Optimal Expediting Policies for an Inventory System with Expiry Dates

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Abstract

We study serial stochastic lead time supply chains with a single supplier and a single manufacturing facility, where outstanding orders can be expedited to the manufacturing facility to avoid high backlogging costs in addition to the regular delivery channel. Furthermore, goods in transit are assumed to be perishable and have deterministic expiry dates, thus the outstanding orders close to the expiry date have to be *mandatorily expedited* in order to avoid high scrapping or spoilage costs. Under a particular expediting cost structure, we derive an optimal policy which minimizes the holding and backlogging costs, and the costs of expediting and regular ordering. The optimal expediting policy identifies a number of expediting base stock levels, which are monotone at any point in time. Because of this *dynamic monotonicity*, the optimal expediting policy is simple and well-structured. On the other hand, the optimal regular ordering policy is base stock with a single base stock level with respect to the inventory position.

1 Introduction

Inventory management is to find an optimal tradeoff among various costs such as holding, backlogging, spoilage, and delivery under various uncertainties including stochastic lead time and demand. Uncertainties and costs are closely related and have to be considered altogether in controlling the inventory. A frequent issue is high backlogging cost due to demand spikes and/or elongated lead time, thus many firms employ an expedited delivery channel, e.g., by air, in addition to its regular delivery channel, e.g., by ground. Though expediting of outstanding orders may incur additional expediting cost, it can reduce unexpected high backlogging cost if used effectively, and thus reduce the total costs. Furthermore, the utilization of expediting options adds agility to the supply chain

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and results in less safety stock on average in the system, hence less holding cost. Another frequent issue is managing time-bound goods or raw materials. Supply chain participants are now very concerned about the "freshness" of goods to guarantee a certain quality level. Setting expiry dates on orders in transit to manufacturers is becoming a common business practice. If such a time-bound item is ordered and not delivered within the set expiry date due to the stochastic nature of lead time, a high scrapping cost occurs. Again expediting can be used in order to avoid this high unexpected scrapping cost. This paper studies the optimal use of expediting options in a single supplier and manufacturing facility setting where there are multiple intermediate installations with stochastic lead time and goods in transit have deterministic expiry dates.

In practice, outstanding orders often do not cross in time. We also impose this observation in our problem by confining systems so that outstanding orders do not cross in time for both expediting and regular ordering by imposing certain conditions on the expediting costs. We call such systems *sequential systems* because they keep the sequence of orders in time within the *optimal* expediting practice. For these systems we provide analytical results on optimal policies, which also provide a valuable insight or a valid starting point in finding well-performing heuristics for general systems.

In our model, the manufacturer places an order to the supplier, and it takes stochastic time to deliver it through multiple intermediate installations. The expiry date is imposed on outstanding orders in transit. If an order is not delivered within the expiry date, we assume that the manufacturer must expedite it just before its expiry date to avoid much higher scrapping costs, and we call this process *mandatory expediting*. Since expediting is instantaneous relative to the scale of a time period, it implies that orders always arrive at the manufacturer before the expiry date. The manufacturer could also scrap the order instead of mandatory expediting, however this is not considered here. Another important modeling assumption is that we do not impose any expiry dates on delivered orders at the manufacturing facility. Once an order is delivered, we assume that the order is immediately processed and transformed into a nonperishable product. For example, canned products of meats, fish, or produce are possible applications of the model. After canning, the expiry date becomes much longer than the expiry date of raw materials, usually ranging up to several years. Similarly, any processed product with preservatives falls within this application category.

For sequential systems, the optimal policy for expediting identifies base stock levels for every age - time in the supply chain - and location of outstanding orders. In order for an optimal policy to be simple and practical, it is usually required to have a certain relationship among these base stock levels, for example monotonicity. We find two types of monotonicity: first, monotonicity with respect to a location for a fixed age, and second, the monotonicity with respect to age for a fixed location. Considering these two types of monotonicity together does not provide overall monotonicity of all the base stock levels (monotonicity is essential to get a simple optimal policy). Therefore, we introduce a concept of *dynamic monotonicity* for a subset of base stock levels, which is determined dynamically according to the current state of the system. We find that for sequential systems, dynamic monotonicity is always guaranteed and thus a simple optimal policy which is a variant of the base stock policy is obtained. At the same time, we find that the optimal policy for regular ordering is the base stock policy with respect to the inventory position.

We make several contributions. First, this is the seminal work considering expiry dates in a stochastic lead time setting with expediting. Second, the optimal policy for sequential systems is obtained. The optimal policy for expediting is a variant of the base stock policy with a number of base stock levels for each time period. The optimal regular ordering policy is also the base stock policy with a single base stock level for each time period. Third, the concept of *dynamic monotonicity* is introduced revealing a simple optimal policy.

In Section 2, we describe the model. Sequential systems are defined in Section 3. Optimal policies are derived and illustrated in Section 4. We conclude the introduction with the literature review.

Literature Review

Multi-modal supply problems are studied by Barankin (1961), Neuts (1964), Daniel (1963), Fukuda (1964), Veinott (1966), Lawson and Porteus (2000), Muharremoglu and Tsitsiklis (2003), Kim et al. (2009), and Kim et al. (2012). All of these multi-modal supply models assume non-order-crossing, and the optimal policy is usually the base stock type policy. The concept of sequential systems, which prevents order-crossing in the optimal expediting practice is first introduced in Kim et al. (2009) and extended in Kim et al. (2012). Kim et al. (2009) study the performance of systems with deterministic lead times under order-crossing numerically by using heuristics based on the optimal policy obtained for sequential systems.

Kim et al. (2012) consider a model with stochastic lead time, as apposed to deterministic as is the case in Kim et al. (2009), through multiple intermediate installations but no expiry date constraints. The added control constraints of expiry dates and mandatory expediting bring additional complexity to the solution structure, and a careful treatment is required. For instance, because of the expiry date, we have to keep track of the ages of all outstanding orders, which means many more dimensions in the state space and thus increased complexity. To manage the increased state space dimensionality, we have to carefully structure the state space. To optimally capture control with the expiry dates, the control scheme is much more sophisticated.

There are two major directions in the literature that consider both the expiry dates and exact optimal policies. One thread considers deterministic expiry dates while the other one considers random expiry dates. Most of the random expiry models assume continuous review, and we refer to Nahmias (1982) for a complete review. For models with deterministic expiry dates and periodic review, there are only a few notable works. For convenience, let us denote the deterministic lead time by l and the expiry date on shelf (*shelf life*) by m. The model with arbitrary deterministic shelf life $m \geq 1$ and lead time $l = 0$, where unmet demand is backlogged, is studied by Nahmias (1975) and Fries (1975) independently. These works are analytical, and they found that the optimal policy is a threshold policy, but the order-up-to level is a complex function of system states. Furthermore, the optimal policy depends on the initial amount of stock in time period 1. Nahmias (1982) states that the actual computation is impractical for $m \geq 3$. On the other hand, the model with $m = 2$ and arbitrary deterministic lead time $l \geq 0$, where unmet demand is lost, is studied by Williams and Patuwo (1999). Their work is computational and also shows that the optimal policy is complex. All other publications consider special cases either of Nahmias (1975), Fries (1975), or Williams and Patuwo (1999). For a review of approximation models, we refer to Nahmias (1982). All these differ with our work by not having stochastic lead time and not allowing expediting.

2 Model Statement

We consider a single supplier with a single-item manufacturing facility facing random demand D_t which has a compact support with known distribution, and $\bar{K} - 1$ serial intermediate installations between them. The supplier is denoted as installation $\bar{K}$ and the manufacturing facility is denoted as installation 0. The intermediate installations are numbered from 1 (next to the manufacturing facility) to $\bar{K} - 1$ (next to the supplier). The manufacturer periodically reviews the inventory on hand and places a regular order at the supplier by paying per unit procurement cost c_t in time period t. Unsatisfied demand is backlogged and excessive inventory at the manufacturing facility is penalized. Let us denote by $r(\cdot)$ the convex holding and backlogging cost function with respect to the on-hand inventory at the manufacturing facility, and $L_t(x) = E[r(x - D_t)]$. The amount of inventory at installation i for $0 \leq i \leq \bar{K}$ is denoted by v_i. The manufacturer may expedite any of the outstanding orders with an expediting cost d_i for expediting a unit from installation i. Every new order that the manufacturing facility places has an expiry date of R time periods. If an order is not delivered within R time periods, the manufacturer must expedite it from its current location just before it expires, and this is called *mandatory expediting*. Once the order arrives at $\bar{K}$ or less time periods, it is assumed that it is immediately processed and thus the expiry date no longer applies. The planning horizon consists of T time periods. For simplicity, we assume that the system is stationary with respect to demand distribution and procurement, holding, backlogging, and expediting costs. These assumptions are without loss of generality.

A movement pattern w defines the destination installation of outstanding orders for each installation in a time period. Multiple possible movement patterns may exist to describe the stochastic behavior of a delivery. Let us denote by $\mathbf{W}$ the set of all movement patterns, i.e., $\mathbf{W} = \{w_1, w_2, w_3, \cdots\}$. There is an exogenous random variable W with a known distribution that selects a movement pattern in $\mathbf{W}$. In each time period, W realizes, and according to the realized movement pattern w, the outstanding orders at installation $i, 1 \leq i \leq \bar{K}$, move to installation $j = M(i,w)$, $0 \leq j \leq i$, where M is a function that takes the origin installation i and the realized movement pattern w as arguments. Note that orders are not allowed to go backward to the upstream installations based on this definition though they may stay at the current location (installation). We define $M(0,w) = 0$, and before W is realized we denote the corresponding random variable by $M(i,W)$. Let us denote by $M^n(i,W)$ the n-period random movement function that represents the location after n regular movements of the outstanding orders at installation i. Formally, $M^1(i,W) = M(i,W)$ and $M^n(i,W) = M(M^{n-1}(i,W),W)$. We denote the stochastic lead time of an order at installation i by $l(i,W) = \min\{n : M^n(i,W) = 0, n \geq 1\}$. The lead time of a regular order is stochastic and determined by multiple realized movement patterns until delivery. This definition does not consider mandatory expediting, thus it is possible that it can be greater than R.

We make the following assumptions. Assumption 1 states that orders do not cross in time in movement patterns. This assumption is necessary for analytical tractability and is standard in treatment of stochastic lead times.

Assumption 1 (Orders not crossing in time). $M(i,w) \geq M(i-1,w)$ *for all* i *and* $w \in \mathbf{W}$.

The next assumption states that all regular orders must be eventually delivered, which is a natural assumption.

Assumption 2 (Eventual delivery of regular orders). $Prob[\cup_{n=1}^{\infty}\{w : M^n(i,w) = 0\}] = 1$ *for every installation* i.

Age of an order is the number of time periods since it was placed. When it has just been placed the age is 0, and there can be multiple outstanding orders with different ages in an installation. In Figure 1, we show physical installations with age bins for orders with different ages. Note that

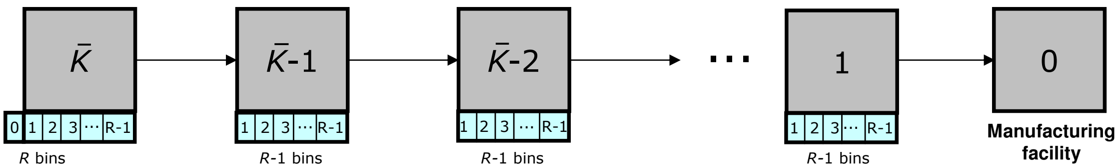

Figure 1: Physical installations with age bins

the bin for age 0 is only in installation $\bar{K}$ because fresh orders are always placed to installation $\bar{K}$ (The figure for consistency actually shows all of them). Also, there are no bins for age R because of mandatory expediting: if an order with age $R-1$ is not expedited at time period t, then it will move to a certain downstream installation according to the regular movement for time period t and will be expedited mandatorily at the beginning of time period $t+1$. Therefore we do not need age bins for age R. If an order about to expire is delivered to the manufacturing facility by a regular movement, we do not have to mandatorily expedite it. For convenience the cost of mandatory expediting at time period $t+1$ is accounted in time period t. We note that a higher aged order is closer to a lower aged order.

The sequence of events in a time period is as follows. At the beginning of the time period, the state information is given. First, mandatory expediting from the current location happens for goods that have been in age bin $R-1$ in the previous time period. Then the manufacturer places a regular order with the supplier (installation $\bar{K}$). Next, the manufacturer makes decisions on expediting for each installation, and the expedited orders arrive at the manufacturing facility instantaneously. After that, demand D realizes in the current time period. Inventory holding or backlogging cost is accounted for at the manufacturing facility after demand realization. Next, W realizes and regular delivery occurs just before the end of the time period. Then the next time period begins.

Let us denote the order that has j remaining time periods until mandatory expediting as a *stage* j order: stage = R–age. Again, we say that on order is closer to the manufacturing facility than a different order in the same physical location if its stage is lower. We express the state of the system by the stage inventory level and location as

$$(v_0, v_1, \cdots, v_{R-1}, l_1, l_2, \cdots, l_{R-1}),$$

where v_0 is the on-hand inventory at the manufacturing facility, v_j, $j \geq 0$, is the amount of stock in stage j, and l_j is the corresponding physical location. In other words, installation l_j contains the order in stage j. Note that an order in stage j can have at most one location, though there can be multiple stage orders at a location. If there is no order in stage j or $v_j = 0$, then we assume $l_j = 0$. We define $l_0 = 0$, $l_R = \bar{K}$, and v_R as the amount of fresh order. Also, if $v_i > 0$ for $i > 0$, then l_i must be positive as well. Figure 2 shows the stage inventory levels and location representations. For notational convenience, let $\bar{v}_i = (v_i, v_{i+1}, v_{i+2}, \cdots, v_{R-1})$, $\bar{l}_i = (l_i, l_{i+1}, l_{i+2}, \cdots, l_{R-1})$, and

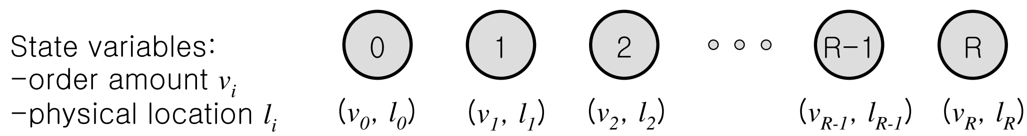

Figure 2: Stage inventory levels and location representations

$\bar{0}^i = (0, 0, \cdots, 0)$ of dimension i. Then, the state can be written in a compact way as $(v_0, \bar{v}_1, \bar{l}_1)$. Finally, let us denote by $\bar{l}(j, l_j, W)$ the lead time including possible mandatory expediting of the order at stage j, installation l_j. Note that $\bar{l}(j, l_j, W) = \min\{j, l(l_j, W)\}$ and thus $\bar{l}(j, l_j, W) \leq j$ and $\bar{l}(j, l_j, W) \leq l(l_j, W)$.

3 Sequential Systems

Due to the complexity of the problem, we confine our problem under the following cost structure on expediting costs. We call such systems sequential because they preserve the sequence of orders under the optimal expediting practice including mandatory expediting. Theorem 1 presents this result.

Sequential systems A system is *sequential*, if expediting cost coefficients d_i's satisfy $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ for all i, where $d_0 = 0$.

Quantity $d_i - E[d_{M(i,W)}]$ represents the time value of delayed expediting because it is the cost that can be saved if we delay expediting an order at installation i for a time period. Therefore, the definition states that the time value of delayed expediting is increasing as orders move further away from the manufacturing facility.

Theorem 1. *Sequential systems preserve the sequence of orders in time when operated optimally.*

Proof. Consider two nonempty stages i and j, $i > j$, which contain $unit_i$ and $unit_j$, respectively. We compare the following two strategies.

- Strategy 1: Expedite $unit_j$ from stage j, and then expedite $unit_i$ after (realized) $\bar{l}(j, l_j, w)$ time periods from the corresponding position. This strategy is feasible by keeping track of the position of a fictitious unit in stage j, since it takes (realized) $\bar{l}(j, l_j, w)$ time periods for the fictitious unit to arrive at the manufacturing facility. The expected cost of strategy 1 is $d_{l_j} + E[d_{M^{\bar{l}(j,l_j,W)}(l_i,W)}]$.
- Strategy 2: Expedite $unit_i$, and then do not do anything on $unit_j$ except mandatory expediting. The expected cost of strategy 2 is $d_{l_i} + E[d_{M^{\bar{l}(j,l_j,W)}(l_j,W)}]$.

These two strategies are identical in the view point of the manufacturer because they yield the same inventory level at the manufacturer. However, the cost implication is different.

Sequential systems satisfy $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ for all i, where $d_0 = 0$. From Lemma 2 in Appendix, we have $d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}]$, for any i and j, $i \geq j$, and $n \geq 1$ because M is defined independently from the expiry date of R time periods. By considering $n = \bar{l}(j, l_j, w)$, we have

$$d_{l_i} + E[d_{M^{\bar{l}(j,l_j,W)}(l_j,W)}] \geq d_{l_j} + E[d_{M^{\bar{l}(j,l_j,W)}(l_i,W)}]. \tag{1}$$

This implies that Strategy 1 is always better than or equal to Strategy 2 in terms of costs. In other words, any policy that starts with Strategy 2 cannot be optimal. Therefore, if expediting is necessary in sequential systems, it is optimal to expedite from the nonempty installation that is closest to the manufacturing facility. Therefore, orders preserve sequence in time under an optimal expediting policy for sequential systems. The proof is completed. □

Theorem 1 provides an important fact of the optimal expediting policy for sequential systems: it is best to expedite the closest order to the manufacturing facility if we need to expedite any outstanding orders. We build the optimality equation based on this theorem. To do so, let us define restricted cost-to-go functions: for $1 \leq j \leq R$, let $J_t^j(\cdot)$ be the optimal cost-to-go from time period t onwards that is achieved by a restricted control space, in which expediting from *stages* $j+1, j+2, \cdots, R$ in time period t is not allowed. The control space for J_t^j is restricted in time period t, but unrestricted after time period t. We mainly utilize J_t^j with respect to special states $A_i = (x, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i)$ for $1 \leq i \leq R-1$. This state represents x units at the manufacturing facility, no inventory from stage 1 up to stage $i-1$, v_i units in stage i, v_{i+1} units in stage $i+1$, and so on. The corresponding locations are zero up to stage $i-1$ by definition, l_i for stage i, l_{i+1} for stage $i+1$, and so on. Also, let us denote by NS_i the next state in the next time period from the current state A_i under the restriction that expediting only from stage i is allowed in the current time period.

Let J_t be the optimal cost-to-go without any restrictions and let the echelon stock be $x^i = v_0 + v_1 + \cdots + v_i$. For a sequential system, we have the following optimality equation

$$\begin{aligned} J_t(v_0, \bar{v}_1, \bar{l}_1) = \min\{ & J_t^1(x^0, \bar{v}_1, \bar{l}_1), \\ & d_{l_1} v_1 + J_t^2(x^1, 0, \bar{v}_2, 0, \bar{l}_2), \\ & d_{l_1} v_1 + d_{l_2} v_2 + J_t^3(x^2, \bar{0}^2, \bar{v}_3, \bar{0}^2, \bar{l}_3), \\ & \quad \cdots, \\ & \sum_{i=1}^{R-1} d_{l_i} v_i + J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1})\}, \end{aligned}$$

where

$$\begin{aligned} J_t^1(x^0, \bar{v}_1, \bar{l}_1) = & \min_{\substack{x^0 \le y_1 \le x^1 \\ z \ge x^{R-1}}} \{ d_{l_1}(y_1 - x^0) + c(z - x^{R-1}) + L(y_1) \\ & + E[d_{M(l_1,W)}](x^1 - y_1) + E[J_{t+1}(NS_1)]\}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} J_t^i(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) = & \min_{\substack{x^{i-1} \le y_i \le x^i \\ z \ge x^{R-1}}} \{ d_{l_i}(y_i - x^{i-1}) + c(z - x^{R-1}) + L(y_i) \\ & + E[J_{t+1}(NS_i)]\}, \end{aligned} \tag{3}$$

for $i > 1$, where y_i and z are decision variables: $y_i - x^{i-1}$ corresponds to the expediting amount from stage i, and $z - x^{L-1}$ corresponds to the regular ordering amount. In (2), $E[d_{M(l_1,W)}](x^1 - y_1)$ represents the mandatory expediting cost. To facilitate the analysis of the system dynamics, let us define the following set of probabilities:

- $p(l_i) = prob(M(l_i, W) > 0)$, and
- $p(l_i, l_{i+1}) = prob\{M(l_i, W) = 0 \text{ and } M(l_{i+1}, W) > 0\}$ for all i.

Note that p's are deterministic functions, and if $l_i = l_{i+1}$, then $p(l_i, l_{i+1}) = 0$. We first present the following lemma with its proof in Appendix.

Lemma 1. *We have $p(l_i) + p(l_i, l_{i+1}) = p(l_{i+1})$ for all i.*

For ease of exposition, let $M(\bar{l}_i, W) = (M(l_i, W), M(l_{i+1}, W), \cdots, M(l_R, W))$. Then, the future cost $E[J_{t+1}(NS_i)]$ is

$$\begin{aligned}
E[J_{t+1}(NS_i)] = & \, p(l_i)E[J_{t+1}(y_i - D, \bar{0}^{i-2}, x^i - y_i, \bar{v}_{i+1}, u, \bar{0}^{i-2}, M(\bar{l}_i, W))|M(l_i, W) > 0] \\
& + p(l_i, l_{i+1})E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0] \\
& + p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^{i}, \bar{v}_{i+2}, u, \bar{0}^{i}, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0] \\
& \quad \vdots \\
& + p(l_{R-1}, l_R)E[J_{t+1}(x^{R-1} - D, \bar{0}^{R-2}, u, \bar{0}^{R-2}, M(l_R, W))|M(l_R, W) > 0] \\
& + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],
\end{aligned} \tag{4}$$

where u is the amount of the regular order: i.e. $u = z - x^{R-1}$. Note the conditional expectations in (4) to distinguish different cases. We also assume that all terminal functions are 0.

4 Optimal Policies for Sequential Systems

In this section, we focus on identifying optimal policies for sequential systems with mandatory expediting. We start with preliminary results.

4.1 Preliminaries

In analyzing the optimality equation, an important result is that

$$J_t(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) - J_k(x^i, \bar{0}^i, \bar{v}_{i+1}, \bar{0}^i, \bar{l}_{i+1}) \text{ is only a function of } i, t, l_i, x^{i-1}, \text{ and } x^i.$$

Furthermore, this function has the form of $S^0_{i,t}(l_i) + S^1_{i,t}(x^{i-1}, l_i) + S^2_{i,t}(x^i, l_i)$ for recursively defined functions $S^0_{i,t}, S^1_{i,t}$ and $S^2_{i,t}$. Another important result is that the minimization with respect to y_i in (2) and (3) can be isolated from the minimization with respect to z, and it has the form of $\min f_{i,t}(y_i, l_i)$ for a function $f_{i,t}$ defined by using $S^0_{i,t}, S^1_{i,t}$ and $S^2_{i,t}$. These key functions greatly simply the analysis of the optimal policy. The definition of these key functions is as follows.

$$\begin{aligned}
f_{1,t}(y_1, l_1) &= d_{l_1}y_1 + L(y_1) - p(l_1)E[d_{M(l_1,W)}(y_1 - D)|M(l_1, W) > 0] \\
&= d_{l_1}y_1 + L(y_1) - E[d_{M(l_1,W)}(y_1 - D)] \\
f_{i,t}(y_i, l_i) &= d_{l_i}y_i + L(y_i) + p(l_i)E[S^1_{i-1,t+1}(y_i - D, M(l_i, W))|M(l_i, W) > 0] \\
&= d_{l_i}y_i + L(y_i) + E[S^1_{i-1,t+1}(y_i - D, M(l_i, W))] \text{ for } i > 1
\end{aligned}$$

$$
\begin{aligned}
S^0_{i,t}(l_i) &= p(l_i)E[S^0_{i-1,t+1}(M(l_i,W))|M(l_i,W)>0] + a_{i,t}(l_i) \\
&= E[S^0_{i-1,t+1}(M(l_i,W))] + a_{i,t}(l_i) \\
S^1_{i,t}(x^{i-1},l_i) &= g_{i,t}(x^{i-1},l_i) - d_{l_i}x^{i-1} \\
S^2_{1,t}(x^1,l_1) &= h_{1,t}(x^1,l_1) - L(x^1) + p(l_1)E[d_{M(l_1,W)}|M(l_1,W)>0]x^1 \\
&= h_{1,t}(x^1,l_1) - L(x^1) + E[d_{M(l_1,W)}]x^1 \\
S^2_{i,t}(x^i,l_i) &= h_{i,t}(x^i,l_i) - L(x^i) + p(l_i)E[S^2_{i-1,t+1}(x^i-D,M(l_i,W))|M(l_i,W)>0] \\
&= h_{i,t}(x^i,l_i) - L(x^i) + E[S^2_{i-1,t+1}(x^i-D,M(l_i,W))] \text{ for } 1<i\le R-1
\end{aligned}
$$

We also define $S^0_{0,t}(\cdot) = S^1_{0,t}(\cdot) = S^2_{0,t}(\cdot) = 0$ for all t and $S^0_{i,T+1}(\cdot) = S^1_{i,T+1}(\cdot) = S^2_{i,T+1}(\cdot) = 0$ for all i. Here, $a_{i,t}$, $g_{i,t}$, and $h_{i,t}$ are defined according to Lemma 3 in Appendix with respect to $f_{i,t}$. This lemma is applied for any fixed l_i and thus $a_{i,t}$, $g_{i,t}$, and $h_{i,t}$ depend on l_i. Starting from the last time period T, functions $f_{i,t}$ and $S^j_{i,t}$ can be obtained recursively. It is easy to check for all i and t that $f_{i,t}(y_i,l_i)$ is convex in y_i for a given l_i, if the system is sequential. Also, $S^0_{i,t}(l_i) + S^1_{i,t}(x,l_i) + S^2_{i,t}(x,l_i) = 0$ for every x and l_i.

Let us denote by $y^*_{i,t}(l_i)$ a minimizer of $f_{i,t}(y_i,l_i)$: $y^*_{i,t}(l_i) \in \arg\min_{y_i} f_{i,t}(y_i,l_i)$. The following theorem is an important property of $f_{i,t}(y_i,l_i)$ for sequential systems, and the proof is given in Appendix.

Theorem 2. *For sequential systems, the following holds:*

a. for any given j, $y^*_{i,t}(j)$ is nonincreasing in i, and

b. for any given i, $y^*_{i,t}(j)$ is nonincreasing in j.

4.2 Optimal Policies

For sequential systems we have the following theorem, which is the key result in this paper.

Theorem 3. *a. The base stock policy with respect to the corresponding echelon stock x^{i-1} is optimal for expediting from stage i. Also, the base stock policy with respect to the inventory position x^{R-1} is optimal for regular ordering.*

b. Function $p(l_R)E[S^2_{R-1,t}(z-D,M(l_R,w))|M(l_R,w)>0] + E[J_t(z-D,\bar{0}^{R-1},\bar{0}^{R-1})]$ is convex in z.

c. For $1 \le i \le R-1$, we have $J_t(x^{i-1},\bar{0}^{i-1},\bar{v}_i,\bar{0}^{i-1},\bar{l}_i) - J_k(x^i,\bar{0}^i,\bar{v}_{i+1},\bar{0}^i,\bar{l}_{i+1}) = S^0_{i,t}(l_i) + S^1_{i,t}(x^{i-1},l_i) + S^2_{i,t}(x^i,l_i)$.

Part a is the most important as it describes the optimal policies. Part b and c are needed in the inductive proof. We postpone the proof of this theorem to later in the document and first provide an illustrative example.

4.3 Illustration of the Optimal Policy

Part (a) of Theorem 3 states that the optimal regular ordering policy follows the base stock policy with respect to the inventory position. Compared to the result in Kim et al. (2012), though the base stock level is different, the optimal regular ordering policy remains the same regardless of mandatory expediting. However, the optimal expediting policy is quite different with the introduction of mandatory expediting. We explain the optimal expediting policy described in Theorems 2 and 3 through the following illustrative example.

Part (a) of Theorem 2 states that there is monotonicity of expediting base stock levels across installations for the same age bins, see Figure 3. On the other hand, part (b) of Theorem 2 states

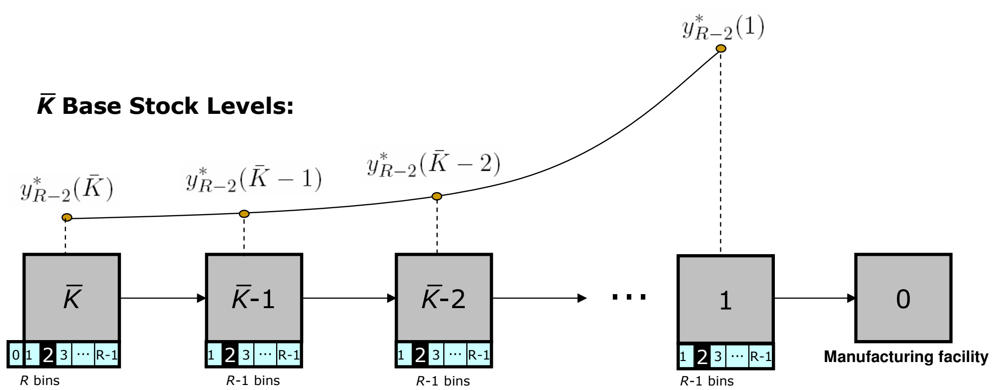

Figure 3: Monotonicity across installations of the same age

that there is monotonicity of expediting base stock levels across all age bins in an installation, see Figure 4. Considering both parts of Theorem 2, we do not have overall monotonic base stock levels for expediting as shown in Figure 5. At first, it appears that this non-monotonicity contradicts the definition of sequential systems or Theorem 1, since order crossing in time might happen for expedited orders. However, if we consider only the expediting base stock levels for nonempty age bins in all installations, the monotonicity is established, and the reason is the following. Assumption 1 guarantees that outstanding orders under regular movements do not cross in time, thus an order that is placed earlier should be closer to the manufacturing facility. Also, the order that is placed earlier has higher age, or in other words, lower stage. Therefore, if an order is closer to the

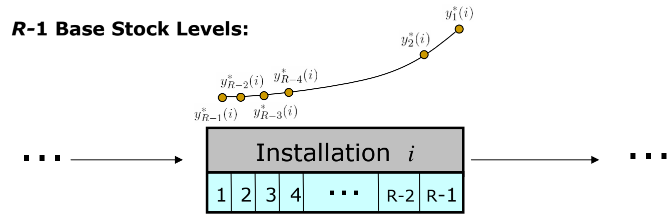

Figure 4: Monotonicity within an installation

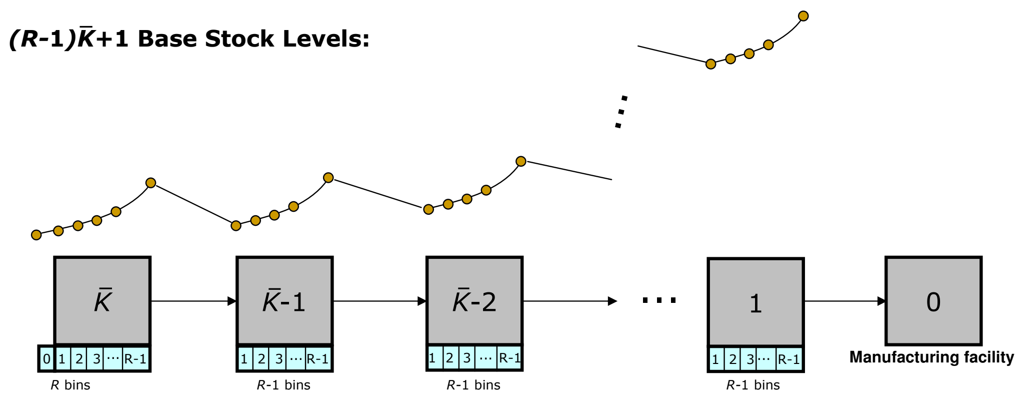

Figure 5: Non-monotonicity considering all age bins in all installations

manufacturing facility, it must have higher age or lower stage. Now parts (a) and (b) of Theorem 2 combined indicate that there is monotonicity of the expediting base stock levels for nonempty age bins at any moment, as shown in Figure 6. Because the nonempty age bins are changing in every time period, the set of corresponding base stock levels is also changing, but the monotonicity in the set is guaranteed, and this in essence is *dynamic monotonicity*. It is obvious that order crossing does not happen for sequential systems, hence there is no contradiction.

Finally, the dynamic monotonicity and part (a) of Theorem 3 reveal the simple structure of the optimal expediting policy for sequential systems as follows. The echelon stock is nondecreasing as the stage increases, since it is the sum of nonnegative numbers. At the same time, we have dynamic monotonicity of the expediting base stock levels. Therefore, there can be at most one intersection point between the echelon stock and the base stock level profiles. Part (a) of Theorem 3 implies to expedite everything up to this intersection, see Figure 7.

As the expiry date increases to infinity, the optimal expediting policy just illustrated reduces to the optimal expediting policy in Kim et al. (2012) because the probability of mandatory expediting becomes smaller. This implies that the expediting base stock levels are getting closer to each other

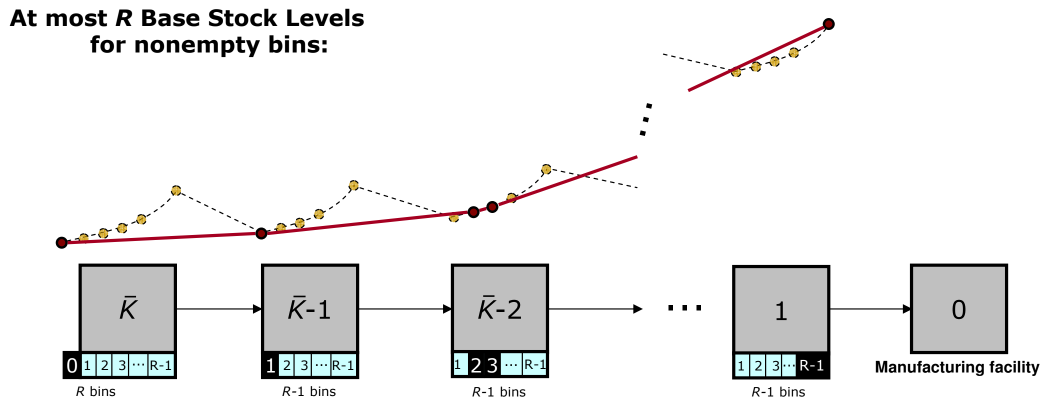

Figure 6: Dynamic monotonicity of nonempty age bins in all installations

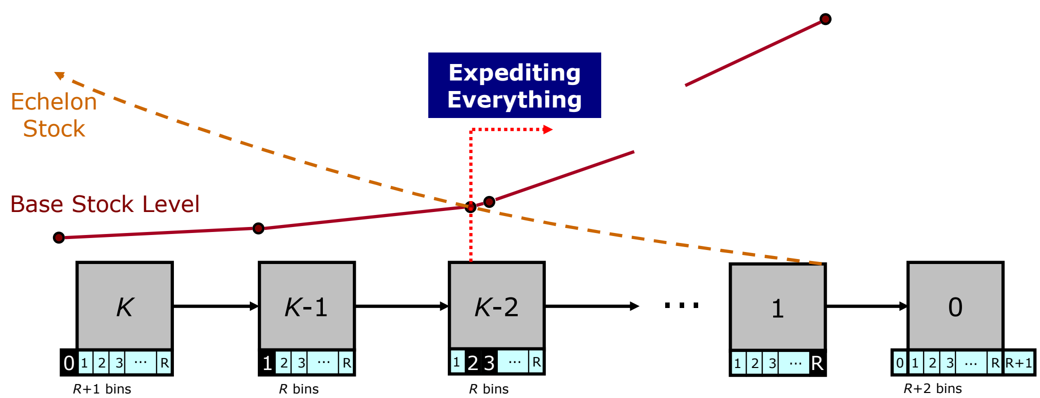

Figure 7: The simple structure of the optimal expediting policy

as the expiry date increases, and we conjecture that they eventually converge to a single value for each installation, and we have a unique expediting base stock level for each installation. Since they are monotonic, we must have the optimal expediting policy as described in Kim et al. (2012).

4.4 Proof of Optimality

Proof of Theorem 3. We prove parts (a), (b), and (c) concurrently by induction on t. In the base case $t = T + 1$, the optimal expediting and regular ordering policies are null. Also (b) and (c) hold obviously when $t = T + 1$. Now we proceed to the induction step.

We prove that part (a) holds at time period t. First consider (4). By repeatedly applying part

(c) with time period $t+1$, which holds by the induction hypothesis, we have

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S^0_{i-1,t+1}(M(l_i,W)) + S^1_{i-1,t+1}(y_i - D, M(l_i,W)) \\
& + S^2_{i-1,t+1}(x^i - D, M(l_i,W)) | M(l_i,W) > 0] \\
& + \cdots \\
& + p(l_R)E[J_{t+1}(x^{R-1} - D, \bar{0}^{R-2}, u, \bar{0}^{R-2}, M(l_R,W)) | M(l_R,W) > 0] \\
& + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],
\end{aligned}
$$

where Lemma 1 is used. This is again rearranged to the following by using part (c):

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S^0_{i-1,t+1}(M(l_i,W)) + S^1_{i-1,t+1}(y_i - D, M(l_i,W)) \\
& + S^2_{i-1,t+1}(x^i - D, M(l_i,W)) | M(l_i,W) > 0] \\
& + \cdots \\
& + p(l_R)E[S^0_{R-1,t+1}(M(l_R,W)) + S^1_{R-1,t+1}(x^{R-1} - D, M(l_R,W)) \\
& + S^2_{R-1,t+1}(z - D, M(l_R,W)) | M(l_R,W) > 0] + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})].
\end{aligned}
$$

Let us denote by OT the terms that contain only state variables. Then,

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S^1_{i-1,t+1}(y_i - D, M(l_i,W)) | M(l_i,W) > 0] \\
& + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R,W)) | M(l_R,W) > 0] \\
& + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] + OT.
\end{aligned} \tag{5}
$$

Plugging (5) into (2) and (3) yields

$$
\begin{aligned}
J^i_t(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) = {} & \min_{x^{i-1} \le y_i \le x^i} f_{i,t}(y_i, l_i) \\
& + \min_{z \ge x^{R-1}} \{cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R,W)) | M(l_R,W) > 0] \\
& + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} + OT,
\end{aligned} \tag{6}
$$

for $i < R$. For $i = R$, we have

$$
\begin{aligned}
J^R_t(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = {} & \min_{x^{R-1} \le y_R \le z} \{f_{R,t}(y_R, l_R) + cz \\
& + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R,W)) | M(l_R,W) > 0] \\
& + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} \\
& + p(l_R)E[S^0_{R-1,t+1}(M(l_R,W)) | M(l_R,W) > 0] \\
& - d_{l_R} x^{R-1} - c x^{R-1}.
\end{aligned} \tag{7}
$$

Note that $l_R = \bar{K}$. By applying Lemma 9 in Appendix, we have

$$\begin{aligned} J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = \min_{x^{R-1} \le z} \{ & h_{R,t}(z, l_R) + cz \\ & + p(l_R) E[S^2_{R-1,t+1}(z - D, M(l_R, W)) | M(l_R, W) > 0] \\ & + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] \} + a_{R,t}(l_R) \\ & + p(l_R) E[S^0_{R-1,t+1}(M(l_R, W)) | M(l_R, W) > 0] \\ & + g_{R,t}(x^{R-1}, l_R) - d_{l_R} x^{R-1} - c x^{R-1}, \end{aligned} \tag{8}$$

which is equal to

$$\begin{aligned} J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = \min_{x^{R-1} \le z} \{ & h_{R,t}(z, l_R) + cz \\ & + p(l_R) E[S^2_{R-1,t+1}(z - D, M(l_R, W)) | M(l_R, W) > 0] \\ & + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] \} - S^2_{R,t}(x^{R-1}, l_R) - c x^{R-1}. \end{aligned} \tag{9}$$

Therefore, optimal expediting follows the base stock policy from (6) or (7) with the base stock level given by

$$y_i^*(l_i) = \arg\min_{y_i} f_{i,t}(y_i, l_i).$$

The optimal regular ordering policy is the base stock policy with the base stock level z_t^* determined from (9) by

$$\begin{aligned} z*_t = \arg \min_{z \ge x^{R-1}} \{ & h_{R,t}(z, l_R) + cz + p(l_R) E[S^2_{R-1,t+1}(z - D, M(l_R, W)) | M(l_R, W) > 0] \\ & + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] \}, \end{aligned}$$

for any i. Since $p(l_R) E[S^2_{R-1,t+1}(z - D, M(l_R, W)) | M(l_R, W) > 0] + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]$ is convex by the induction hypothesis of part (b), the optimal regular ordering policy is well defined.

Note that we use (9) instead of (6) in determining the optimal regular ordering quantity for the following reason. Recall that $h_{R,t}(z, l_R)$ is nonincreasing convex, and $h_{R,t}(z, l_R) = 0$ for $z \ge y^*_{R,t}$. Therefore, if $z_t^* \ge y^*_{R,t}(l_R)$, then (6) and (9) lead to the same minimizer z_t^*. On the other hand, if $z_t^* < y^*_{R,t}(l_R)$, then the minimizer from (6) is also smaller than $y^*_{R,t}(l_R)$. In this case, either there is no regular ordering (when $x^R > y^*_{R,t}(l_R)$) or we expedite everything in the supply chain including the fresh regular order (when $x^R \le y^*_{R,t}(l_R)$) because of Theorem 2 in Appendix, where we have $z_t^* < y^*_{i,t}(l_i)$ for all i. When there is no regular ordering, (9) and (6) lead to the same result again. When expediting everything, (9) determines the regular ordering quantity since we are expediting from the supplier. As a result, (9) always determines the optimal regular ordering. This completes the induction step of part (a).

Next, we prove that part (b) holds at time period t. Adding

$$p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0]$$

to both sides of (8), we have

$$\begin{aligned}
&p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0] + J^R_t(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) \\
&\quad = \min_{x^{R-1} \le z} \{h_{R,t}(z, l_R) + cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, W))|M(l_R, W) > 0] \\
&\quad + J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} + p(l_R)E[S^0_{R-1,t+1}(M(l_R, W))|M(l_R, W) > 0] \\
&\quad + g_{R,t}(x^{R-1}, l_R) + p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0] \\
&\quad + a_{R,t}(l_R) - d_{l_R} x^{R-1} - cx^{R-1}.
\end{aligned}$$

By the induction hypothesis,

$$p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, w))|M(l_R, w) > 0] + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]$$

is convex in z. Therefore,

$$\begin{aligned}
&\min_{x^{R-1} \le z} \{h_{R,t}(z, l_R) + cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, W))|M(l_R, W) > 0] \\
&\quad + J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\}
\end{aligned}$$

is convex in x^{R-1}. Also, $g_{R,t}(x^{R-1}, l_R) + p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0]$ is convex in x^{R-1} by Lemma 8 in Appendix. All the other terms are either constant or linear in x^{R-1}. Therefore,

$$p(l_R)E[S^2_{R-1,t+1}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0] + J^R_t(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1})$$

is convex in x^{R-1}, and the proof of part (b) is completed.

The proof of part (c) is provided in Appendix. □

5 Concluding Remarks

This paper addresses operational issues resulting from both stochastic demand and lead time with expiry dates by considering expediting. It presents the optimal policy of regular ordering and expediting. The mandatory expediting brings complexity to the optimal policy but in a manageable and well-structured fashion due to the dynamic monotonicity of expediting base stock levels. This result certainly broadens our understanding and intuition when coping with demand and lead time related emergencies in supply chains.

An important but unaddressed situation, though, is the expiry constraint within the manufacturing facility. There exists literature on the shelf life of various models that have deterministic lead times. We deem that an important task in the future is to extend our results to include the shelf life of the delivered orders in the manufacturing facility. Figure 8 summarized the previous research, the position of this paper, and the future direction. We do not think that this extension

		Expiry on Delivery	Shelf Life	
			At most 2 time periods	Multiple time periods
Stochastic Lead Time		Our research	No previous literature	No previous literature
Deterministic Lead Time	Nonzero	Not meaningful	Williams and Patuwo (1999) - Computational	No previous literature
	Zero	Not meaningful	Not meaningful	Nahmias(1975), Fries (1975) - Analytical

Figure 8: Future research

will be immediate, since the previous literature on the shelf life suggests inherent complexity of the optimal policy in simpler cases. However, we believe that there can be theoretical or practical solutions for this extension with the advancement of our understanding in complex supply chain systems.

Appendix

Lemma 2. *In sequential systems,* $d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}]$*, for any* i *and* j*,* $i \geq j$*, and* $n \geq 1$*.*

Proof. The proof can be found in Kim et al. (2012). $\square$

Proof of Lemma 1. By assumption, regular orders do not cross in time. From $p(l_i) = prob(M(l_i, W) > 0) = prob\{M(l_i, W) > 0 \text{ and } M(l_{i+1}, W) > 0\}$, we have $p(l_i) + p(l_i, l_{i+1}) = prob\{M(l_i, W) > 0 \text{ and } M(l_{i+1}, W) > 0\} + prob\{M(l_i, W) = 0, \text{ and } M(l_{i+1}, W) > 0\} = prob(M(l_{i+1}, W) > 0) = p(l_{i+1})$ for all i and any value of l_i. $\square$

Lemma 3. *Let f be convex and have a finite minimizer on $\mathbb{R}$. Let $y^* = \arg\min f(x)$. Then, $\min_{x_1 \le x \le x_2} f(x) = a + g(x_1) + h(x_2)$, where $a = f(y^*)$, and penalty functions $g(x_1)$ and $h(x_2)$ are*

$$g(x_1) = \begin{cases} 0 & x_1 \le y^* \\ f(x_1) - a & x_1 > y^* \end{cases} \quad \text{and} \quad h(x_2) = \begin{cases} f(x_2) - a & x_2 \le y^* \\ 0 & x_2 > y^* \end{cases}.$$

For a nondecreasing convex f, we define $a = 0$, $g(x) = f(x)$, and $h(x) = 0$. On the other hand, for a nonincreasing convex f, we define $a = 0$, $g(x) = 0$, and $h(x) = f(x)$.

Proof. The proof can be found in Karush (1959). □

For a convex function $f : \mathbb{R} \to \mathbb{R}$, let $\partial f(x)$ be its subdifferential at x, which is a set. For two sets S_1 and S_2, we denote $S_1 \le S_2$ if there exists $s_2 \in S_2$ such that $s_1 \le s_2$ for any $s_1 \in S_1$, and there exists $s_1 \in S_1$ such that $s_1 \le s_2$ for any $s_2 \in S_2$. The following lemmas are from Kim et al. (2012).

Lemma 4. *Let f_1 and f_2 be convex functions. If $\partial f_1(x) \le \partial f_2(x)$ for all $x \in \mathbb{R}$, then*

$$\arg\min_x f_1(x) \ge \arg\min_x f_2(x).$$

Lemma 5. *Let f_1 and f_2 be convex functions, and let g_1 and g_2 be their penalty functions as in Lemma 3. If $\partial f_1(x) \le \partial f_2(x)$, then $\partial g_1(x) \le \partial g_2(x)$.*

Lemma 6. *Let f_1, f_2, $\tilde{f}_1$, and $\tilde{f}_2$ be convex functions. If $\partial f_1(x) \le \partial f_2(x)$ and $\partial \tilde{f}_1(x) \le \partial \tilde{f}_2(x)$, then $\partial\{f_1 + \tilde{f}_2\}(x) \le \partial\{f_2 + \tilde{f}_2\}(x)$.*

Lemma 7. *Let f_1 and f_2 be convex functions, and let $F_1(x) = E[f_1(x - D)]$ and $F_2(x) = E[f_2(x - D)]$. If $\partial f_1(x) \le \partial f_2(x)$, then $\partial F_1(x) \le \partial F_2(x)$.*

Proof of Theorem 2. In this proof, we use Lemmas 4, 5, 6, and 7. Let us first consider part (a). We first rewrite

$$\begin{aligned} f_{1,t}(y_1, j) &= d_j y_1 + L(y_1) - E[d_{M(j,W)}]y_1 + E[d_{M(j,W)}D] \\ &= (d_j - E[d_{M(j,W)}])y_1 + L(y_1) + E[d_{M(j,W)}D], \end{aligned}$$

and

$$\begin{aligned} f_{i,t}(y_i, j) &= d_j y_i + L(y_i) + E[S^1_{i-1,t+1}(y_i - D, M(j, W))] \\ &= d_j y_i + L(y_i) + E[g_{i-1,t+1}(y_i - D, M(j, W)) - d_{M(j,W)}(y_i - D)] \\ &= (d_j - E[d_{M(j,W)}])y_i + L(y_i) + E[g_{i-1,t+1}(y_i - D, M(j, W))] \\ &\quad + E[d_{M(j,W)}D], \end{aligned}$$

for $i > 1$. Also,

$$f_{i+1,t}(y_{i+1}, j) = (d_j - E[d_{M(j,W)}])y_{i+1} + L(y_{i+1}) + E[g_{i,t+1}(y_{i+1} - D, M(j, W))] \\ + E[d_{M(j,W)} D].$$

We prove that $\partial f_{i,t}(y, j) \leq \partial f_{i+1,t}(y, j)$ for all j. We use induction on t. The base case is when $t = T$ where $f_{i,T}(y, j) = f_{i+1,T}(y, j) = d_j y + L(y)$ for all i. Assuming $\partial f_{i,t+1}(y, j) \leq \partial f_{i+1,t+1}(y, j)$ for a fixed $t < T$, we obtain $\partial g_{i,t+1}(y, j) \leq \partial g_{i+1,t+1}(y, j)$. Therefore, we have $\partial f_{i,t}(y, j) \leq \partial f_{i+1,t}(y, j)$, and $y^*_{i,t}(j) \geq y^*_{i+1,t}(j)$. The proof of part (a) is completed.

Next, we prove part (b) that $\partial f_{i,t}(y, j) \leq \partial f_{i,t}(y, j+1)$ for all i. We use induction on t. The base case is when $t = T$, where $f_{i,T}(y, j) = d_j y + L(y)$. Therefore, we have $\partial f_{i,T}(y, j) \leq \partial f_{i,T}(y, j + 1)$ for all i due to $d_j \leq d_{j+1}$. Now assume $\partial f_{i,t+1}(y, j) \leq \partial f_{i,t+1}(y, j + 1)$ for all i and a fixed $t < T$. We have

$$f_{i,t}(y, j) = (d_j - E[d_{M(j,W)}])y + L(y) + E[g_{i-1,t+1}(y - D, M(j, W))] + E[d_{M(j,W)} D],$$
$$f_{i,t}(y, j + 1) = (d_{j+1} - E[d_{M(j+1,W)}])y + L(y) + E[g_{i-1,t+1}(y - D, M(j + 1, W))] \\ + E[d_{M(j+1,W)} D].$$

Because of the induction hypothesis and the definition of sequential systems, we have $\partial f_{i,t}(y, j) \leq \partial f_{i,t}(y, j + 1)$ and thus $y^*_{i,t}(j) \geq y^*_{i,t}(j + 1)$. Note that $M(j, w) \leq M(j + 1, w)$ for any realization w of W, and monotonicity of $\partial g_{i-1,t+1}$ in the second variable follows from the induction hypothesis on $\partial f_{i-1,t+1}$. In other words, if $\partial f_{i-1,t+1}$ is monotone in the second variable, then $\partial g_{i-1,t+1}$ is also monotone. The proof is completed. □

Lemma 8. *For any j and $w \in \mathbf{W}$ with $M(j, w) > 0$, function $p(j)S^2_{i-1,t}(x, M(j, w)) + g_{i,t}(x, j)$ is convex in x for all i.*

Proof. We prove convexity of

$$\begin{aligned} & g_{i,t}(x, j) + p(j)S^2_{i-1,t}(x, M(j, w)) \\ & \quad = g_{i,t}(x, j) + p(j)\{h_{i-1,t}(x, M(j, w)) - L(x) \\ & \qquad + p(M(j, w))E[S^2_{i-2,t+1}(x - D, M^2(j, W))|M^2(j, W) > 0]\}, \end{aligned} \tag{10}$$

for all w. We use induction. The base case of $t = T + 1$ is obvious. We assume convexity at $t + 1$ as the induction hypothesis. First, we show convexity of (10) when $x \leq y^*_{i,t}(M(j, w))$ and $x \geq y^*_{i,t}(j)$.

- When $x \leq y^*_{i,t}(M(j, w))$, we have $h_{i-1,t}(x, M(j, w)) = f_{i-1,t}(x, M(j, w)) - a_{i-1,t}(M(j, w)) = d_{M(j,w)}x + L(x) + p(M(j, w))E[S^1_{i-2,t+1}(x - D, M^2(j, W))|M^2(j, W) > 0] - a_{i-1,t}(M(j, w))$.

By using the fact that $S^1(x,j) + S^2(x,j) = -S^0(j)$, it is easy to see that $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex in x for $x \leq y^*_{i,t}(M(j,w))$.

- When $x \geq y^*_{i,t}(j)$, we have $g_{i,t}(x,j) = f_{i,t}(x,j) - a_{i,t}(j) = d_j x + L(x) + p(j)E[S^1_{i-1,t+1}(x - D, M(j,w))] - a_{i,t}(j)$. Since $S^1_{i-1,t+1}(x-D, M(j,w)) = g_{i-1,t+1}(x-D, M(j,w)) - d_{M(j,w)}(x - D)$, and $p(M(j,w))E[S^2_{i-2,t+1}(x-D, M^2(j,W))|M^2(j,W) > 0] + g_{i-1,t+1}(x-D, M(j,w))$ is convex by induction hypothesis, it is also easy to see convexity for $x \geq y^*_{i,t}(j)$.

Since $y^*_{i,t}(j) \leq y^*_{i,t}(M(j,w))$, we consider the following two cases: $y^*_{i,t}(j) < y^*_{i,t}(M(j,w))$ and $y^*_{i,t}(j) = y^*_{i,t}(M(j,w))$. If $y^*_{i,t}(j) < y^*_{i,t}(M(j,w))$, then $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex since it is convex for two partially overlapping intervals. Otherwise, if $y^* = y^*_{i,t}(j) = y^*_{i,t}(M(j,w))$, then

$$
\begin{aligned}
&g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w)) \\
&= g_{i,t}(x,j) + p(j)\{-L(x) + p(M(j,w))E[S^2_{i-2,t+1}(x-D, M^2(j,W))|M^2(j,W) > 0]\}
\end{aligned}
$$

for $x \geq y^*$, and

$$
\begin{aligned}
&g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w)) \\
&= p(j)\{h_{i-1,t}(x, M(j,w)) - L(x) + p(M(j,w))E[S^2_{i-2,t+1}(x-D, M^2(j,W))|M^2(j,W) > 0]\}
\end{aligned}
$$

for $x \leq y^*$. Given the convexity of the function for each interval and the convexity of $g_{i,t}(x,j) + p(j)h_{i-1,t}(x, M(j,w))$, we conclude that $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex also in this case. □

Lemma 9. *Let f_1 be convex and $b \in \mathbb{R}$. We have $\min_{b \leq x \leq y}\{f_1(x) + f_2(y)\} = a_1 + g_1(b) + \min_{b \leq y}\{h_1(y) + f_2(y)\}$, where a_1, h_1, and g_1 are defined as in Lemma 3 with respect to f_1.*

Proof. The proof can be found in Kim et al. (2012). □

Proof of part (c) of Theorem 3. Consider the following two states: $A_i = (x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i)$ and $A_{i+1} = (x^i, \bar{0}^i, \bar{v}_{i+1}, \bar{0}^i, \bar{l}_{i+1})$. Because of the sequential property of the system, we have to first expedite from stage i according to the base stock policy of part (a).

We now examine the following three cases for $i = 1$. Note that NS_1 and NS_2 are the same when we only expedite from stage 1, because of mandatory expediting.

Case 1. Let first $y_1^*(l_1) < x^0$. In this case, no expediting is necessary for both A_1 and A_2. Because of mandatory expediting with probability $p(l_1)$, we have

$$J_t(A_1) = L(x^0) + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0](x^1 - x^0) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_1)]\}$$

and

$$J_t(A_2) = L(x^1) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_2)]\}.$$

Thus, $J_t(A_1) - J_t(A_2) = L(x^0) + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0](x^1 - x^0) - L(x^1)$.

Case 2. If $x^0 \leq y_1^*(l_1) < x^1$, we have

$$J_t(A_1) = d_{l_1}(y_1^*(l_1) - x^0) + L(y_1^*(l_1)) + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0](x^1 - y_1^*(l_1)) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_1)]\}$$

and

$$J_t(A_2) = L(x^1) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_2)]\}.$$

Thus,

$$J_t(A_1) - J_t(A_2) = d_{l_1}(y_1^*(l_1) - x^0) + L(y_1^*(l_1)) + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0](x^1 - y_1^*(l_1)) - L(x^1).$$

Case 3. Finally, let $y_1^*(l_1) \geq x^1$. In this case, we have to expedite everything in stage 1, thus $J_t(A_1) - J_t(A_2) = d_{l_1}(x^1 - x^0)$.

The three cases can be summarized as

$$\begin{aligned} J_t(A_1) - J_t(A_2) &= a_{1,t}(l_1) + g_{1,t}(x^0, l_1) + h_{1,t}(x^1, l_1) \\ &\quad - d_{l_1}x^0 + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0]x^1 - L(x^1) \\ &= S_{1,t}^0(l_1) + S_{1,t}^1(x^0, l_1) + S_{1,t}^2(x^1, l_1). \end{aligned}$$

Next, consider the following three cases for $i > 1$.

Case 1. Let $y_i^*(l_i) < x^{i-1}$, and thus no expediting is necessary for both A_i and A_{i+1}. Therefore $J_t(A_i) = L(x^{i-1}) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_i)]\}$ and $J_t(A_{i+1}) = L(x^i) +$

$\min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_{i+1})]\}$. Let us examine NS_i by replacing y_i with x^{i-1} in (4). We obtain

$$
\begin{aligned}
&E[J_{t+1}(NS_i)] \\
&= p(l_i)E[S^0_{i-1,t+1}(M(l_i, W)) + S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) \\
&\qquad + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0] \\
&\quad + (p(l_i) + p(l_i, l_{i+1}))E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0] \\
&\quad + p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^i, \bar{v}_{i+2}, u, \bar{0}^i, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0] \\
&\qquad \vdots \\
&\quad + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] \\
&= p(l_i)E[S^0_{i-1,t+1}(M(l_i, W)) + S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) \\
&\quad + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0] \\
&\quad + p(l_{i+1})E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0] \\
&\quad + p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^i, \bar{v}_{i+2}, u, \bar{0}^i, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0] \\
&\qquad \vdots \\
&\quad + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],
\end{aligned}
$$

where we applied Lemma 1.

Therefore, we have

$$
\begin{aligned}
&E[J_{t+1}(NS_i)] - E[J_{t+1}(NS_{i+1})] = p(l_i)E[S^0_{i-1,t+1}(M(l_i, W)) \\
&\quad + S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&J_t(A_i) - J_t(A_{i+1}) = L(x^{i-1}) - L(x^i) + p(l_i)E[S^0_{i-1,t+1}(M(l_i, W)) \\
&\quad + S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].
\end{aligned}
$$

Case 2. If $x^{i-1} \leq y_i^*(l_i) < x^i$, we obtain

$$J_t(A_i) = d_{l_i}(y_i^*(l_i) - x^{i-1}) + L(y_i^*(l_i)) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_i)]\}$$

and

$$J_t(A_{i+1}) = L(x^i) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_{i+1})]\}.$$

Similarly to the previous case, we have

$$E[J_{t+1}(NS_i)] - E[J_{t+1}(NS_{i+1})] = p(l_i)E[S^0_{i-1,t+1}(M(l_i,W)) + S^1_{i-1,t+1}(y_i(l_i) - D, M(l_i,W)) + S^2_{i-1,t+1}(x^i - D, M(l_i,W))|M(l_i,W) > 0].$$

Therefore,

$$J_t(A_1) - J_t(A_2) = d_{l_i}(y_i^*(l_i) - x^{i-1}) + L(y_i^*(l_i)) - L(x^i) + p(l_i)E[S^0_{i-1,t+1}(M(l_i,W)) + S^1_{i-1,t+1}(y_i(l_i) - D, M(l_i,W)) + S^2_{i-1,t+1}(x^i - D, M(l_i,W))|M(l_i,W) > 0].$$

Case 3. If $y_1^*(l_i) \geq x^i$, then we simply have $J_t(A_1) - J_t(A_2) = d_{l_i}(x^i - x^{i-1})$.

The three cases can be summarized as

$$\begin{aligned} J_t(A_i) - J_t(A_{i+1}) &= a_{i,t}(l_i) + g_{i,t}(x^{i-1}, l_i) + h_{i,t}(x^i, l_i) - d_{l_i}x^{i-1} - L(x^i) \\ &\quad + p(l_i)E[S^0_{i-1,t+1}(M(l_i,W)) \\ &\quad + S^2_{i-1,t+1}(x^i - D, M(l_i,W))|M(l_i,W) > 0] \\ &= S^0_{i,t}(l_i) + S^1_{i,t}(x^{i-1}, l_i) + S^2_{1,t}(x^i, l_i). \end{aligned}$$

The proof of part (c) is thus completed. □

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