refueling problem, it assumes that fuel prices at each station are static and deterministic. Approaches for finding optimal refueling policies when fuel prices are stochastic are given by Klampfl et al. (2008) and Suzuki (2009). Klampfl et al. (2008) use a forecasting model for predicting future fuel prices to generate parameters for a deterministic mixed integer program, and Suzuki (2009) presents a dynamic programming framework that is designed to grant drivers greater autonomy to select the stations where they refuel. These models are difficult to solve analytically, and the authors develop heuristics for obtaining reasonable solutions. In addition, just like the other models of the vehicle refueling problem, these ones do not include any costs that are analogous to battery overcharging costs for EVs. Sweda and Klabjan (2012) address this issue by introducing generalized charging cost functions, but some restrictive assumptions are required in order to perform insightful analysis. In this work, we use a tractable yet realistic charging cost function that enables us to easily find optimal recharging policies and develop a deeper understanding of such policies.

Overcharging costs, incurred when an EV's battery is charged near its maximum capacity, are important to consider when creating EV recharging policies for a number of reasons. Recharging an EV battery while it is already at a high state of charge takes place at a slower rate than when it is more depleted, and storing high levels of charge for prolonged periods of time can shorten the lifespan of the battery. A couple of models describing this relation can be found in Millner (2010) and Serrao et al. (2011). Overcharging also causes battery degradation due to greater stresses from being charged near full capacity and excess heat generated during recharging. In the refueling problem for conventional vehicles, the only main disadvantage of traveling with a full tank of fuel is the limited ability to take advantage of lower fuel prices further along the route. Optimal solutions tend to favor filling large quantities and making fewer stops (assuming that stopping costs are considered), but the opposite is true for optimal EV recharging policies.

To solve the problem of finding a path for an EV within a network with recharging considerations, a recent thread of research has taken an entirely different approach, having vehicles recharge via regenerative braking rather than by recharging at stations along their paths. As an EV decelerates, it can recapture some of its lost kinetic energy as electrical energy, which can then be used to recharge the battery. It is therefore possible in some cases for an EV's state of charge to *increase* while traveling rather than decrease, such as when the vehicle is coasting and braking downhill. Artmeier et al. (2010) model the problem of finding the most energy-efficient path for an EV in a network as a shortest path problem with constraints on the charge level of the vehicle, such that the charge level can never be negative and cannot exceed the maximum charge level of the battery. Edge weights are permitted to be negative to represent energy recapturing from regenerative braking, but no negative cycles exist. A simple algorithm for solving the problem is provided, and more efficient algorithms are presented by Eisner et al. (2011) and Sachenbacher et al. (2011). Eisner et al. (2011) show that the battery capacity constraints can be modeled as cost functions on the edges, and a transformation of the edge cost functions permits the application of Dijkstra's algorithm. The approach described by Sachenbacher et al. (2011) avoids the use of preprocessing techniques so that edge costs can be calculated dynamically, and it achieves an order of magnitude reduction in the time complexity of the algorithm from Artmeier et al. (2010). In practice, however, the amount of energy recovered by regenerative braking is insignificant compared with the amount that must be recharged at charging stations, and these papers do not model recharging decisions at nodes. Consequently, they also do not capture overcharging costs considered in the presented work.

One model type that is well suited for capturing overcharging costs is the inventory model with a convex ordering cost function. If the inventory corresponds to a vehicle's charge level and the ordering cost corresponds to the cost of recharging, then the convexity of the ordering cost function can be interpreted as the result of overcharging costs. Unfortunately, few papers in the literature have studied inventory models with convex ordering costs. The earliest of such models appeared in Bellman et al. (1955) and Karlin (1958), albeit with limited discussion. Three different models with general convex ordering costs are analyzed by Bulinskaya (1967) and include assumptions such as random delivery of orders, perishable inventory, and multiple orders with different delivery times, although closed-form optimal policies are not given. A model with a piecewise linear convex ordering cost function is studied in Henig et al. (1997), and Bhaskaran et al. (2010) characterize optimal inventory policies for a model with convex ordering costs in which excess demand may either be accepted (and backlogged) or rejected. However, these models all have stationary random demands, whereas the demands in our model (i.e., the energy consumptions between each pair of nodes) are deterministic and non-stationary. The models also assume that inventory levels are uncapacitated, which is not useful for modeling vehicle recharging policies since batteries are limited in the amount of energy that they can store. Atamtürk and Küçükyavuz (2005) study an inventory-capacitated lot-sizing model, but their model does not consider convex inventory ordering cost functions.

3. Vehicle recharging problem

In this paper, we study the following recharging problem for EVs. Consider an EV with battery capacity q_{max} that must travel along a fixed path $P = (1, ..., n+1)$ consisting of a sequence of $n+1$ nodes. Charging stations are available at each of the first n nodes (let $S_P = (1, ..., n)$ denote the sequence of nodes in P that have charging stations), and the driver must decide how much to recharge at each station. The vehicle's charge level can never exceed the maximum capacity of the battery, and it can never drop below zero. We also do not allow the vehicle to discharge energy back to the grid. We let q_i denote the charge level of the vehicle when it arrives at node i and $h_i > 0$ denote the amount of charge required to travel from node i to node $i+1$. Then the set of feasible charging amounts, which we denote $\mathcal{A}_i(q_i)$, is $\mathcal{A}_i(q_i) = [(h_i - q_i)^+, q_{max} - q_i]$. We assume that $h_i \leq q_{max}$ for all $i \in S_P$ so that a feasible recharging policy exists.

Each time that the vehicle stops to recharge, it incurs a fixed *stopping cost* s. It also incurs a *recharging cost* at a rate of γ per unit of energy recharged ($\gamma \geq 0$) plus an additional *overcharging cost* if the vehicle's charge level rises above αq_{max}, where $0 < \alpha < 1$. This threshold represents the point at which the charging voltage reaches its maximum value and the charging current begins to decrease. Thus, the overcharging cost takes into account both the additional time per unit of energy recharged (due to the decreasing current) and wear on the battery, each converted to the same units as the stopping and recharging costs. We denote the overcharging cost as $f(x)$, where $x \in [0, (1-\alpha)q_{max}]$ is the amount by which the vehicle's charge level exceeds αq_{max} after recharging and $f(\cdot)$ is convex and increasing with $f(0) = 0$. If the vehicle's charge level already exceeds αq_{max} when it stops to recharge, we discount the overcharging cost by $f(q - \alpha q_{max})$. Therefore, if we let

$c(r,q)$ denote the cost of recharging r at a node when the vehicle arrives at the node with charge level q, then we have

$$c(r,q) = sI_{\{r>0\}} + \gamma r + \left[f\left((q+r-\alpha q_{max})^+\right) - f\left((q-\alpha q_{max})^+\right)\right], \tag{1}$$

where $I_{\{r>0\}}$ is the indicator function that equals 1 if $r > 0$ and 0 otherwise (see Figure 1 for an example illustration of $c(r,0)$). Note that $c(r,q) = 0$ whenever $r = 0$. We assume that all charging stations are identical, which is reasonable since most public charging stations in existence today have similar hardware configurations (most recharge at 220 volts, with the exception of a few "fast charging" stations that recharge at 440 volts) and regional variations in electricity rates are minimal.

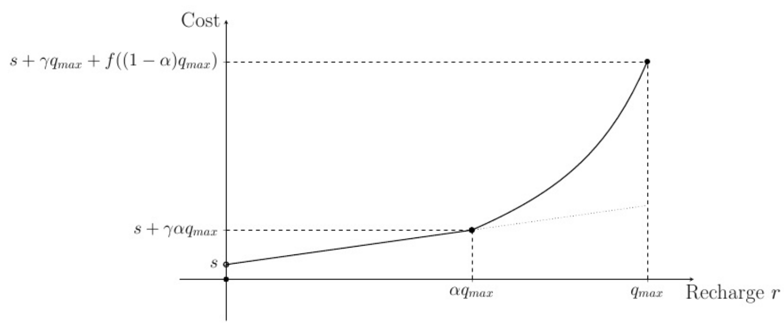

Figure 1 Sample illustration of $c(r,q)$ for $q = 0$

Our objective is to minimize the total cost of recharging along the path P. We let $V_i(q_i)$ denote the value function, which represents the minimum cost of traveling to the end of P from node i starting with charge level q_i, and it can be defined recursively as

$$V_i(q_i) = \min_{r_i \in \mathcal{A}_i(q_i)} \{c(r_i,q_i) + V_{i+1}(q_i + r_i - h_i)\}. \tag{2}$$

The first term within the brackets is the cost of recharging r_i at node i and the second term is the optimal recharging cost from the next node to the end of the path. We set $V_{n+1}(\cdot) = 0$ and seek to evaluate $V_1(0)$, the minimum total cost when the vehicle's initial charge level at the beginning of the path is 0.

3.1. Properties of optimal recharging policies

In this section, we establish properties of optimal recharging policies. We first define a recharging policy and its optimality. Let $\pi_P = [r_1, ..., r_n]$ be an ordered sequence, where r_i represents the amount to recharge at node i ($i = 1, ..., n$). From (2), it follows that these recharging amounts must satisfy

$$r_i \in \mathcal{A}_i\left(\sum_{j=1}^{i-1}(r_j - h_j)\right)$$

for all $i \in S_P$ in order for π_P to be a (feasible) recharging policy for P.

We denote by

$$C(\pi_P) = \sum_{i \in S_P} c\left(r_i, \sum_{j=1}^{i-1}(r_j - h_j)\right)$$

the total cost of policy $\pi_P = [r_1, ..., r_n]$. Our goal is to find an optimal policy for P that minimizes the total cost. However, the action space $\mathcal{A}_i(\cdot)$ at each node is a continuous interval in $\mathbb{R}$. This can be problematic from an optimization perspective since there are infinitely many actions that must be considered. Thus, it would be beneficial to show that the action space can be reduced to a finite set without increasing the optimal cost. The following lemma establishes that there exists an optimal policy in which the vehicle only recharges when its charge level is zero (see Figure 2). (In the inventory theory, these policies are known as zero-inventory-ordering policies. It is interesting to point out that such policies are not optimal in the presence of a replenishment upper bound, which is another indication that our problem is quite different.)

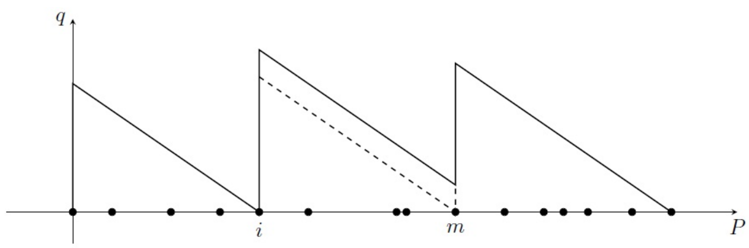

Figure 2 Original recharging policy (solid line) and alternate policy that only recharges when $q = 0$ (dotted line), with charging station locations identified along the horizontal axis

LEMMA 1. *Suppose $\pi_P^* = [r_1^*, ..., r_n^*]$ is an optimal recharging policy for P in which*

$$m = \min\left\{\ell \in S_P : r_\ell^* > 0,\ \sum_{j=1}^{\ell-1}(r_\ell^* - h_\ell) > 0\right\}$$

is the first node where the vehicle stops to recharge with a non-zero charge level and

$$i = \max\{\ell \in \{1, ..., m-1\} : r_\ell^* > 0\}$$

is the previous node with a positive recharging amount in the policy. Then policy $\pi_P = [r_1, ..., r_n]$ defined as

$$r_\ell = \begin{cases} \sum_{j=i}^{m-1} h_j, & \ell = i \\ r_m^* + \left(r_i^* - \sum_{j=i}^{m-1} h_j\right), & \ell = m \\ r_\ell^*, & \ell \in S_P \backslash \{i, m\} \end{cases}$$

is also optimal.

Proof of Lemma 1. Note that π_P is feasible because $r_i = \sum_{j=i}^{m-1} h_j$ ensures that the vehicle will reach node m (albeit with 0 charge remaining), $r_m = r_m^* + \left(r_i^* - \sum_{j=i}^{m-1} h_j\right)$ ensures that the vehicle departs node m with the same charge level as it would have in π_P^*, and all other recharging amounts are the same as in π_P^*. We have

$$\begin{aligned} C(\pi_P^*) - C(\pi_P) &= \sum_{\ell=1}^{n} \left[c\left(r_\ell^*, \sum_{j=1}^{\ell-1}(r_j^* - h_j)\right) - c\left(r_\ell, \sum_{j=1}^{\ell-1}(r_j - h_j)\right)\right] \\ &= c\left(r_i^*, \sum_{j=1}^{i-1}(r_j^* - h_j)\right) - c\left(r_i, \sum_{j=1}^{i-1}(r_j - h_j)\right) + c\left(r_m^*, \sum_{j=1}^{m-1}(r_j^* - h_j)\right) \\ &\quad - c\left(r_m, \sum_{j=1}^{m-1}(r_j - h_j)\right), \end{aligned}$$

since $r_\ell^* = r_\ell$ and $\sum_{j=1}^{\ell-1}(r_j^* - h_j) = \sum_{j=1}^{\ell-1}(r_j - h_j)$ for all $\ell \in \{1, ..., i-1, m+1, ..., n\}$ and

$$c\left(r_\ell^*, \sum_{j=1}^{\ell-1}(r_j^* - h_j)\right) = c\left(r_\ell, \sum_{j=1}^{\ell-1}(r_j - h_j)\right) = 0$$

for all $\ell \in \{i+1, ..., m-1\}$ due to $r_\ell^* = r_\ell = 0$ and $c(0, \cdot) = 0$. We also have

$$\sum_{j=1}^{i-1}(r_j^* - h_j) = \sum_{j=1}^{i-1}(r_j - h_j) = \sum_{j=1}^{m-1}(r_j - h_j) = 0$$

by definition of i and m. It follows that

$$\begin{aligned} C(\pi_P^*) - C(\pi_P) &= c(r_i^*, 0) - c(r_i, 0) + c\left(r_m^*, r_i^* - \sum_{j=i}^{m-1} h_j\right) - c(r_m, 0) \\ &= \left(s + \gamma r_i^* + f\left((r_i^* - \alpha q_{max})^+\right)\right) - \left(s + \gamma r_i + f\left(\left(\sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right)\right) \\ &\quad + \left(s + \gamma r_m^* + \left[f\left(\left(r_i^* - \sum_{j=i}^{m-1} h_j + r_m^* - \alpha q_{max}\right)^+\right)\right.\right. \\ &\quad \left.\left. - f\left(\left(r_i^* - \sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right)\right]\right) \\ &\quad - \left(s + \gamma r_m + f\left(\left(r_m^* + r_i^* - \sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right)\right) \\ &= f\left((r_i^* - \alpha q_{max})^+\right) - f\left(\left(\sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right) - f\left(\left(r_i^* - \sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right) \end{aligned}$$

$$\geq f\left((r_i^* - \alpha q_{max})^+\right) - f\left(\left(\sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+ + \left(r_i^* - \sum_{j=i}^{m-1} h_j - \alpha q_{max}\right)^+\right)$$
$$\geq f\left((r_i^* - \alpha q_{max})^+\right) - f\left((r_i^* - \alpha q_{max})^+\right)$$
$$= 0$$

since $\sum_{j=i}^{m-1} h_j > 0$, $r_i^* - \sum_{j=i}^{m-1} h_j > 0$, and $f(\cdot)$ is convex and increasing. Because π_P^* is an optimal policy, $C(\pi_P^*) = C(\pi_P)$ and π_P is also an optimal policy.

As a consequence of this lemma, the feasible action space at each node can be reduced to a discrete set without affecting the optimal cost. Thus, at any stop, the vehicle can recharge just enough to reach some later node, of which there are finitely many, and one such recharging policy is optimal. Without loss of generality, we introduce a property based on this result that will be used in our later analysis.

PROPERTY 1. *Let $\pi_P = [r_1, ..., r_n]$ be a recharging policy for P. Then for all $i \in S_P$, $r_i > 0$ if and only if $\sum_{j=1}^{i-1} r_j = \sum_{j=1}^{i-1} h_j$.*

A vehicle obeying a recharging policy that satisfies this property only recharges when its charge level is zero, and therefore never recharges more than necessary to reach its next stop. The following two lemmas show that further reduction of the action space is possible, providing a lower bound on the amount that the vehicle recharges at each stop that it makes.

LEMMA 2. *Let P' denote the path consisting of P with an additional node at the beginning (node 0), and let $S_{P'} = (0, S_P)$. If π_P^* and $\pi_{P'}^*$ are optimal recharging policies for P and P', respectively, then $C(\pi_{P'}^*) \geq C(\pi_P^*) + \gamma h_0$.*

Proof of Lemma 2. Suppose r'_0 and r'_1 are the amounts to recharge at nodes 0 and 1, respectively, specified by policy $\pi_{P'}^*$. Let $\pi_P = [r_1, ..., r_n]$ denote the recharging policy for P that satisfies

$$r_\ell = \begin{cases} r'_0 + r'_1 - h_0, & \ell = 1 \\ r'_\ell, & \ell \in S_P \backslash \{1\}. \end{cases}$$

This policy is feasible because $r_1 = r'_0 + r'_1 - h_0$ ensures that the vehicle departs node 1 with the same charge level as in $\pi_{P'}^*$, and all other recharging amounts are the same as in $\pi_{P'}^*$ (with the exception of node 0, which is not part of P and therefore has no corresponding recharging amount in π_P). Then the total cost of $\pi_{P'}^*$ exceeds that of π_P by

$$\begin{aligned} C(\pi_{P'}^*) - C(\pi_P) &= c(r'_0, 0) + c(r'_1, r'_0 - h_0) - c(r'_0 + r'_1 - h_0, 0) \\ &= \left[s + \gamma r'_0 + f((r'_0 - \alpha q_{max})^+)\right] + \left[s I_{\{r'_1 > 0\}} + \gamma r'_1 + f((r'_0 - h_0 + r'_1 - \alpha q_{max})^+)\right. \\ &\quad \left. - f((r'_0 - h_0 - \alpha q_{max})^+)\right] - \left[s + \gamma(r'_0 + r'_1 - h_0) + f((r'_0 + r'_1 - h_0 - \alpha q_{max})^+)\right] \\ &= s I_{\{r'_1 > 0\}} + \gamma h_0 + f((r'_0 - \alpha q_{max})^+) - f((r'_0 - h_0 - \alpha q_{max})^+) \\ &\geq \gamma h_0 \end{aligned}$$

because $f(\cdot)$ is increasing. Since π_P^* is an optimal recharging policy for P, it follows that

$$C(\pi_{P'}^*) \geq C(\pi_P) + \gamma h_0 \geq C(\pi_P^*) + \gamma h_0,$$

as desired.

One consequence of this lemma is that when an optimal recharging policy specifies an amount to recharge at a node that is less than αq_{max}, and increasing that amount by just enough to reach some further node along the path does not cause it to exceed αq_{max} (see Figure 3), the total cost of the new policy is no greater than that of the original policy. This is formally stated in the following lemma.

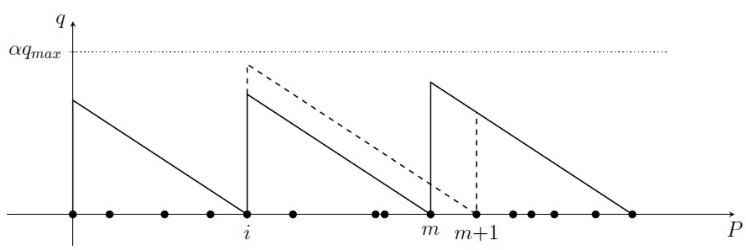

Figure 3 Original recharging policy (solid line) and alternate policy that recharges an additional amount at node i without exceeding αq_{max} (dotted line), with charging station locations identified along the horizontal axis

Lemma 3. *Let $\pi_P^* = [r_1^*, ..., r_n^*]$ be an optimal recharging policy for P where Property 1 holds. Suppose that $r_i^* > 0$ for some node $i \in S_P$ and that*

$$m = \min\{\ell \in \{i+1, ..., n\} : r_\ell^* > 0\}$$

is the next node where the vehicle is recharged. If $r_i^* + h_m \leq \alpha q_{max}$, then the policy $\pi_P = [r_1, ..., r_n]$ with

$$r_\ell = \begin{cases} r_i^* + h_m, & \ell = i \\ 0, & \ell = m \\ r_m^* + r_{m+1}^* - h_m, & \ell = m+1 \\ r_\ell^*, & \ell \in S_P \backslash \{i, m, m+1\}, \end{cases}$$

where $r_{n+1}^* = r_{n+1} = 0$, is also optimal.

Proof of Lemma 3. Note that π_P is feasible since $r_i = r_i^* + h_m$ ensures that the vehicle departs node i with just enough charge to reach node $m+1$ (and $r_i \leq \alpha q_{max}$ by supposition), and $r_{m+1} = r_m^* + r_{m+1}^* - h_m$ ensures that the vehicle departs node $m+1$ with the same charge level as in π_P^*. We define the following subpaths of P along with corresponding recharging policies:

$$P_1 = (1, ..., m), \ \pi_{P_1}^* = [r_1^*, ..., r_{m-1}^*];$$

$$P_2 = (m, ..., n+1),\ \pi^*_{P_2} = [r^*_m, ..., r^*_n];$$
$$P'_1 = (1, ..., m+1),\ \pi_{P'_1} = [r_1, ..., r_m];$$
$$P'_2 = (m+1, ..., n+1),\ \pi_{P'_2} = [r_{m+1}, ..., r_n];$$

where $\pi^*_{P_1}$ and $\pi^*_{P_2}$ are subpolicies of π^*_P, and $\pi_{P'_1}$ and $\pi_{P'_2}$ are subpolicies of π_P. These policies are all feasible due to Property 1, which ensures that the recharging policies can be subdivided at any node where the vehicle stops to recharge since the charge level there will be 0. Then we have

$$\begin{aligned}
C(\pi^*_P) - C(\pi_P) &= (C(\pi^*_{P_1}) + C(\pi^*_{P_2})) - (C(\pi_{P'_1}) + C(\pi_{P'_2})) \\
&\geq C(\pi^*_{P_1}) - C(\pi_{P'_1}) + \gamma h_m \quad \text{(by Lemma 2)} \\
&= c(r^*_i, 0) - c(r^*_i + h_m, 0) + \gamma h_m \\
&= -\gamma h_m + f((r^*_i - \alpha q_{max})^+) - f((r^*_i + h_m - \alpha q_{max})^+) + \gamma h_m \\
&= 0
\end{aligned}$$

since $r^*_i + h_m \leq \alpha q_{max}$. Because π^*_P is an optimal policy, $C(\pi^*_P) = C(\pi_P)$ and π_P is also an optimal policy.

We have now established several properties of optimal recharging policies that restrict the range of possible recharging amounts at each node to a discrete set. In the next section, we show how to apply these properties in order to find recharging policies under various sets of assumptions.

4. Solution methods

This section describes methods for obtaining recharging policies for EVs along a fixed path. We first find optimal recharging policies for general paths and two specific path types. We then analyze two heuristic methods and derive bounds on the costs of the resulting policies.

4.1. Optimal policy algorithm for a general path

The properties established in the previous section for optimal recharging policies can be used to design an algorithm capable of finding an optimal recharging policy for a given path P efficiently. Although it is possible to solve the recursive expression given in equation (2) by imposing restrictions on the amount that can be recharged at each node, a disadvantage to this approach is that the value function must be calculated for every node and for multiple different charge levels. We instead propose a modified reaching procedure (see Algorithm 1) that computes the value function only when $q_i = 0$ and only for nodes where the vehicle stops whenever the properties from Section 3.1 apply. A naive dynamic programming algorithm would use the fact that the vehicle recharges only when $q_i = 0$ and scan all nodes along the path. Here, we present a more efficient version relying on Lemma 3.

In Algorithm 1, U_j is the total cost of traveling from the beginning of the path to node j and N_j is the node from which the vehicle reaches node j in an optimal recharging policy for the subpath $(1,...,j)$ (i.e., the previous node where the vehicle stops to recharge in order to arrive at node j with charge level $q_j = 0$) for every $j \in S_P$. Beginning in line 2, for all nodes m satisfying $\alpha q_{max} - h_m < \sum_{j=i}^{m-1} h_j \leq q_{max}$ (i.e., the set of nodes requiring a charge level between $\alpha q_{max} - h_m$ and q_{max} in order to be reached from i), if the sum of U_i and the cost of recharging the exact amount required to reach node m is less than U_m, then U_m and N_m are updated. The procedure is repeated for each possible value of i, and at the end, U_{n+1} gives the total cost of an optimal policy while the N_j values allow the optimal recharging stops to be determined.

It is important to note that when the algorithm terminates, there may be some nodes i for which U_i (and also N_i) is not updated and therefore not equal to the optimal cost of traveling from the beginning of the path to node i. This is especially the case near the beginning of the path. For example, in the first iteration, if $m > 1$ is the smallest index for which U_m and N_m are updated, then $U_2 = ... = U_{m-1} = \infty$ at the end of the algorithm. The fact that $U_i = \infty$ at these nodes does not imply that stopping to recharge at any of them is infeasible, but rather that no optimal recharging policy (at least among the policies we consider) has recharging stops at those nodes.

4.2. Optimal policy for a path with continuous charging capability

Next we consider the special case in which charging capability is available continuously along a path. In other words, a vehicle can stop to recharge anywhere along the path, not just at prespecified nodes. Because of this alteration, we redefine $r_i > 0$ as the amount recharged at the i^{th} stop for $i = 1,...,\xi$, where ξ is the total number of stops. We also use H to denote the total energy required to traverse the entire path P, and we redefine the encoding of path P as the function $P(\cdot) : [0,1] \to [0,H]$, with $P(x) = Hx$, since P is no longer a sequence of nodes. Because of this new definition, we no longer need to use h_i. We now explicitly define how continuous recharging policies differ from the recharging policies described previously.

Definition 1. Let $\rho_{P,\xi} = ([\lambda_1,...,\lambda_\xi],[r_1,...,r_\xi])$ be a pair of ordered sequences satisfying $\lambda_1 = 0$ and

(i) $\lambda_i < \lambda_{i+1}$,
(ii) $\sum_{j=1}^{i} r_j \geq P(\lambda_{i+1})$, and
(iii) $\left(P(\lambda_{i+1}) - \sum_{j=1}^{i-1} r_j\right)^+ \leq r_i \leq q_{max} - \left(\sum_{j=1}^{i-1} r_j - P(\lambda_i)\right)$

for all $i \in \{1,...,\xi\}$ (where $\lambda_{\xi+1} = 1$). Then $\rho_{P,\xi}$ is a (feasible) **continuous recharging policy** with ξ stops for a path P, where λ_i is the location of the i^{th} recharging stop and r_i represents the amount to recharge at the i^{th} stop $(i = 1,...,\xi)$.

Algorithm 1 Finding an optimal recharging policy for a path P

Input: number of nodes in P (i.e., $n+1$); inter-node energy requirements (h_j for every $j \in S_P$)

Output: an optimal recharging policy for P, $\pi_P^* = [r_1^*, ..., r_n^*]$

Initialize: $h_{n+1} = \infty$; $U_1 = 0$, $U_2 = ... = U_{n+1} = \infty$; $N_j = 1$ for every $j \in P$; $r_j^* = 0$ for every $j \in S_P$

for $i = 1, ..., n$ **do**
  $m = i + 1$
  while $\sum_{j=i}^{m-1} h_j \leq q_{max}$ **do**
    if $\sum_{j=i}^{m} h_j > \alpha q_{max}$ **and** $U_i + c\left(\sum_{j=i}^{m-1} h_j, 0\right) < U_m$ **then**
      $U_m = U_i + c\left(\sum_{j=i}^{m-1} h_j, 0\right)$
      $N_m = i$
    end if
    $m = m + 1$
  end while
end for
$i = n + 1$
while $i > 1$ **do**
  $r_{N_i}^* = \sum_{j=N_i}^{i-1} h_j$
  $i = N_i$
end while

We denote by

$$D(\rho_{P,\xi}) = \sum_{i=1}^{\xi} c\left(r_i, \sum_{j=1}^{i-1} r_j - P(\lambda_i)\right)$$

the total cost of policy $\rho_{P,\xi} = ([\lambda_1, ..., \lambda_\xi], [r_1, ..., r_\xi])$. Here, i indexes the actual stops that the vehicle makes to recharge (as opposed to the nodes where the vehicle may stop in the previous definition of $c(\cdot)$).

By allowing the vehicle to recharge anywhere along the path instead of only at prespecified nodes, the total cost of an optimal continuous recharging policy is no greater than when the vehicle may only recharge at discrete intervals and provides a lower bound on the cost in the discrete case. The following theorems show how to easily find an optimal policy in the continuous case and compute such a bound for the discrete case. We first show that a continuous recharging policy of the form

$$\rho_{P,\xi}^* = ([\lambda_\ell^* = (\ell - 1)H/\xi : \ell = 1, ..., \xi], [r_\ell^* = H/\xi : \ell = 1, ..., \xi]) \tag{3}$$

is optimal for given P and ξ.

THEOREM 1. *For given P and ξ, the continuous recharging policy $\rho_{P,\xi}^* = ([\lambda_\ell^* = (\ell - 1)H/\xi : \ell = 1, ..., \xi], [r_\ell^* = H/\xi : \ell = 1, ..., \xi])$ is optimal.*

Proof of Theorem 1. Suppose for some continuous recharging policy $\rho_{P,\xi} = ([\lambda_1,...,\lambda_\xi],[r_1,...,r_\xi])$ that $\sum_{\ell=1}^{i} r_\ell = P(\lambda_{i+1}) = H\lambda_{i+1}$ for all $i \in \{1,...,\xi\}$. Note that this condition is the continuous analog of Property 1 and there exists an optimal recharging policy that satisfies it. Also suppose that $r_i < r_j$ for some $i,j \in \{1,...,\xi\}$. Let $\rho'_{P,\xi} = ([\lambda'_1,...,\lambda'_\xi],[r'_1,...,r'_\xi])$ be a policy with

$$r'_\ell = \begin{cases} r_i+\varepsilon, & \ell = i \\ r_j-\varepsilon, & \ell = j \\ r_\ell, & \ell \in \{1,...,\xi\}\backslash\{i,j\} \end{cases}$$

for some $\varepsilon \in (0, r_j - r_i)$ and $\sum_{j=1}^{i} r'_j = P(\lambda'_{i+1}) = H\lambda'_{i+1}$ for all $i \in \{1,...,\xi\}$. That is, the locations of recharging stops i through j (or j through i) are adjusted such that $\sum_{\ell=1}^{i} r'_\ell = P(\lambda_{i+1})$ for all $i \in \{1,...,\xi\}$. Note that this policy is feasible because the total amount recharged along the path is still H and the vehicle only recharges when its charge level is 0. Then we have

$$\begin{aligned} D(\rho'_{P,\xi}) - D(\rho_{P,\xi}) &= \sum_{\ell=1}^{\xi}(c(r'_\ell,0) - c(r_\ell,0)) \\ &= c(r_i+\varepsilon,0) - c(r_i,0) + c(r_j-\varepsilon,0) - c(r_j,0) \\ &= \left[s+\gamma(r_i+\varepsilon) + f((r_i+\varepsilon-\alpha q_{max})^+)\right] - \left[s+\gamma r_i + f((r_i-\alpha q_{max})^+)\right] \\ &\quad + \left[s+\gamma(r_j-\varepsilon) + f((r_j-\varepsilon-\alpha q_{max})^+)\right] - \left[s+\gamma r_j + f((r_j-\alpha q_{max})^+)\right] \\ &= \left(f((r_i+\varepsilon-\alpha q_{max})^+) - f((r_i-\alpha q_{max})^+)\right) \\ &\quad - \left(f((r_j-\alpha q_{max})^+) - f((r_j-\varepsilon-\alpha q_{max})^+)\right) \\ &\le 0 \end{aligned}$$

by the convexity of $f(\cdot)$ since $r_i+\varepsilon < r_j$. No such improvement exists when $r_1 = ... = r_\xi$, in which case we must have $r_\ell = H/\xi$ and $\lambda_\ell = (\ell-1)H/\xi$ for all $\ell \in \{1,...,\xi\}$ in order to satisfy the condition $\sum_{\ell=1}^{i} r_\ell = P(\lambda_{i+1}) = H\lambda_{i+1}$ for all $i \in \{1,...,\xi\}$. Therefore, the continuous recharging policy $\rho^*_{P,\xi}$ is optimal.

In addition to the structure of an optimal policy for a given number of stops, it is also desirable to know the number of stops that minimizes the total cost of an optimal recharging policy. To this end, we must evaluate the expression $\text{argmin}_\xi\{D(\rho^*_{P,\xi})\}$ to determine the optimal number of stops. This requires calculating $D(\rho^*_{P,\xi})$ for every integer value of ξ between $\lceil H/q_{max}\rceil$ (the minimum feasible number of stops) and $\lceil H/\alpha q_{max}\rceil$ (the minimum number of stops with no overcharging), inclusive. Note that for any $\xi > \lceil H/\alpha q_{max}\rceil$, we have $D(\rho^*_{P,\xi}) = D(\rho^*_{P,\lceil H/\alpha q_{max}\rceil}) + (\xi - \lceil H/\alpha q_{max}\rceil)s$ since neither policy has any overcharging cost and there is no difference in the total amount recharged. In fact, for optimal policies of the form (3), the total amount recharged is the same for any ξ. To obtain a closed-form expression for the optimal number of stops, the following assumption must be made.

Assumption 1. *In the expression for the cost of recharging given in (1), let $\gamma = 0$ and $f(x) = kx$, where $k \geq 0$ is a constant.*

Setting $\gamma = 0$ is without loss of generality because every feasible recharging policy incurs a fixed cost of at least γH, with optimal policies of the form (3) having a fixed cost of exactly γH, and thus optimal recharging policies are not affected by adjusting γ. The use of a linear function for the overcharging cost $f(\cdot)$ simplifies some of our later calculations by allowing us to determine the optimal number of recharging stops as a function of the model parameters without having to create multiple different policies and evaluate their costs.

When Assumption 1 holds, the optimal number of stops depends on the ratio between the stopping cost, s, and the overcharging cost rate, k, of charging above the level αq_{max}. If the ratio is sufficiently small, then the minimum number of stops such that the amount recharged at each stop is less than αq_{max} is optimal. As the ratio increases towards αq_{max}, it becomes optimal to recharge at least αq_{max} at each stop, and if the ratio is equal to αq_{max} or greater, then the amount recharged at each stop should be maximized in order to minimize the total number of stops. The following theorem states this in rigorous terms.

Theorem 2. *Let Assumption 1 hold. For a path P, the value of ξ that minimizes the total cost is*

$$\underset{\xi}{\operatorname{argmin}}\left\{D(\rho^*_{P,\xi})\right\} = \begin{cases} \left\lceil \dfrac{H}{\alpha q_{max}} \right\rceil, & \frac{s}{k} < H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor & \text{(4a)} \\ \left\lfloor \dfrac{H}{\alpha q_{max}} \right\rfloor, & H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor \leq \frac{s}{k} < \alpha q_{max} \text{ and } \left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor \geq \left\lceil \frac{H}{q_{max}} \right\rceil & \text{(4b)} \\ \left\lceil \dfrac{H}{q_{max}} \right\rceil, & \text{otherwise.} & \text{(4c)} \end{cases}$$

Proof of Theorem 2. For any ξ, let $\rho^*_{P,\xi} = ([\lambda^*_\ell = (\ell-1)H/\xi : \ell = 1,...,\xi], [r^*_\ell = H/\xi : \ell = 1,...,\xi])$. By Theorem 1, this policy is optimal for a given ξ. The total cost of the policy $\rho^*_{P,\xi}$ is thus

$$D(\rho^*_{P,\xi}) = \xi s + k(H - \xi\alpha q_{max})^+. \tag{5}$$

We consider the following two cases that describe the possible relations of the recharging amount at each stop, which is the same for all stops, to αq_{max}:

Case 1: $\alpha q_{max} < \frac{H}{\xi}$ (i.e., the vehicle always overcharges whenever it stops) and

Case 2: $\alpha q_{max} \geq \frac{H}{\xi}$ (i.e., the vehicle never overcharges when it stops).

Then for a given case, increasing the value of ξ by one (i.e., adding an extra recharging stop) decreases the amount recharged at each stop to $\frac{H}{\xi+1}$. This results in a transition from Case 1 to Case 2 if $\frac{H}{\xi+1} < \alpha q_{max} \leq \frac{H}{\xi}$ or no change otherwise. We define the possible transitions as follows:

Transition 1: Case 1 to Case 1,

Transition 2: Case 1 to Case 2, and

Transition 3: Case 2 to Case 2.

The difference in cost when increasing the number of stops by one is

$$D(\rho^*_{P,\xi+1}) - D(\rho^*_{P,\xi}) = s - \begin{cases} k\alpha q_{max}, & \xi \le \frac{H}{\alpha q_{max}} - 1 & \text{(6a)} \\ k\left(H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor\right), & \frac{H}{\alpha q_{max}} - 1 < \xi \le \frac{H}{\alpha q_{max}} & \text{(6b)} \\ 0, & \xi > \frac{H}{\alpha q_{max}} & \text{(6c)} \end{cases}$$

for feasible ξ (i.e., $\xi \ge H/q_{max}$). The subcases (6a), (6b), and (6c) correspond to the cost differences of Transitions 1, 2, and 3, respectively. As ξ increases, note that Transition 1 cannot occur after Transition 2, which in turn cannot occur after Transition 3. We can therefore order the transitions and compare them sequentially. Comparing Transitions 1 and 2 first, we find that

$$s - k\alpha q_{max} < s - k\left(H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor\right)$$

since $H - \alpha q_{max}\lfloor H/\alpha q_{max} \rfloor < \alpha q_{max}$, and comparing Transitions 2 and 3 reveals that

$$s - k\left(H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor\right) \le s$$

because $H - \alpha q_{max}\lfloor H/\alpha q_{max} \rfloor \ge 0$. The function $D(\rho^*_{P,\xi})$ is therefore convex and also piecewise linear with respect to ξ and has up to three segments. We are interested in the value of ξ that minimizes $D(\rho^*_{P,\xi})$, or the smallest feasible integer value of ξ such that the cost difference is non-negative. There are three different cases to consider.

- If the expression in (6b) is negative, then only the third segment of the function has a non-negative slope. (It is also implied that $\frac{H}{\alpha q_{max}}$ is non-integer in this case because otherwise we would have $H - \alpha q_{max}\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor = 0$ and consequently $s < 0$, which is not possible since the stopping time is assumed to be non-negative.) That segment includes values of ξ that are greater than $\frac{H}{\alpha q_{max}}$, and the smallest integer value of ξ such that $\xi > \frac{H}{\alpha q_{max}}$, where $\frac{H}{\alpha q_{max}}$ is non-integer, is $\left\lceil \frac{H}{\alpha q_{max}} \right\rceil$. This result corresponds to the case (4a), and because $\left\lceil \frac{H}{\alpha q_{max}} \right\rceil \ge \left\lceil \frac{H}{q_{max}} \right\rceil$, where $\left\lceil \frac{H}{q_{max}} \right\rceil$ is the minimum possible number of stops, it is feasible.
- If the expression in (6a) is negative and the one in (6b) is not, then both the second and third segments of the function have non-negative slopes. Thus, the optimal number of stops is included in the second segment (if feasible), which consists of values satisfying $\frac{H}{\alpha q_{max}} - 1 < \xi \le \frac{H}{\alpha q_{max}}$. The only integer value of ξ that satisfies the inequalities is $\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor$, but it cannot be less than the minimum possible number of stops, $\left\lceil \frac{H}{q_{max}} \right\rceil$. This result corresponds to the the case (4b).
- If the expression in (6a) is non-negative, then all three segments of the function have non-negative slopes. Therefore, the function is minimized when ξ is minimized, and the smallest feasible integer value of ξ is $\left\lceil \frac{H}{q_{max}} \right\rceil$. This is also the number of stops made when $\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor$ is infeasible for ξ in the previous case, and the result corresponds to the case (4c).

Therefore, the stated expression for $\text{argmin}_\xi \left\{D(\rho^*_{P,\xi})\right\}$ holds.

In this section, we have shown how to find an optimal continuous recharging policy for a given path. By calculating the ratio of the stopping cost parameter to the overcharging cost rate parameter, we can determine the optimal number of stops analytically, which we then use to determine the corresponding stopping locations and recharging amounts. The total cost of an optimal continuous recharging policy can therefore be computed quickly and in closed form. We use this solution in later sections to bound and compare the costs of other recharging policies.

4.3. Optimal policy for a path with equidistant charging locations

We now return to our original definition of a path, P, as a sequence of nodes, where $P = (1, ..., n+1)$. The vehicle may once again only recharge at nodes, but in this section we suppose that the nodes in P are equidistant such that the distance between any pair of adjacent nodes is $h_i = h$ for all $i \in S_P$. We motivate this scenario as an intermediate case between general paths and paths with continuous charging capability. Although we enforce that recharging can only occur at nodes, the uniform spacing between nodes makes finding an optimal recharging policy easier than in the case of general paths. In fact, we show later in this section that a path with continuous charging capability is a limiting case of a path with equidistant charging locations, and thus the optimal policies in both cases are related.

Rather than use the same notation as before for optimal recharging policies for general paths, we define a new type of policy specifically for paths with equidistant charging locations, mimicking the continuous case.

Definition 2. Let $\sigma_{P,\xi} = ([\mu_1, ..., \mu_\xi], [r_1, ..., r_\xi])$ be a pair of ordered sequences satisfying $\mu_1 = 1$ and

(*i*) $\mu_i < \mu_{i+1}$ (for all $\mu_i \in S_P$),
(*ii*) $\sum_{j=1}^{i} r_j \geq (\mu_{i+1} - 1)h$, and
(*iii*) $\left((\mu_{i+1} - 1)h - \sum_{j=1}^{i-1} r_j\right)^+ \leq r_i \leq q_{max} - \left(\sum_{j=1}^{i-1} r_j - (\mu_i - 1)h\right)$

for all $i \in \{1, ..., \xi\}$ (where $\mu_{\xi+1} = n+1$). Then $\sigma_{P,\xi}$ is a (feasible) **equidistant recharging policy** with ξ stops for a path P, where μ_i is the node index of the i^{th} recharging stop and r_i represents the amount to recharge at the i^{th} stop $(i = 1, ..., \xi)$.

We let

$$E(\sigma_{P,\xi}) = \sum_{i=1}^{\xi} c\left(r_i, \sum_{j=1}^{i-1} r_j - (\mu_i - 1)h\right)$$

denote the total cost of a policy $\sigma_{P,\xi} = ([\mu_1, ..., \mu_\xi], [r_1, ..., r_\xi])$. An optimal policy over all policies with ξ stops is defined in the same way as in the continuous case.

When the number of recharging stops is fixed at ξ, then there exists a feasible recharging policy $\sigma_{P,\xi} = ([\mu_1, ..., \mu_\xi], [r_1, ..., r_n])$ with $r_i = \left\lceil \frac{n}{\xi} \right\rceil h$ at $n - \xi \left\lfloor \frac{n}{\xi} \right\rfloor$ of the stops and $r_i = \left\lfloor \frac{n}{\xi} \right\rfloor h$ at each of the remaining stops. The following theorem shows that this policy is also optimal.

Theorem 3. *For given P and ξ, where $h_i = h$ for all $i \in S_P$, the equidistant recharging policy $\sigma^*_{P,\xi} = ([\mu^*_1, ..., \mu^*_\xi], [r^*_1, ..., r^*_\xi])$ with $r^*_1 = ... = r^*_{n-\xi\lfloor \frac{n}{\xi} \rfloor} = \left\lceil \frac{n}{\xi} \right\rceil h$, $r^*_{n-\xi\lfloor \frac{n}{\xi} \rfloor + 1} = ... = r^*_\xi = \left\lfloor \frac{n}{\xi} \right\rfloor h$, and corresponding $[\mu^*_1, ..., \mu^*_\xi]$ that satisfy Property 1 is optimal.*

Proof of Theorem 3. Suppose for some equidistant recharging policy $\sigma_{P,\xi} = ([\mu_1, ..., \mu_\xi], [r_1, ..., r_\xi])$ satisfying Property 1 that $r_i + h < r_j$ for some $i, j \in \{1, ..., \xi\}$. Let $\sigma'_{P,\xi} = ([\mu'_1, ..., \mu'_\xi], [r'_1, ..., r'_\xi])$ be the policy in which

$$r'_\ell = \begin{cases} r_i + h, & \ell = i \\ r_j - h, & \ell = j \\ r_\ell, & \ell \in \{1, ..., \xi\} \setminus \{i, j\} \end{cases}$$

and corresponding $[\mu'_1, ..., \mu'_\xi]$ are defined based on Property 1. Note that this policy is feasible because the total amount recharged along the path is still nh and the vehicle only recharges when its charge level is 0. Then we have

$$\begin{aligned} E(\sigma'_{P,\xi}) - E(\sigma_{P,\xi}) &= \sum_{\ell=1}^{\xi} (c(r'_\ell, 0) - c(r_\ell, 0)) \\ &= c(r_i + h, 0) - c(r_i, 0) + c(r_j - h, 0) - c(r_j, 0) \\ &= \left[s + \gamma(r_i + h) + f((r_i + h - \alpha q_{max})^+)\right] - \left[s + \gamma r_i + f((r_i - \alpha q_{max})^+)\right] \\ &\quad + \left[s + \gamma(r_j - h) + f((r_j - h - \alpha q_{max})^+)\right] - \left[s + \gamma r_j + f((r_j - \alpha q_{max})^+)\right] \\ &= \left(f((r_i + h - \alpha q_{max})^+) - f((r_i - \alpha q_{max})^+)\right) \\ &\quad - \left(f((r_j - \alpha q_{max})^+) - f((r_j - h - \alpha q_{max})^+)\right) \\ &\leq 0 \end{aligned}$$

by the convexity of $f(\cdot)$ since $r_i + h < r_j$. No such improvement exists when $\max_{i,j \in \{1,...,\xi\}} \{r_i - r_j\} \leq h$, and the total cost cannot be further reduced by violating Property 1. Therefore, the equidistant recharging policy $\sigma^*_{P,\xi}$ is optimal.

As with continuous optimal recharging policies, the optimal number of stops depends on the ratio between the stopping cost and overcharging cost rate when Assumption 1 holds. If the ratio is sufficiently small, then the minimum number of stops such that the amount recharged at each stop is less than αq_{max} is optimal. The optimal number of stops decreases as the ratio increases, crossing different thresholds until it equals the minimum possible number of stops. The next theorem shows how to find the number of stops that minimizes the total cost of an optimal equidistant recharging policy in rigorous terms.

THEOREM 4. *Let Assumption 1 hold. For a path P with $h_i = h$ for all $i \in S_P$, if $\sigma^*_{P,\xi}$ is an optimal equidistant recharging policy with ξ stops and $y = \lfloor \alpha q_{max}/h \rfloor$ (i.e., y is the largest integer multiple of h such that $yh \le \alpha q_{max}$), then the value of ξ that minimizes the total cost of such a policy is*

$$\operatorname*{argmin}_{\xi}\{E(\sigma^*_{P,\xi})\} = \begin{cases} \left\lceil \frac{n}{y} \right\rceil, & \frac{s}{k} < nh - \left\lfloor \frac{n}{y} \right\rfloor \alpha q_{max} \quad (7a) \\ \left\lfloor \frac{n}{y} \right\rfloor, & nh - \left\lfloor \frac{n}{y} \right\rfloor \alpha q_{max} \le \frac{s}{k} < \alpha q_{max} \text{ and } \left\lfloor \frac{n}{y} \right\rfloor \ge \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil \quad (7b) \\ \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil, & \text{otherwise} \quad (7c) \end{cases}$$

if $\frac{n}{y+1} > \left\lfloor \frac{n}{y} \right\rfloor$ *and*

$$\operatorname*{argmin}_{\xi}\{E(\sigma^*_{P,\xi})\} = \begin{cases} \left\lceil \frac{n}{y} \right\rceil, & \frac{s}{k} < \left(n - y\left\lfloor \frac{n}{y} \right\rfloor\right)((y+1)h - \alpha q_{max}) \quad (8a) \\ \left\lfloor \frac{n}{y} \right\rfloor, & \left(n - y\left\lfloor \frac{n}{y} \right\rfloor\right)((y+1)h - \alpha q_{max}) \le \frac{s}{k} < y((y+1)h - \alpha q_{max}) \\ & \text{and } \left\lfloor \frac{n}{y} \right\rfloor \ge \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil \quad (8b) \\ \left\lceil \frac{n}{y+1} \right\rceil, & y((y+1)h - \alpha q_{max}) \le \frac{s}{k} < \alpha q_{max} - \left((y+1)\left\lceil \frac{n}{y+1} \right\rceil - n\right) \cdot \\ & (\alpha q_{max} - yh) \text{ and } \left\lceil \frac{n}{y+1} \right\rceil \ge \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil \quad (8c) \\ \left\lfloor \frac{n}{y+1} \right\rfloor, & \alpha q_{max} - \left((y+1)\left\lceil \frac{n}{y+1} \right\rceil - n\right)(\alpha q_{max} - yh) \le \frac{s}{k} < \alpha q_{max} \\ & \text{and } \left\lfloor \frac{n}{y+1} \right\rfloor \ge \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil \quad (8d) \\ \left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil, & \text{otherwise} \quad (8e) \end{cases}$$

if $\frac{n}{y+1} \le \left\lfloor \frac{n}{y} \right\rfloor$.

Proof of Theorem 4. Let $\sigma^*_{P,\xi} = ([\mu^*_1, \ldots, \mu^*_\xi], [r^*_1, \ldots, r^*_\xi])$ with $r^*_1 = \ldots = r^*_{n-\xi\lfloor \frac{n}{\xi} \rfloor} = \left\lceil \frac{n}{\xi} \right\rceil h$ and $r^*_{n-\xi\lfloor \frac{n}{\xi} \rfloor + 1} = \ldots = r^*_\xi = \left\lfloor \frac{n}{\xi} \right\rfloor h$. By Theorem 3, this policy is optimal for a given ξ. The total cost of policy $\sigma^*_{P,\xi}$ is thus

$$E(\sigma^*_{P,\xi}) = \xi s + k \sum_{i=1}^{\xi} (r^*_i - \alpha q_{max})^+. \quad (9)$$

If we relate the two possible values for the amount to recharge at each stop to αq_{max}, we see that there are three possible cases to consider:

Case 1: $\alpha q_{max} < \left\lfloor \frac{n}{\xi} \right\rfloor h \le \left\lceil \frac{n}{\xi} \right\rceil h$ (i.e., the vehicle always overcharges whenever it stops),

Case 2: $\left\lfloor \frac{n}{\xi} \right\rfloor h \le \alpha q_{max} < \left\lceil \frac{n}{\xi} \right\rceil h$ (i.e., the vehicle overcharges at the first $n - \xi\left\lfloor \frac{n}{\xi} \right\rfloor$ stops but not at the remaining stops), and

Case 3: $\left\lfloor \frac{n}{\xi} \right\rfloor h \leq \left\lceil \frac{n}{\xi} \right\rceil h \leq \alpha q_{max}$ (i.e., the vehicle never overcharges when it stops).

Then for a given case, increasing the value of ξ by one (i.e., adding an extra recharging stop) decreases the amounts recharged to $\left\lfloor \frac{n}{\xi+1} \right\rfloor h$ and $\left\lceil \frac{n}{\xi+1} \right\rceil h$ for the corresponding stops. This could result in a new relationship of the recharging amounts to αq_{max} and transition us to a different case. For example, we transition from Case 1 to Case 2 when $\alpha q_{max} < \left\lfloor \frac{n}{\xi} \right\rfloor h \leq \left\lceil \frac{n}{\xi} \right\rceil h$ and $\left\lfloor \frac{n}{\xi+1} \right\rfloor h \leq \alpha q_{max} < \left\lceil \frac{n}{\xi+1} \right\rceil h$. The possible transitions are the following:

$$\begin{aligned}
&\text{Transition 1: Case 1 to Case 1,}\\
&\text{Transition 2: Case 1 to Case 2,}\\
&\text{Transition 3: Case 2 to Case 2,}\\
&\text{Transition 4: Case 1 to Case 3,}\\
&\text{Transition 5: Case 2 to Case 3, and}\\
&\text{Transition 6: Case 3 to Case 3.}
\end{aligned}$$

Note that $\frac{n}{y+1}$ is the maximum (possibly fractional) number of stops such that the amount recharged at each stop is greater than αq_{max}, and $\left\lfloor \frac{n}{y} \right\rfloor$ is the maximum feasible number of stops such that not all of the amounts recharged at each stop are less than αq_{max}. If $\frac{n}{y+1} > \left\lfloor \frac{n}{y} \right\rfloor$, then there is no feasible value for ξ such that $\left\lfloor \frac{n}{\xi} \right\rfloor h \leq \alpha q_{max} < \left\lceil \frac{n}{\xi} \right\rceil h$. Thus, we only need to consider Transitions 1, 4, and 6. Alternatively, if $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, then all transitions except Transition 4 are possible. We now compute the costs for each of the six possible transitions.

The difference in cost when increasing the number of stops by one is

$$E\left(\sigma^*_{P,\xi+1}\right) - E\left(\sigma^*_{P,\xi}\right) =$$

$$s - \begin{cases}
k\alpha q_{max}, & \xi < \left\lfloor \frac{n}{y+1} \right\rfloor \quad \text{(10a)}\\
k\left(\alpha q_{max} - \left((y+1)\left\lceil \dfrac{n}{y+1} \right\rceil - n\right)(\alpha q_{max} - yh)\right), & \xi = \left\lfloor \frac{n}{y+1} \right\rfloor < \frac{n}{y+1} \text{ and } \frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor \quad \text{(10b)}\\
ky((y+1)h - \alpha q_{max}), & \left\lceil \frac{n}{y+1} \right\rceil \leq \xi < \left\lfloor \frac{n}{y} \right\rfloor \text{ and } \frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor \quad \text{(10c)}\\
k\left(nh - \left\lfloor \dfrac{n}{y} \right\rfloor \alpha q_{max}\right), & \xi = \left\lfloor \frac{n}{y} \right\rfloor < \frac{n}{y} \text{ and } \frac{n}{y+1} > \left\lfloor \frac{n}{y} \right\rfloor \quad \text{(10d)}\\
k\left(n - y\left\lfloor \dfrac{n}{y} \right\rfloor\right)((y+1)h - \alpha q_{max}), & \xi = \left\lfloor \frac{n}{y} \right\rfloor < \frac{n}{y} \text{ and } \frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor \quad \text{(10e)}\\
0, & \xi \geq \left\lceil \frac{n}{y} \right\rceil \quad \text{(10f)}
\end{cases}$$

for feasible ξ (i.e., $\xi \geq H/q_{max}$). The six subcases in (10) correspond to the cost differences of Transitions 1 through 6, in order.

We first consider the case where $\frac{n}{y+1} > \left\lfloor \frac{n}{y} \right\rfloor$. As ξ increases, note that Transition 1 cannot occur after Transition 4, which in turn cannot occur after Transition 6. We can therefore order the transitions and compare them sequentially. Beginning with Transitions 1 and 4, we compare (10a) and (10d) and find that

$$\begin{aligned} s-k\alpha q_{max} &\le s-k\left(\left(n-y\left\lfloor\frac{n}{y}\right\rfloor\right)h-\left\lfloor\frac{n}{y}\right\rfloor(\alpha q_{max}-yh)\right)\\ &= s-k\left(nh-\left\lfloor\frac{n}{y}\right\rfloor\alpha q_{max}\right)\end{aligned}$$

since $\left(n-y\left\lfloor\frac{n}{y}\right\rfloor\right)h<yh\le\alpha q_{max}$ and $\left\lfloor\frac{n}{y}\right\rfloor(\alpha q_{max}-yh)\ge 0$. Comparing Transitions 4 and 6, we note that

$$\begin{aligned} s-k\left(nh-\left\lfloor\frac{n}{y}\right\rfloor\alpha q_{max}\right) &< s-k\left(nh-\frac{n}{y+1}(y+1)h\right)\\ &= s\end{aligned}$$

because $\frac{n}{y+1}>\left\lfloor\frac{n}{y}\right\rfloor$ and $(y+1)h>\alpha q_{max}$. Thus, $E(\cdot)$ is convex and piecewise linear with respect to ξ in this case.

We next consider the case where $\frac{n}{y+1}\le\left\lfloor\frac{n}{y}\right\rfloor$. As in the previous case, the transitions can be ordered and compared sequentially. We first compare (10a) and (10b) (Transitions 1 and 2) and see that

$$s-k\alpha q_{max}\le k\left(\alpha q_{max}-\left((y+1)\left\lceil\frac{n}{y+1}\right\rceil-n\right)(\alpha q_{max}-yh)\right)$$

because $(y+1)\left\lceil\frac{n}{y+1}\right\rceil-n\ge 0$ and $\alpha q_{max}-yh\ge 0$. We next compare Transitions 2 and 3 and find that

$$\begin{aligned} s-k\left(\alpha q_{max}-\left((y+1)\left\lceil\frac{n}{y+1}\right\rceil-n\right)(\alpha q_{max}-yh)\right) &\le s-k\left(\alpha q_{max}-(y+1)(\alpha q_{max}-yh)\right)\\ &= s-ky((y+1)h-\alpha q_{max}).\end{aligned}$$

Comparing (10c) and (10e), or Transitions 3 and 5, gives

$$s-ky((y+1)h-\alpha q_{max})<s-k\left(n-y\left\lfloor\frac{n}{y}\right\rfloor\right)((y+1)h-\alpha q_{max}),$$

and the expression in (10e) is no greater than s, the cost of Transition 6. Therefore, in each of the cases $\frac{n}{y+1}>\left\lfloor\frac{n}{y}\right\rfloor$ and $\frac{n}{y+1}\le\left\lfloor\frac{n}{y}\right\rfloor$, the function $E(\cdot)$ is convex and also piecewise linear with respect to ξ. We are interested in the value of ξ that minimizes $E(\cdot)$, or the smallest feasible integer of ξ such that the cost difference is non-negative. There are several cases to consider.

- If the expression in (10d) is negative when $\frac{n}{y+1}>\left\lfloor\frac{n}{y}\right\rfloor$ or the expression in (10e) is negative when $\frac{n}{y+1}\le\left\lfloor\frac{n}{y}\right\rfloor$, then only the last segment of the function has a non-negative slope. That segment includes values of ξ that are at least $\left\lceil\frac{n}{y}\right\rceil$, and the smallest integer value of ξ such that $\xi\ge\left\lceil\frac{n}{y}\right\rceil$ is $\left\lceil\frac{n}{y}\right\rceil$. This result corresponds to cases (7a) and (8a), and because $\left\lceil\frac{n}{y}\right\rceil\ge\left\lceil\frac{n}{\lfloor q_{max}/h\rfloor}\right\rceil$, where $\left\lceil\frac{n}{\lfloor q_{max}/h\rfloor}\right\rceil$ is the minimum possible number of stops, it is feasible.

- If the expression in (10a) is negative and the one in (10d) is not when $\frac{n}{y+1} > \left\lfloor \frac{n}{y} \right\rfloor$, or if the expression in (10c) is negative and the one in (10e) is not when $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, then if $\left\lfloor \frac{n}{y} \right\rfloor$ is a feasible number of stops, the last two segments of the function have non-negative slopes. Thus, the optimal number of stops is included in the second-to-last segment, which consists of values satisfying $\xi = \left\lfloor \frac{n}{y} \right\rfloor < \frac{n}{y}$. The only integer value of ξ that satisfies the expression is $\left\lfloor \frac{n}{y} \right\rfloor$, where $\frac{n}{y}$ must be non-integer. This result corresponds to cases (7b) and (8b).
- If the expression in (10b) is negative and the one in (10c) is not when $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, then the third, fourth, and fifth segments of the function have non-negative slopes. The optimal number of stops is therefore included in the third segment (if feasible) when $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, which consists of values satisfying $\left\lceil \frac{n}{y+1} \right\rceil \leq \xi < \left\lfloor \frac{n}{y} \right\rfloor$. The smallest integer value of ξ satisfying the expression is $\left\lceil \frac{n}{y+1} \right\rceil$, but it must be feasible and cannot be less than $\left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil$. This result corresponds to case (8c).
- If the expression in (10a) is negative and the one in (10b) is not when $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, and if $\left\lfloor \frac{n}{y+1} \right\rfloor$ is a feasible number of stops, then the second through fifth segments of the function have non-negative slopes. Thus, the optimal number of stops is included in the second segment when $\frac{n}{y+1} \leq \left\lfloor \frac{n}{y} \right\rfloor$, which consists of values satisfying $\xi = \left\lfloor \frac{n}{y+1} \right\rfloor < \frac{n}{y+1}$. The only integer value of ξ that satisfies the expression is $\left\lfloor \frac{n}{y+1} \right\rfloor$, where $\frac{n}{y+1}$ must be non-integer. This result corresponds to case (8d).
- If the expression in (10a) is non-negative, then all segments of the function have non-negative slopes. Therefore, the function is minimized when ξ is minimized, and the smallest feasible integer value of ξ is $\left\lceil \frac{n}{\lfloor q_{max}/h \rfloor} \right\rceil$. This is also the number of stops that is made if the value in any of the previous cases is infeasible, and the result corresponds to cases (7c) and (8e).

Therefore, the stated expressions for $\operatorname{argmin}_\xi \left\{ E(\sigma^*_{P,\xi}) \right\}$ hold.

In a similar manner as for optimal continuous recharging policies, we have shown how to find an optimal equidistant recharging policy as well as the number of stops that minimizes the total cost of such a policy. It is also useful to compare the optimal policies of the continuous and equidistant charging location scenarios. The continuous case can be viewed as a limiting instance of the equidistant case in which $h \to 0$. This relationship is verified in the next lemma.

Lemma 4. *Let P be a path with $n+1$ equidistant nodes such that $h_i = h$ for all $i \in S_P$ and $nh = H$, and let $P'(x) = Hx$ be a path with continuous charging capability. Also, let $\sigma^*_{P,\xi}$ be an optimal equidistant recharging policy for P and $\rho^*_{P',\xi}$ be an optimal continuous recharging policy for P'. Then $\lim_{h\to 0} E(\sigma^*_{P,\xi}) = D(\rho^*_{P',\xi})$.*

Proof of Lemma 4. Let $\sigma^*_{P,\xi} = ([\mu^*_1, ..., \mu^*_\xi], [r^*_1, ..., r^*_\xi])$, and let $\max_{i,j \in S_P} \{r^*_i - r^*_j\} \leq h$ since by Lemma 3, such a policy cannot be improved by adjusting the recharging amounts at each stop in increments of h. Then in the limit as $h \to 0$, the amounts recharged at each stop approach the same value, namely $r_1^* = ... = r_\xi^* = H/\xi$. This gives

$$\begin{aligned}
\lim_{h\to 0} E(\sigma^*_{P,\xi}) &= \lim_{h\to 0} \left\{ \xi s + k \sum_{i=1}^{\xi} (r_i^* - \alpha q_{max})^+ \right\} \\
&= \xi s + k\xi \left(\frac{H}{\xi} - \alpha q_{max} \right)^+ \\
&= \xi s + k(H - \xi \alpha q_{max})^+ \\
&= D(\rho^*_{P',\xi}).
\end{aligned}$$

Therefore, in the limit as $h \to 0$, the costs of an optimal recharging policy in both the equidistant and continuous cases are equal.

We have thus established that the case of continuous charging capability along the path is a limiting instance of the case with equidistant charging locations in which the distance between nodes goes to 0. Because of the consistent path structure in each case, manageable closed-form threshold recharging policies exist, but this is not necessarily true for general paths. In the next section, we discuss heuristic methods that build on the special cases discussed here and impose consistency in their recharging policies despite possible irregularities in the distances between nodes for an arbitrary path.

4.4. Heuristic solution methods

So far, we have examined methods for finding optimal recharging policies. We have shown how to obtain optimal continuous and equidistant recharging policies by first solving a closed-form expression to determine the optimal number of stops, and then using that number of stops to construct an optimal policy. For general paths, we have presented an algorithm that uses forward recursion to identify the nodes where the vehicle should stop to recharge. Although this last approach can be useful for minimizing the total cost, the resulting policy can be unstructured and difficult to understand at a glance. An EV driver may prefer a simpler recharging policy that can be determined quickly with less computation. For that reason, we have developed two heuristic methods for creating reasonable recharging policies with intuitive interpretation and limited computational effort.

In Section 3.1, we established a range of possible actions at each recharging stop. Subsequently, from our analysis of optimal continuous and equidistant recharging policies in Sections 4.2 and 4.3, respectively, we found that there exist thresholds when the vehicle should never overcharge and when it should overcharge to minimize its total number of stops. These observations motivate the two heuristics presented below.

4.4.1. Heuristic 1: Avoid overcharging Our first heuristic, outlined in Algorithm 2, yields a policy in which the vehicle minimizes its total number of stops while avoiding overcharging whenever possible. In the algorithm, $\hat{r}_j$ denotes the recharging amount at node j. The procedure finds the maximum value m such that $\sum_{j=i}^{m-1} h_j \leq \alpha q_{max}$ (i.e., the charge level required at node i to reach node m is no greater than αq_{max}), or if no such value exists, then $m = i+1$.

Note that the vehicle overcharges only when consecutive charging stations are sufficiently far apart (i.e., when $h_i > \alpha q_{max}$ for some $i \in S_P$). Otherwise, it will never overcharge but still try to minimize the number of times that it stops to recharge. This heuristic method is most preferable when a driver is primarily concerned with preserving battery health and would rather stop more frequently to recharge than incur overcharging costs.

Algorithm 2 Heuristic method for finding a recharging policy for a path P that never overcharges

Input: number of nodes in P (i.e., $n+1$); inter-node energy requirements (h_j for every $j \in S_P$)

Output: a recharging policy for P, $\hat{\pi}_P = [\hat{r}_1, ..., \hat{r}_n]$

Initialize: $\hat{r}_j = 0$ for every $j \in S_P$; $i = 1$; $h_{n+1} = \infty$

while $i < n+1$ **do**
  $m = i + 1$
  while $\sum_{j=i}^{m} h_j \leq \alpha q_{max}$ **do**
    $m = m + 1$
  end while
  $\hat{r}_i = \sum_{j=i}^{m-1} h_j$
  $i = m$
end while

In the following two lemmas, we derive bounds on the number of stops and total cost of the policy obtained using Algorithm 2. We let $H = \sum_{i \in S_P} h_i$ denote the total energy required to travel along the path P.

LEMMA 5. *Let Assumption 1 hold, and let $\hat{\pi}_P = [\hat{r}_1, ..., \hat{r}_n]$ be the recharging policy obtained using Algorithm 2 for the path P. Also let $\hat{\xi} = \sum_{j \in S_P} I_{\{\hat{r}_j > 0\}}$ be the number of recharging stops in $\hat{\pi}_P$. Then*

$$\hat{\xi} \leq \left\lceil \frac{2H}{\alpha q_{max}} \right\rceil,$$

and this bound is tight.

Proof of Lemma 5. Let μ_i denote the index of the i^{th} recharging stop where $\hat{r}_{\mu_i} > 0$ for all $i \in \{1,...,\hat{\xi}\}$. Due to the construction of $\hat{\pi}_P$, it must be true that $\alpha q_{max} < \hat{r}_{\mu_i} + \hat{r}_{\mu_{i+1}}$ for any $i \in \{1,...,\hat{\xi}-1\}$. Then we have

$$(\hat{\xi}-1)\alpha q_{max} < \sum_{i=1}^{\hat{\xi}-1}(\hat{r}_{\mu_i} + \hat{r}_{\mu_{i+1}}) < 2\sum_{i=1}^{\hat{\xi}}\hat{r}_{\mu_i} = 2H,$$

which implies

$$\hat{\xi} - 1 < \frac{2H}{\alpha q_{max}},$$

and

$$\hat{\xi} \leq \left\lceil \frac{2H}{\alpha q_{max}} \right\rceil.$$

In the extreme case where $h_i = \frac{1}{2}\alpha q_{max} + \varepsilon$ with $\varepsilon < 1$ for all $i \in S_P$, the above bound is tight since

$$\hat{\xi} = \frac{H}{\frac{1}{2}\alpha q_{max} + \varepsilon} < \lim_{\varepsilon \to 0}\left(\frac{H}{\frac{1}{2}\alpha q_{max} + \varepsilon}\right) = \frac{2H}{\alpha q_{max}}.$$

We thus have

$$\hat{\xi} \leq \left\lfloor \frac{2H}{\alpha q_{max}} \right\rfloor,$$

which is also equal to $\lceil 2H/\alpha q_{max} \rceil$ if $2H/\alpha q_{max}$ is an integer, and equality holds when

$$\varepsilon = \frac{H}{\lfloor 2H/\alpha q_{max} \rfloor} - \frac{1}{2}\alpha q_{max}.$$

Therefore, the lemma statement is valid.

We use this result in the next theorem to calculate the maximum ratio between the cost of the policy from Algorithm 2 and the cost of an optimal recharging policy.

THEOREM 5. *Let Assumption 1 hold. Let $\pi_P^* = [r_1^*,...,r_n^*]$ be an optimal recharging policy for the path P, and let $\hat{\pi}_P = [\hat{r}_1,...,\hat{r}_n]$ be the recharging policy obtained using Algorithm 2. Then the cost of $\hat{\pi}_P$ relative to that of π_P^* is bounded by*

$$\frac{C(\hat{\pi}_P)}{C(\pi_P^*)} \leq \frac{2}{\alpha} + \frac{q_{max}}{H}. \tag{11}$$

Proof of Theorem 5. Let K denote the overcharging cost incurred by the policy $\hat{\pi}_P$, where

$$K = k\sum_{i=1}^{n}(h_i - \alpha q_{max})^+$$

since the policy only overcharges whenever $h_i > \alpha q_{max}$. Then the total cost of the policy $\hat{\pi}_P$ satisfies

$$\begin{aligned} C(\hat{\pi}_P) &= \hat{\xi}s + K \\ &\leq \left\lceil \frac{2H}{\alpha q_{max}} \right\rceil s + K \quad \text{(by Lemma 5).} \end{aligned}$$

It follows that the ratio of the cost of the heuristic policy to the cost of an optimal policy must satisfy

$$\frac{C(\hat{\pi}_P)}{C(\pi_P^*)} \leq \frac{\left\lceil \frac{2H}{\alpha q_{max}} \right\rceil s + K}{\left\lceil \frac{H}{q_{max}} \right\rceil s + k\sum_{i=1}^{n}(r_i^* - \alpha q_{max})^+}$$
$$\leq \frac{\left\lceil \frac{2H}{\alpha q_{max}} \right\rceil s + K}{\frac{H}{q_{max}} s + K}$$

because any feasible recharging policy must incur overcharging costs of at least K. Furthermore, since $\left\lceil \frac{2H}{\alpha q_{max}} \right\rceil s > \frac{H}{q_{max}} s$, it follows that

$$\frac{C(\hat{\pi}_P)}{C(\pi_P^*)} \leq \frac{\left\lceil \frac{2H}{\alpha q_{max}} \right\rceil}{\frac{H}{q_{max}}} < \frac{\frac{2H}{\alpha q_{max}} + 1}{H/q_{max}} = \frac{2}{\alpha} + \frac{q_{max}}{H},$$

as desired.

In the result of Theorem 5, the term q_{max}/H represents a correction factor for the ceiling function in the numerator. As $H \to \infty$, the effect of the ceiling function becomes negligible and the ratio simply becomes $2/\alpha$, which is also equal to the maximum ratio between the number of stops in the two policies.

4.4.2. Heuristic 2: Minimize number of stops Our second heuristic seeks to minimize the total number of stops. As shown in Sections 4.2 and 4.3 for optimal continuous and equidistant recharging policies, as the stopping cost increases relative to the cost rate of overcharging, it becomes desirable to minimize the number of times that the vehicle stops to recharge. A method for obtaining such a policy is detailed in Algorithm 3. The heuristic finds the maximum value m such that $\sum_{j=i}^{m-1} h_j \leq q_{max}$ (i.e., the charge level required at node i to reach node m is no greater than q_{max}).

In a recharging policy produced by this procedure, the vehicle minimizes its total number of recharging stops and, whenever it stops, always recharges enough to travel as far as it can without having to stop again. This heuristic performs best when overcharging costs are low or the driver is averse to stopping more frequently to recharge.

We next derive an upper bound on the total cost of a policy obtained using Algorithm 3 relative to the cost of an optimal recharging policy in the following theorem. As with Lemma 5 and Theorem 5, we use Assumption 1 in our analysis.

Algorithm 3 Heuristic method for finding a recharging policy for a path P that minimizes the number of recharging stops

Input: number of nodes in P (i.e., $n+1$); inter-node energy requirements (h_j for every $j \in S_P$)

Output: a recharging policy for P, $\tilde{\pi}_P = [\tilde{r}_1, ..., \tilde{r}_n]$

Initialize: $\tilde{r}_j = 0$ for every $j \in S_P$; $i = 1$; $h_{n+1} = \infty$

while $i < n+1$ **do**
  $m = i + 1$
  while $\sum_{j=i}^{m} h_j \leq q_{max}$ **do**
    $m = m + 1$
  end while
  $\hat{r}_i = \sum_{j=i}^{m-1} h_j$
  $i = m$
end while

Theorem 6. *Let Assumption 1 hold. Let $\pi_P^* = [r_1^*, ..., r_n^*]$ be an optimal recharging policy for the path P, and let $\tilde{\pi}_P = [\tilde{r}_1, ..., \tilde{r}_n]$ be the recharging policy obtained using Algorithm 3. Then the cost of $\tilde{\pi}_P$ relative to that of π_P^* is bounded by*

$$\frac{C(\tilde{\pi}_P)}{C(\pi_P^*)} \leq \begin{cases} 1 + (1-\alpha)\left(\frac{H/\alpha q_{max}}{\lfloor H/\alpha q_{max} \rfloor}\right)\left(\frac{\alpha q_{max}}{s/k}\right), & s/k < \alpha q_{max} \\ 1/\alpha, & s/k \geq \alpha q_{max} \text{ and } H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} \leq 0 \\ 1 + \frac{\alpha q_{max}}{H}, & s/k \geq \alpha q_{max} \text{ and } H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} > 0. \end{cases} \tag{12}$$

Proof of Theorem 6. By extension of Lemma 3, it can be seen that the policy $\tilde{\pi}_P$ minimizes the number of recharging stops along P. For $\alpha = 1$, the lemma states that the policy $\tilde{\pi}_P$ is optimal, and its corresponding cost would be $\tilde{\xi}s$, where $\tilde{\xi}$ is the number of stops in the policy. Therefore, $\tilde{\xi}$ must be the minimum feasible number of stops. For the case where $\alpha < 1$, however, the policy may not be optimal due to the possibility of overcharging costs. At most, the amount overcharged along the entire path is $(1-\alpha)H$, where $H = \sum_{j=1}^{n} h_j$ is the total energy requirement of the path P. If we let ξ^* denote the number of stops in the optimal policy π_P^*, then we have

$$\frac{C(\tilde{\pi}_P)}{C(\pi_P^*)} \leq \frac{\tilde{\xi}s + k(1-\alpha)H}{\xi^* s + k(H - \xi^* \alpha q_{max})^+}.$$

For $s/k < \alpha q_{max}$, the cost of an optimal policy is bounded below by $\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor s$, which comes from the continuous result given in Theorem 2. It follows that

$$\frac{C(\tilde{\pi}_P)}{C(\pi_P^*)} \leq 1 + \frac{k(1-\alpha)H}{\xi^* s + k(H - \xi^* \alpha q_{max})^+} \quad (\text{since } \tilde{\xi} \leq \xi^*)$$

$$\leq 1 + \frac{k(1-\alpha)H}{\left\lfloor \frac{H}{\alpha q_{max}} \right\rfloor s}$$
$$= 1 + (1-\alpha)\left(\frac{H/\alpha q_{max}}{\lfloor H/\alpha q_{max} \rfloor}\right)\left(\frac{\alpha q_{max}}{s/k}\right).$$

When $s/k \geq \alpha q_{max}$, the cost of an optimal policy is bounded below by $\left\lceil \frac{H}{q_{max}} \right\rceil s + k\left(H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}\right)^+$ (from Theorem 2), and therefore

$$\begin{aligned}
\frac{C(\tilde{\pi}_P)}{C(\pi_P^*)} &\leq 1 + \frac{k(1-\alpha)H - k\left(H - \xi^* \alpha q_{max}\right)^+}{\xi^* s + k\left(H - \xi^* \alpha q_{max}\right)^+} \quad (\text{since } \tilde{\xi} \leq \xi^*) \\
&\leq 1 + \frac{k(1-\alpha)H - k\left(H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}\right)^+}{\left\lceil \frac{H}{q_{max}} \right\rceil s + k\left(H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}\right)^+} \\
&\leq 1 + \frac{(1-\alpha)H - \left(H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}\right)^+}{\left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} + \left(H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}\right)^+} \quad (\text{since } s/k \geq \alpha q_{max}) \\
&= 1 + \begin{cases} \frac{(1-\alpha)H}{\left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max}}, & H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} \leq 0 \\ \alpha\left(\left\lceil \frac{H}{q_{max}} \right\rceil \left(\frac{q_{max}}{H}\right) - 1\right), & H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} > 0 \end{cases} \\
&\leq \begin{cases} 1/\alpha, & H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} \leq 0 \\ 1 + \frac{\alpha q_{max}}{H}, & H - \left\lceil \frac{H}{q_{max}} \right\rceil \alpha q_{max} > 0. \end{cases}
\end{aligned}$$

Thus, the stated bound for $C(\tilde{\pi}_P)/C(\pi_P^*)$ is valid.

When the stopping cost is sufficiently high and $s/k \geq \alpha q_{max}$, the goals of the optimal and heuristic policies are both to minimize the total number of stops. Because their goals are aligned, the two policies perform similarly and the total cost of the heuristic policy is close to that of an optimal recharging policy. The bounding ratio $1/\alpha$ applies for larger values of α whereas the ratio $1 + \alpha q_{max}/H$ applies for smaller values of α, and both bounds are nearly equal to 1 for their respective α values. If the stopping cost is low, then the heuristic method performs quite poorly, and the value of the bounding ratio approaches ∞ as $s/k \to 0$.

5. Numerical results

To compare the actual performance of the two heuristic methods against the exact algorithm for finding an optimal recharging policy, we implemented Algorithms 1, 2, and 3 using data for U.S. Interstate 90. We consider the portion in the eastbound direction beginning at the Wisconsin/Illinois state border and ending at the New York/Massachusetts state border, which spans approximately 950 miles. This stretch of the road passes through Chicago, Cleveland, Buffalo, and other big cities and stretches along the north side of the country. The 256 node locations were determined by identifying existing exits along the highway leading to rest areas, towns, or other places where charging stations might feasibly be located. (Exits leading to other interstates, for example, are not included as nodes.)

We compare the recharging policies of the three methods for all combinations of the following parameter settings, where Assumption 1 applies.

$$q_{max} \in \{70, 75, 80, 85, 90, 95, 100\}$$
$$\alpha \in \{0.75, 0.76, 0.77, 0.78, 0.79.0.80, 0.81, 0.82, 0.83, 0.84, 0.85\}$$
$$s/k \in \{0, 10, 20, 30, 40, 50, 60, 70\}$$

The range of values for q_{max} corresponds to the maximum driving range of the Nissan Leaf under various weather and driving conditions. Rather than use traditional units of kilowatt-hours, we assume that the energy consumed per mile traveled is constant and instead use the amount of energy required to travel one mile as our unit of measure for q_{max} (e.g., $q_{max} = 70$ implies that the vehicle can travel 70 miles on a full charge). For α, the most commonly used value in practice is 0.8, but we consider the interval $[0.75, 0.85]$ to account for possible error. It is more difficult to assign a realistic value to the ratio s/k because it depends on a number of different factors, and as a result we include a wide range of values. If the overcharging cost rate is high (i.e., when charging stations use higher voltages to recharge vehicles and thus cause greater battery degradation), then the ratio can be low. On the other hand, if k is small or s is large, such as when the driver has an aversion to stopping to recharge, then the ratio can take on larger values.

5.1. Recharging policy analysis

The number of times that the vehicle stops to recharge in the optimal and heuristic recharging policies is shown in Figure 4. Because Heuristic 1 avoids overcharging, its policies have the greatest number of stops, and Heuristic 2's policies achieve the minimum number of stops. The number of stops in an optimal policy is always between those of the two heuristic policies, but it can also equal one or the other. Thus, the range of the numbers of stops in optimal policies among all parameter settings studied spans the ranges of both heuristics.

The solution quality of the recharging policies generated by both heuristics is illustrated in Figure 5. Recall from the expression in (11) that the bound on the ratio of the cost of a policy generated by Heuristic 1 to that of an optimal policy is $2/\alpha + q_{max}/H$. However, we see in the figure that all of the Heuristic 1 policies are within 20% of the optimal value. This suggests that the calculated bound may be skewed towards extreme cases and that, in practice, the heuristic performs much better than the worst case.

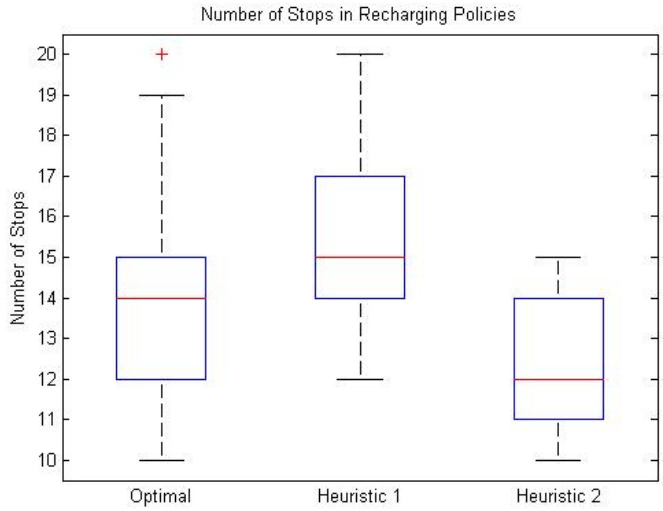

Figure 4 Number of stops in optimal and heuristic recharging policies

Policies generated by Heuristic 2 achieved a median ratio below 1.2, albeit with much variability overall and the worst instances having ratios above 2. Based on the bound given in (12), this result is to be expected. Heuristic 2 performs best when $s/k \geq \alpha q_{max}$ but does much poorer as $s/k \rightarrow 0$, and some of the parameter settings used in our numerical study have low s/k ratios. It should therefore be used more selectively for obtaining recharging policies in order to achieve the best results. This also confirms our earlier statement that minimizing the number of recharging stops is not beneficial if the stopping cost is relatively low.

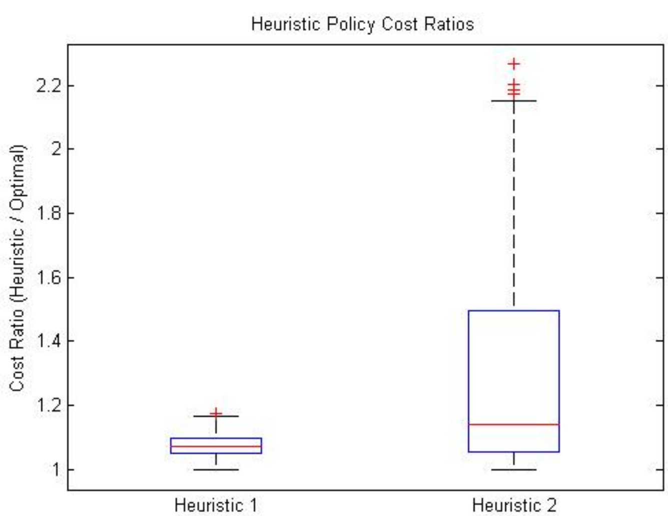

Figure 5 Heuristic recharging policy costs, measured relative to optimal recharging policy costs

5.2. Recharging stop locations

In Figure 6, the fraction of policies that recharge at each node, or stop frequencies, are shown for each of the different solution methods. (Note that recharging policies obtained from Heuristic 2 only depend on q_{max}, so there is little variability in the stop frequencies among the nodes.) The plot for optimal policies reveals one node near the center of the path that is used frequently as a recharging stop (by approximately 40% of policies), even though other nodes are nearby, which suggests that this particular node is important for optimal recharging policies along the studied path. The higher frequencies for some of the nodes between Chicago and Cleveland are simply due to the lower number of nodes within that region and do not highlight the importance of any one node in specific. These types of observations can be important to policy makers to suggest locations for building EV charging infrastructure.

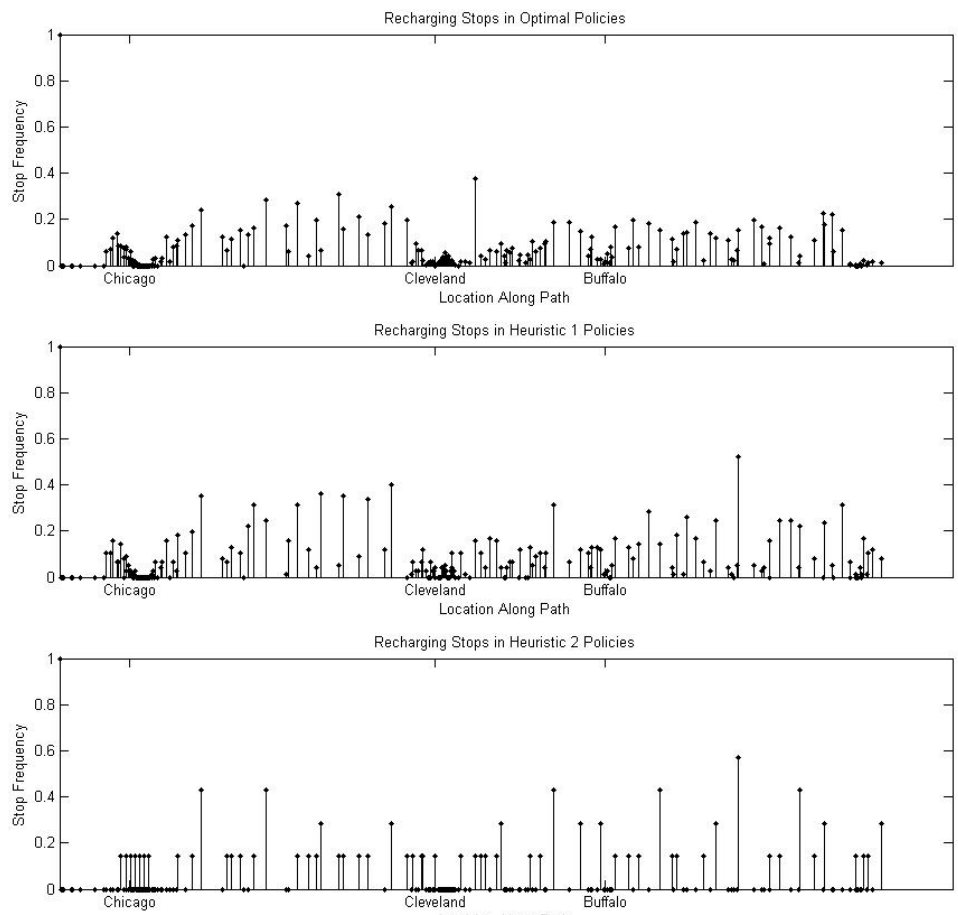

Figure 6 Recharging stop frequencies in optimal and heuristic policies

The plots for the two heuristic methods also show one node as being significant – approximately halfway between Buffalo and the New York/Massachusetts state border – although this one is different than the most frequent stopping location for optimal recharging policies. The portion of the route between Chicago and Cleveland again has several important nodes, and thus the nodes in regions without many other possible stopping locations are more likely to be stops in a recharging policy regardless of how the policy is determined. Other nodes that happen to be important for recharging policies obtained using one method may not be as important for policies created with another method.

One way to use the stop frequencies for locating charging stations is to first locate charging stations at the nodes with the greatest stop frequencies, and then distribute the remaining stations so that they are approximately evenly spaced along the rest of the path. A sample recommendation using this method is shown in Figure 7, where the stop frequencies for optimal policies are used. It includes 31 charging stations with an average distance of 29 miles, minimum distance of 17 miles, and maximum distance of 40 miles between adjacent stations. Placing stations based strictly on the stop frequencies is not appropriate because regions with fewer nodes would tend to have more charging stations than necessary, whereas the regions near major cities with high node densities would have few to none. However, by locating the first few charging stations at the nodes with the greatest stop frequencies, it becomes easier to place the remaining stations since the regions without charging stations yet will tend to have higher node densities and therefore fewer restrictions on where stations may be located.

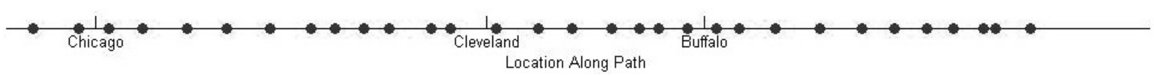

Figure 7 Sample recommendation for locating 31 charging stations based on stop frequencies for optimal policies

The analysis performed in this section assumes that traffic volume is constant along the entire route studied and that all drivers recharge their vehicles at the beginning of the route. More detailed traffic data that captures drivers entering the route at different exits along the way and departing before the end of the path would help to provide a more accurate picture of where charging stations are most needed along the highway. Also, better parameter estimation would reduce some of the noise present in the stop frequency plots and allow policy makers to more easily recognize the charging station locations with the most potential.

6. Conclusions and future work

In this paper, we study the problem of finding an optimal recharging policy for an EV along a fixed path. We first determine properties of optimal policies, and based on those properties, we design methods for obtaining optimal recharging policies for general paths, paths with continuous charging capability, and paths with equidistant charging locations. We also develop two heuristics and analyze the quality of their recharging policies, both theoretically and empirically for a designated stretch of U.S. Interstate 90. We find that each heuristic performs well, yielding recharging policies within 20% of optimal on average, with one of the heuristics producing policies within 20% of optimal for all considered parameter sets.

An important aspect of our work is the inclusion of a realistic recharging model for electric vehicles. While most of the literature on vehicle refueling policies has focused primarily on the limited range of the vehicles, we show in this paper that the costs associated with battery overcharging can significantly influence recharging decisions, and thus, they should be taken into consideration when determining recharging policies for EVs. The simpler models of vehicle refueling used for conventional gasoline-powered vehicles are not suitable for EVs and require major enhancements before they can be used to improve our understanding of the range of influences that battery dynamics have on EV recharging decisions.

Our current model considers recharging policies along a given route. A more detailed model would simultaneously find a path within a network, as well as a recharging policy for that path in order to minimize the total travel cost. In our future work, we plan to extend our current analysis and results in order to design algorithms that also determine the best path for the vehicle to take. Furthermore, we intend to integrate additional features into our model that capture such aspects of EV recharging as the impacts of overcharging and temperature on battery storage capacity, the ability to recover energy via regenerative braking, and time-of-use energy prices.

Acknowledgments

The authors would like to thank Scott Moura from the University of California, San Diego for his insightful discussions.

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