A Polyhedral Study of Integer Variable Upper Bounds

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Abstract

We study the polyhedron of the single node capacitated network design model with integer variable upper bounds. We first give a characterization of valid inequalities that is useful in proving the validity of several classes of inequalities. Next we derive several classes of valid inequalities and we give conditions for them to be facet-defining. Sequence independent lifting is used to obtain additional facets. We conclude by reporting computational results with a branch-and-cut algorithm.

1 Introduction

The single node capacitated network design model with variable upper bounds has been studied extensively since valid inequalities, called flow covers, derived from it can be used in branch-and-cut algorithms for mixed integer programs. In this paper we consider a generalization where the binary variables are replaced by integer variables. Valid inequalities for this problem can be used in mixed integer programs with integer variables. The facility location problem with integer variables representing the number of facilities to open at a specified location, which appears in certain circuit design problems, motivated our study, Bauer (1997). Another application is in the network design problem, where the integer variables represent the number of links to open, see e.g. Ahuja et al. (1995).

In this paper we study the set

$$S = \{(y,x) \in \mathbb{R}^{2n}_+ : \sum_{i=1}^{n} y_i \le b, y_i \le a_i x_i, x_i \le v_i, i = 1,\ldots,n, x \text{ integer}\},$$

where $v_i \le \infty$ for all $i \in N = \{1,\ldots,n\}$ are positive integers, b is a positive integer, a_i are positive integers for all $i \in N$, and the associated convex hull $P = conv(S)$.

The case $v_i = 1$ for all $i \in N$ is the usual binary model studied first by Padberg, Van Roy and Wolsey (1985). Several generalizations have been studied since, most of them assuming $v_i = 1$ for all $i \in N$. Variable lower and upper bounds are studied in Van Roy and Wolsey (1986), generalized upper bounds are given in

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Wolsey (1990), and Atamtürk, Nemhauser and Savelsbergh (2001) study additive variable upper bounds. Several lifted inequalities are presented in Gu, Nemhauser and Savelsbergh (1999). Recently Atamtürk (2002) studied the case $v_i = \infty$ for all $i \in N$. He considers the case $a_i = a$ for all $i \in N$ and he also considers a model with additive variable upper bounds, i.e. $y_i \leq a_i x_i$ is replaced by $y_i \leq \sum_{j \in T} a_j x_j^i$. It can be shown that the model with additive variable upper bounds is a relaxation of S. His model is on one hand less general than ours since he assumes $v_i = \infty$ for all $i \in N$, but on the other hand it is more general since he also considers incoming flow.

The rest of the paper is organized as follows. Section 2 presents an alternative condition for checking the validity of an inequality and we give a theorem showing that any vector (y, x) on a facet of P is almost uniquely determined by the integer vector x. Flow cover inequalities are presented in Section 3. Two extentions of S are studied in Section 4. In Section 5 we present a full description of P in terms of linear inequalities in the special case of $a_1 = a_2 = \cdots = a_n$. Computational results with a branch-and-cut algorithm are presented in Section 6.

Basic Properties

Since we also allow the upper bounds v to be infinity, for simplicity of notation we define $F = \{i \in N : v_i < \infty\}$ and let $I = N \setminus F$. We can assume without loss of generality that $a_i \leq b$ for all $i \in N$ as otherwise $y_i \leq a_i x_i$ is dominated by $y_i \leq b x_i$. Similarly if $I = \emptyset$, we assume that $\sum_{i \in F} a_i v_i > b$ since otherwise $\sum_{i=1}^n y_i \leq b$ is redundant. Using these assumptions it is easy to see that $dim(P) = 2n$ and that all the inequalities listed in the description of S are facet-defining. We call these facets the trivial facets.

The following properties of facets can be proven by using the techniques from Padberg, Van Roy and Wolsey (1985) and therefore their proofs are omitted.

Proposition 1. *If $my \leq u_0 + ux$ defines a nontrivial facet of P, then*

1. $0 \leq m, 0 \leq u, 0 < u_0$,
2. *if $u_i > 0$, then $m_i > 0$,*
3. *if $i \in I$, then $m_i > 0$ if and only if $u_i > 0$.*

In the rest of the paper we always assume that $0 \leq u, 0 \leq m, 0 < u_0, m \neq 0$ and that u, m and u_0 are integral. Let $\mathbf{1}$ be the n-dimensional vector whose components are all one and let $M = \max_{i \in N} m_i$.

Throughout the paper we demonstrate our results on the following example.

Example. Consider the following single node capacitated network design model

$$S = \{\sum_{i=1}^{6} y_i \leq 15, y_1 \leq 4x_1, y_2 \leq 3x_2, y_3 \leq 6x_3, y_4 \leq 4x_4, y_5 \leq 6x_5, y_6 \leq 2x_6,$$
$$x_1 \leq 2, x_2 \leq 3, x_3 \leq 3, 0 \leq y, 0 \leq x, x \text{ integer}\}.$$

In this instance we have $a = (4, 3, 6, 4, 6, 2), v = (2, 3, 3, \infty, \infty, \infty)$. □

2 Valid Inequalities and Facial Properties

Define the set

$$V = \{x \in \mathbb{Z}_+^n \colon 1 + u_0 + ux \leq \sum_{i \in N} (m_i - j)^+ a_i x_i + jb \text{ for all } j = 0, \ldots, M,$$
$$x \leq v, \quad x \text{ integer}\}.$$

Here we will first prove that $my \leq u_0 + ux$ is a valid inequality for P if and only if V is empty.

Lemma 1. *Let $A_i \in \mathbb{Q}_+$ for all $i \in N$, $B \in \mathbb{Q}_+$, and let $b \in \mathbb{Q}_+$. Then $\sum_{i=1}^n (m_i - j)^+ A_i + jb \geq B$ for all $j = 0, \ldots, M$ if and only if the polytope*

$$\{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i \leq b, my = B, y_i \leq A_i \text{ for all } i \in N\} \tag{1}$$

is nonempty.

Proof. ($\Longrightarrow$) By Farkas' lemma (1) is nonempty if and only if

$$\{u \in \mathbb{R}_+, v \in \mathbb{R}, x \in \mathbb{R}^n_+ : u - m_i v + x_i \geq 0 \text{ for all } i \in N, bu - vB + \sum_{i=1}^n A_i x_i < 0\} \tag{2}$$

is empty.

We prove the claim by contradiction. Assume that (2) is nonempty and that (u, v, x) is a vector in (2). From the last inequality it follows that $v > 0$. If we divide each constraint in (2) by v and let $z_i = m_i - x_i$, it follows that the polyhedron

$$\{u \in \mathbb{R}_+, z \in \mathbb{R}^n : z_i - u \leq 0 \text{ for all } i \in N, z \leq m, bu + \sum_{i=1}^n A_i(m_i - z_i) < B\} \tag{3}$$

is nonempty. Hence the linear program

$$\min\{bu + \sum_{i=1}^n A_i(m_i - z_i) : z_i - u \leq 0 \text{ for all } i \in N, z \leq m, u \geq 0, z \text{ unrestricted}\},$$

has an optimal solution. Since m is integral, there exists an integral optimal solution and therefore there is an integer vector (u, z) in (3).

If $u > M$, then (M, z) is in (3). Therefore we can assume that $u \leq M$ since otherwise we can replace (u, z) with (M, z). Since $m_i - z_i \geq (m_i - u)^+$, it follows that u satisfies the inequalities $bu + \sum_{i=1}^n A_i(m_i - u)^+ < B$ and $u, 0 \leq u \leq M$ is integral. But this contradicts our assumption in the lemma.

($\Longleftarrow$) For a vector y from (1) and for each integer $j, 0 \leq j \leq M$, we have

$$B = my = \sum_{i=1}^n (m_i - j)^+ y_i + \sum_{k=1}^j \sum_{\substack{i \in N \\ m_i \geq k}} y_i \leq \sum_{i=1}^n (m_i - j)^+ A_i + jb,$$

which proves the other direction. □

Theorem 1. *The inequality $my \leq u_0 + ux$ is valid for P if and only if V is empty.*

Proof. Suppose first that V is nonempty and let $x \in V$. By Lemma 1 with $A_i = a_i x_i$ for all $i \in N$ and $B = 1 + u_0 + ux$, it follows that there exists a vector $(y, x) \in P$ such that $my = 1 + u_0 + ux > u_0 + ux$. Therefore the inequality is not valid.

If the inequality is violated by $(y, x) \in P$, then y is in (1) if we take $B = my$ and $A_i = a_i x_i$ for all $i \in N$. Therefore by Lemma 1, $u_0 + ux < my \leq \sum_{i \in N} (m_i - j)^+ a_i x_i + jb$, which shows that $x \in V$. □

Example (continued). The inequality $y_1 + y_2 \leq 8 + 2x_1 + x_2$ is valid since there do not exist nonnegative integers x_1, x_2 such that

$$\begin{aligned} 9 + 2x_1 + x_2 &\leq 4x_1 + 3x_2 \\ 9 + 2x_1 + x_2 &\leq 15 \\ x_1 \leq 2, x_2 &\leq 3. \end{aligned}$$

On the other hand, the inequality $2y_1 + y_2 \le 8 + x_1 + x_2$ is not valid since $x_1 = x_2 = 1$ satisfies

$$\begin{aligned}
9 + x_1 + x_2 &\le 8x_1 + 3x_2 \\
9 + x_1 + x_2 &\le 4x_1 + 15 \\
9 + x_1 + x_2 &\le 30 \\
x_1 \le 2, x_2 &\le 3.
\end{aligned}$$

□

Unfortunately Theorem 1 does not give a computationally efficient method for testing validity. It is merely a tool for proving validity of various inequalities that are presented in Section 3.

Next we present a theorem that gives some structure of facets. Define the set

$$\begin{aligned}
Q_0 = \{x \in \mathbb{R}^n_+ : u_0 + ux = \sum_{i \in \bar{M}} m_i a_i x_i, \\
\sum_{i \in \bar{M}} a_i x_i \le b - 1, \\
x \le v, \quad x \text{ integer} \},
\end{aligned}$$

where $\bar{M} = \{i \in N : m_i > 0\}$.

Theorem 2. *Let $\mathcal{F} = P \cap \{(y, x) \in \mathbb{R}^{2n} : my = u_0 + ux\}$. If $my \le u_0 + ux$ is a valid inequality for P, then*

1. *there exists $(y, x) \in \mathcal{F}$ such that $\sum_{i \in \bar{M}} y_i < b$ if and only if $Q_0 \neq \emptyset$,*
2. *if $(y, x) \in \mathcal{F}$ and $y_l < a_l x_l$ for an index $l \in \bar{M}$, then there exists an integer $k, m_l \le k \le M$ such that*

$$\begin{aligned}
y_i &= 0 && \text{for all } i \in \bar{M} \text{ such that } m_i < k, \\
\sum_{\substack{j \in \bar{M} \\ m_j = k}} y_j &= b - \sum_{\substack{j \in \bar{M} \\ m_j > k}} a_j x_j, && \qquad (4) \\
y_i &= a_i x_i && \text{for all } i \in \bar{M} \text{ such that } m_i > k.
\end{aligned}$$

Proof. We first prove that if there exists $(y, x) \in \mathcal{F}$ such that $\sum_{i \in \bar{M}} y_i < b$, then $Q_0 = \emptyset$. Let $(y, x) \in \mathcal{F}$ be such that $\sum_{i \in \bar{M}} y_i < b$. Therefore there exists an $\epsilon > 0$ such that

$$\begin{aligned}
\sum_{i \in \bar{M}} y_i &\le b - \epsilon, \\
\sum_{i \in \bar{M}} m_i y_i &= u_0 + ux, \\
y_i &\le a_i x_i \text{ for all } i \in \bar{M}.
\end{aligned}$$

By Lemma 1, for all $j = 0, \ldots, M$ we have $\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + j(b - \epsilon) \ge u_0 + ux$. Therefore

$$\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + jb \ge u_0 + ux + j\epsilon \qquad (5)$$

holds for $j = 0, \ldots, M$. For $j = 1, \ldots, M$ it follows that $\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + jb \ge u_0 + ux + 1$. If (5) is not satisfied at equality for $j = 0$, then $x \in V$, which is a contradiction since by assumption the inequality is valid and therefore by Theorem 1 $V = \emptyset$. Inequality (5) for $j = 0$ reads $u_0 + ux = \sum_{i \in \bar{M}} m_i a_i x_i$. Finally by substituting for $j = 1$, we see that inequality (5) is equivalent to $\sum_{i \in \bar{M}} a_i x_i \le b - 1$. Therefore $x \in Q_0$.

To show the other direction of the first part, let $x \in Q_0$. If we define $y_i = a_i x_i$ for all $i \in \bar{M}$ and $y_i = 0$ otherwise, then $(y, x) \in \mathcal{F}$ and $\sum_{i \in \bar{M}} y_i < b$.

Next we prove the second part of the theorem. Let $(y, x) \in \mathcal{F}$ be such that $y_l < a_l x_l$. Hence there exists an $\epsilon > 0$ such that

$$\begin{aligned}\sum_{i \in \bar{M}} y_i &\leq b, \\ \sum_{i \in \bar{M}} m_i y_i &= u_0 + ux, \\ y_i &\leq a_i x_i \text{ for all } i \in \bar{M} \setminus \{l\} \\ y_l &\leq a_l x_l - \epsilon.\end{aligned}$$

By Lemma 1, $\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + jb \geq u_0 + ux + \epsilon(m_l - j)^+$ holds for $j = 0, \ldots, M$ and therefore $\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + jb \geq u_0 + ux + 1$ for all j such that $j < m_l$. For $j \geq m_l$ it follows that

$$\sum_{i \in \bar{M}} (m_i - j)^+ a_i x_i + jb \geq u_0 + ux. \tag{6}$$

Since $my \leq u_0 + ux$ is a valid inequality, by Theorem 1, $V = \emptyset$. Therefore $x \notin V$ and there is an integer $k, k \geq m_l$, such that (6) is satisfied at equality, i.e. $\sum_{i \in \bar{M}} (m_i - k)^+ a_i x_i + kb = u_0 + ux$. Since $(y, x) \in \mathcal{F}$, it follows that $u_0 + ux = my$. Now we have

$$\begin{aligned}u_0 + ux = my &= \sum_{i \in \bar{M}} (m_i - k)^+ y_i + \sum_{i=1}^{k} \sum_{\substack{j \in N \\ m_j \geq i}} y_j \\ &\leq \sum_{i \in \bar{M}} (m_i - k)^+ a_i x_i + kb = u_0 + ux\,,\end{aligned}$$

where the inequality follows from $y_i \leq a_i x_i$ for i such that $m_i > k$ and $\sum_{\substack{j \in N \\ m_j \geq i}} y_j \leq b$ for $i = 1, \ldots, k$. Since the left hand side is equal to the right hand side, the upper bounds that were made are equalities. We conclude that $y_i = 0$ for $i \in \bar{M}$ such that $m_i < k$ and $y_i = a_i x_i$ for all $i \in \bar{M}$ such that $m_i > k$, and that $\sum_{j : m_j = k} y_j = b - \sum_{j : m_j > k} a_j x_j$. □

If $\mathcal{F}$ is a facet, then Theorem 2 reveals structure on vectors $(y, x) \in \mathcal{F}$. Namely, either $y_i = a_i x_i$ for all $i \in \bar{M}$ or y has the structure described by (4).

Padberg, Van Roy and Wolsey (1985) show for the binary case that if $my \leq u_0 + ux$ defines a nontrivial facet of P, then $\sum_{i \in \bar{M}} a_i \geq b$. The next corollary, in addition to generalizing this result, is also a stronger result.

Corollary 1. *Let $my \leq u_0 + ux$ define a nontrivial facet $\mathcal{F}$ of P. Then $\sum_{i : m_i = M} a_i v_i \geq b$ and there is a vector $(y, x) \in \mathcal{F}$ such that $\sum_{i \in \bar{M}} y_i = b$.*

Proof. If we consider an index l such that $m_l = M$, then both claims follow from Theorem 2 since there is a vector $(y, x) \in \mathcal{F}$ such that $y_l < a_l x_l$. □

3 Flow Cover and Lifted Flow Cover Inequalities

3.1 Unbounded Flow Cover Inequalities

In this section we define flow covers that are subsets of I and we give necessary and sufficient conditions for facet-defining inequalities.

For each $C \subseteq N$, let $\bar{a} = \max_{i \in C} a_i$, $\underline{a} = \min_{i \in C} a_i$, $k = \lceil \frac{b}{\bar{a}} \rceil$, and $\lambda = k\bar{a} - b$. A subset $C_I \subseteq I$ is an unbounded flow cover if $\lambda > 0$ and $\underline{a} \geq \bar{a} - \lambda + 1$.

Proposition 2. *If C_I is an unbounded flow cover, then $\sum_{i\in C_I} y_i \le (k-1)\lambda + (\bar a - \lambda)\sum_{i\in C_I} x_i$ is a valid inequality for P.*

Proof. We prove the validity by using Theorem 1 . Suppose there is an integral x that satisfies the two constraints from V. The inequality with $j=1$, $1+u_0+\sum_{i\in C_I} u_i x_i \le b$ yields that $\sum_{i\in C_I} x_i \le k - \frac{1}{\bar a-\lambda}$ and in turn, since x is integral, $\sum_{\in C_I} x_i \le k-1$. From the inequality with $j=0$, $1+u_0+\sum_{i\in C_I} u_i x_i \le \sum_{i\in C_I} a_i x_i$, it follows by using $\sum_{i\in C_I} a_i x_i \le \bar a \sum_{i\in C_I} x_i$ that $\sum_{i\in C_I} x_i \ge k-1+\frac{1}{\lambda}$. Therefore again by integrality of x we get $\sum_{i\in C_I} x_i \ge k$, which is a contradiction. We conclude that V is empty and hence the inequality is valid. $\square$

Theorem 3. *Let $C_I \subseteq I$ and let t, u_0 be positive integers. The inequality*

$$\sum_{i\in C_I} y_i \le u_0 + t\sum_{i\in C_I} x_i \tag{7}$$

defines a nontrivial facet of P if and only if C_I is an unbounded flow cover, and $u_0 = (k-1)\lambda, t = \bar a - \lambda$.

Proof. ($\Longrightarrow$) Let $a_p = \bar a$ and $a_q = \underline{a}$. If (y,x) is a vector on the facet, then

$$1 + t\sum_{i\in C_I} x_i \le u_0 + t\sum_{i\in C_I} x_i = \sum_{i\in C_I} y_i \le \sum_{i\in C_I} a_i x_i \le \bar a \sum_{i\in C_I} x_i$$

and therefore $t < \bar a$.

We first show that $u_0 + tk = b$. Since $(be_p, ke_p) \in P$ and (7) is valid, it follows that $b \le u_0 + tk$. On the other hand, by Corollary 1, there is a vector $(y,x)\in P$ such that $b = \sum_{i\in C_I} y_i = u_0 + t\sum_{i\in C_I} x_i$. We have $b = \sum_{i\in C_I} y_i \le \sum_{i\in C_I} a_i x_i \le \bar a \sum_{i\in C_I} x_i$. Therefore since x is integral, $\sum_{i\in C_I} x_i \ge \lceil \frac{b}{\bar a}\rceil = k$. It then follows that $b \ge u_0 + tk$.

Next we show that $u_0 = (k-1)(\bar a - t)$. Since V is empty, $x = (k-1)e_p \notin V$ and therefore either $1+u_0+(k-1)t > b$ or $1+u_0+(k-1)t > \bar a(k-1)$. If the first condition holds, then $1+u_0+(k-1)t > b \ge u_0+tk$ and therefore $t<1$. This contradicts our assumption that t is a positive integer. Therefore the second condition, i.e. $1+u_0+(k-1)t > \bar a(k-1)$, is satisfied. This is equivalent to $u_0 \ge (k-1)(\bar a - t)$. Since $\sum_{i\in C_I} y_i \le u_0 + t\sum_{i\in C_I} x_i$ is facet-defining, there exists a vector (y,x) on the facet with $\sum_{i\in N} y_i < b$. By Theorem 2, there exists $x \in Q_0$. For this vector we have $u_0 + t\sum_{i\in C_I} x_i = \sum_{i\in C_I} a_i x_i$ and $\sum_{i\in C_I} a_i x_i \le b-1$. Therefore

$$b - tk + t\sum_{i\in C_I} x_i = u_0 + t\sum_{i\in C_I} x_i = \sum_{i\in C_I} a_i x_i \le b-1$$

and the integrality of x now implies that $\sum_{i\in C_I} x_i \le k-1$. On the other hand from $u_0 + t\sum_{i\in C_I} x_i = \sum_{i\in C_I} a_i x_i$ we get

$$u_0 = \sum_{i\in C_I} (a_i - t)x_i \le (\bar a - t)\sum_{i\in C_I} x_i \le (k-1)(\bar a - t).$$

This proves that $u_0 = (k-1)(\bar a - t)$. Since $u_0 + tk = b$, we have $t = \bar a - \lambda$ and $u_0 = (k-1)\lambda$. Since for any nontrivial facet $u_0 > 0$, it follows that $\lambda > 0$.

Since $\sum_{i\in C_I} y_i \le u_0 + t\sum_{i\in C_I} x_i$ is facet-defining, there exists a vector (y,x) on the facet with $y_q < a_q x_q$; otherwise the facet would have two linearly independent constraints in the equality set. Therefore $1 + u_0 + t\sum_{i\in C_I} x_i \le \sum_{i\in C_I} a_i x_i$ and since $x \notin V$, it follows that $u_0 + t\sum_{i\in C_I} x_i = b$. This yields that $1 + b \ge \sum_{i\in C_I} a_i x_i$ and in turn $k \ge \sum_{i\in C_I} x_i$. Since for this vector $x_q \ge 1$, we get that

$$\bar a - \underline{a} \le (\bar a - \underline{a})x_q \le \sum_{i\in C_I} (\bar a - a_i)x_i \le \bar a k - b - 1.$$

This is equivalent to $\underline{a} \ge \bar a - \lambda + 1$ and therefore C_I is an unbounded flow cover.

($\Longleftarrow$) Next we prove that if $u_0 = (k-1)\lambda, t = \bar a - \lambda$ and C_I is an unbounded flow cover, (7) is facet-defining. The validity has been proven in Proposition 2.

Assume that $\sum_{i\in I}\alpha_i y_i+\sum_{i\in I}\beta_i x_i=\pi_0$ is satisfied by all vectors in $\mathcal{F}=P\cap\{(y,x)\in\mathbb{R}_+^{2\cdot|I|}:\sum_{i\in C_I}y_i=(k-1)\lambda+(\bar a-\lambda)\sum_{i\in C_I}x_i\}$.

Let us denote $y=(k-1)a_pe_p, x=(k-1)e_p$. Together with (y,x), consider the vectors $(y+e_i,x+e_i),(y,x+e_i)$ for each $i\in N\setminus C_I$. Since they are all in $\mathcal{F}$, it follows that $\alpha_i=\beta_i=0$ for all $i\in N\setminus C_I$.

For each $i\in C_I$ let $y^i=\frac{b}{a_i+(k-1)a_r}(a_ie_i+(k-1)a_re_r)$ and $x^i=e_i+(k-1)e_r$. The vectors (y^i,x^i) are in $\mathcal{F}$ and $y_i^i>0, y_r^i<a_rx_r^i$ for each $i\in C_I$. Then there exists an $\epsilon>0$ such that the vectors $(y^i-\epsilon e_i+\epsilon e_r,x^i)$ are in $\mathcal{F}$ for each $i\in C_I$. They yield $\alpha_i=\alpha_r$ for all $i\in C_I$.

Now consider the vectors $(y^i,x^i), i\in C_I$. They yield that $\beta_i=\beta_r$ for all $i\in C_I$. Finally, by considering (y^r,x^r) and (y,x) we conclude that $\beta_r=(\lambda-\bar a)\alpha_r$ and $\pi_0=\lambda(k-1)\alpha_r$. Therefore $\mathcal{F}$ is a facet. □

If C_I is an unbounded flow cover, we call the facet-defining inequality

$$\sum_{i\in C_I}y_i\le(k-1)\lambda+(\bar a-\lambda)\sum_{i\in C_I}x_i \tag{8}$$

the *unbounded flow cover inequality.*

Example (continued). The unbounded flow covers are $\{4\},\{5\},\{6\},\{4,5\}$ and they yield facets defined by

$$y_4\le 3+3x_4 \tag{9}$$
$$y_5\le 6+3x_5 \tag{10}$$
$$y_6\le 7+x_6 \tag{11}$$
$$y_4+y_5\le 6+3x_4+3x_5. \tag{12}$$

□

The unbounded flow cover inequalities yield the following valid inequalities called *simple lifted unbounded flow cover inequalities.* For each $j\in N\setminus C_I$ we define

$$i_j=\begin{cases}\lfloor\frac{a_j}{\bar a}\rfloor & \text{if } a_j\le\lceil\frac{a_j}{\bar a}\rceil\bar a-\lambda,\\ \lceil\frac{a_j}{\bar a}\rceil & \text{otherwise.}\end{cases}$$

Theorem 4. *Let $C_I\subseteq I$ be an unbounded flow cover and let $T\subseteq N\setminus C_I$. For each $j\in T$, let $u_j=a_j-i_j\lambda$ if $i_j\bar a\le a_j\le(i_j+1)\bar a-\lambda$ and let $u_j=i_j(\bar a-\lambda)$ if $i_j\bar a-\lambda<a_j<i_j\bar a$. Then the simple lifted unbounded flow cover inequality*

$$\sum_{i\in C_I\cup T}y_i\le(k-1)\lambda+(\bar a-\lambda)\sum_{i\in C_I}x_i+\sum_{i\in T}u_ix_i \tag{13}$$

is valid for P.

Proof. By Theorem 1 it suffices to prove that V is empty. Suppose that there is an $x\in V$. Then

$$1+\sum_{i\in C_I}(\bar a-\lambda)x_i+\sum_{i\in T}u_ix_i\le b-(k-1)\lambda=k(\bar a-\lambda), \tag{14}$$

$$1+(k-1)\lambda+\sum_{i\in C_I}(\bar a-\lambda)x_i+\sum_{i\in T}u_ix_i\le\sum_{i\in C_I\cup T}a_ix_i. \tag{15}$$

Let $\sum_{i\in T}u_ix_i=\sum_{j\in T_1}(a_j-i_j\lambda)x_j+(\bar a-\lambda)\sum_{j\in T_2}i_jx_j$, where $T_1=\{j\in T: i_j=\lfloor\frac{a_j}{\bar a}\rfloor\}$ and $T_2=\{j\in T: i_j=\lceil\frac{a_j}{\bar a}\rceil\}$. From (14) and since $a_j\ge i_j\bar a\ge i_j\lambda$ for all $j\in T_1$, it follows that $1+(\bar a-\lambda)\sum_{i\in C_I}x_i+(\bar a-\lambda)\sum_{j\in T}i_jx_j\le(\bar a-\lambda)k$ and after dividing by $\bar a-\lambda$ and rounding down we obtain $\sum_{i\in C_I}x_i+\sum_{j\in T}i_jx_j\le k-1$.

From (15) it follows that

$$1+(k-1)\lambda\le\sum_{i\in C_I}(a_i-\bar a+\lambda)x_i+\sum_{i\in T}(a_i-u_i)x_i\le\lambda\sum_{i\in C_I}x_i+\lambda\sum_{j\in T}i_jx_j,$$

where the last inequality follows from $a_j - i_j\bar{a} \le 0$ for all $j \in T_2$. If we divide this inequality by λ and round up, we get that $k \le \sum_{i\in C_I} x_i + \sum_{j\in T} i_j x_j$. Hence $V = \emptyset$. $\square$

The facet-defining simple lifted unbounded flow cover inequalities are given in Section 3.3.1.

Example (continued). With $T = N \setminus C_I$, the facet-defining inequality (9) gives the valid inequality $\sum_{i=1}^{6} y_i \le 3 + 3x_1 + 3x_2 + 5x_3 + 3x_4 + 5x_5 + 2x_6$, (10) yields $\sum_{i=1}^{6} y_i \le 6 + 3\sum_{i=1}^{5} x_i + 2x_6$, (11) produces $\sum_{i=1}^{6} y_i \le 7 + 2x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 + x_6$, and (12) yields the same inequlity as (10). Note that additional valid inequalities are obtained by considering subsets $T \subset N \setminus C_I$ and using the same coefficients. $\square$

Proposition 2 and Theorem 4 can be derived from the multifacility cut-set inequality presented in Atamtürk (2002). However we obtained these results before we knew of Atamtürk's and we use a completely different technique in proving validity. Moreover, since Atamtürk makes the assumption $F = \emptyset$, none of our subsequent results are relevant to his paper.

3.2 Flow Cover Inequalities

In this section we develop flow cover inequalities for subsets of F. We call $C_F \subseteq F$ a flow cover if $\lambda = \sum_{i\in C_F} a_i v_i - b > 0$ and $\bar{a} > \lambda$. We generalize the lifted flow cover inequalities from Gu, Nemhauser and Savelsbergh (1999) and give a short proof that they are valid.

Theorem 5. *Let $C_F \subseteq F$ be a flow cover and let $T \subseteq N \setminus C_F$. For each $j \in T$ let $u_j = a_j - i_j\lambda$ if $i_j\bar{a} \le a_j \le (i_j+1)\bar{a} - \lambda$ and let $u_j = i_j(\bar{a} - \lambda)$ if $i_j\bar{a} - \lambda < a_j < i_j\bar{a}$. Then the simple lifted flow cover inequality*

$$\sum_{i\in C_F\cup T} y_i \le b - \sum_{i\in C_F} (a_j - \lambda)^+ (v_j - x_j) + \sum_{i\in T} u_i x_i \tag{16}$$

defines a valid inequality for P.

Proof. We use Theorem 1. Suppose that $x \in V$ and let $L = \{i \in C_F : a_i > \lambda\}$. Then

$$1 + \sum_{i\in L} (a_i - \lambda) x_i + \sum_{i\in T} u_i x_i \le \sum_{i\in L} v_i (a_i - \lambda)\,, \tag{17}$$

$$1 + b - \sum_{i\in L} v_i (a_i - \lambda) + \sum_{i\in L} (a_i - \lambda) x_i + \sum_{i\in T} u_i x_i \le \sum_{i\in C_F\cup T} a_i x_i\,. \tag{18}$$

For each $i \in L$ we substitute $x_i = v_i - z_i$. As in the proof of Theorem 4, let $\sum_{i\in T} u_i x_i = \sum_{j\in T_1} (a_j - i_j\lambda) x_j + (\bar{a} - \lambda) \sum_{j\in T_2} i_j x_j$, where T_1, T_2 is the partition of T. By the definition of λ and since $\sum_{i\in C_F\setminus L} a_i x_i \le \sum_{i\in C_F\setminus L} a_i v_i$, it follows from (18) that

$$1 + \lambda \sum_{j\in L} z_j - \lambda \sum_{j\in T_1} i_j x_j \le \lambda + \sum_{j\in T_2} (a_j - i_j(\bar{a} - \lambda)) x_j \le \lambda + \lambda \sum_{j\in T_2} i_j x_j,$$

where the last inequality follows from $a_j - i_j\bar{a} \le 0$ for $j \in T_2$. After dividing by λ and rounding up the left hand side since z_i and x_i are integers, we get that

$$\sum_{j\in L} z_j \le \sum_{j\in T} i_j x_j\,. \tag{19}$$

On the other hand, from (17) and since $a_j \ge i_j\bar{a}$ for all $j \in T_1$, we have

$$1 + (\bar{a} - \lambda) \sum_{j\in T} i_j x_j \le \sum_{j\in L} (\bar{a} - \lambda) z_j.$$

After dividing by $\bar{a} - \lambda$ and rounding up the left hand side, we get that $1 + \sum_{j\in T} i_j x_j \le \sum_{i\in L} z_j$. But this contradicts (19), showing that V is empty. $\square$

Next we give some necessary conditions for facet-defining inequalities.

Theorem 6. *Let C_F be a flow cover and assume that $a_r = \bar{a}$. Let $T \subseteq N \setminus C_F$ be such that for each $j \in T$ we have $i_j \leq v_r$ and $i_j\bar{a} - \lambda \leq a_j \leq i_j\bar{a}$. If $|C_F| = 1$, then we require that $i_j < v_r$ for each $j \in T$. Then the inequality*

$$\sum_{i\in C_F\cup T} y_i \leq b - \sum_{i\in C_F}(a_j - \lambda)^+(v_j - x_j) + (\bar{a} - \lambda)\sum_{j\in T} i_j x_j \tag{20}$$

is facet-defining.

Proof. Assume that $\alpha y + \beta x = \pi_0$ is satisfied by all the vectors in $\mathcal{F} = P \cap \{(y,x) \in \mathbb{R}^{2n} : \sum_{i\in C_F\cup T} y_i = b - \sum_{i\in C_F}(a_j - \lambda)^+(v_j - x_j) + (\bar{a} - \lambda)\sum_{j\in T} i_j x_j\}$. We show that (α, β, π_0) represents the same inequality. For simplicity of notation we define $x \circ y$ to be the vector $(x_1y_1, x_2y_2, \ldots, x_ny_n)$. We write n-dimensional vectors as $(z, \bar{z}, \tilde{z})$ where z corresponds to indices in C_F, $\bar{z}$ to indices in T and $\tilde{z}$ to indices in $N \setminus (C_F \cup T)$.

Let $L = \{i \in C_F : a_i > \lambda\}$ and consider an index $j \in L$. Then the vector $(a\circ x, x)$, where $x = (v - e_j, 0, 0)$, is in $\mathcal{F}$. The vectors $(a \circ x, x)$ and $(a \circ x, x + e_i)$ for each $i \in N \setminus (C_F \cup T)$ yield that $\beta_i = 0$. Similarly by considering the vectors $(a \circ x, x)$ and $(a \circ x + e_i, x + e_i)$ we get that $\alpha_i = 0$ for all $i \in N \setminus (C_F \cup T)$.

Next we consider the following integer vectors

$$\begin{aligned}
u^k &= (v - e_k, 0, 0) && k \in C_F \setminus L,\\
z^k &= (v - i_k e_r, e_k, 0) && k \in T,\\
w &= (v, 0, 0).
\end{aligned}$$

For any vector x listed above, we define the corresponding y vector as

$$y(x) = \frac{b}{ax} a \circ x.$$

It is easy to check that $(y(x), x) \in \mathcal{F}$ since $ax > b$ and $\sum_{i\in C_F\cup T} y(x)_i = b$.

For each i, k from C_F consider the vector $(y(w) - \epsilon e_i + \epsilon e_k, w)$. There is an $\epsilon > 0$ such that all these vectors are in $\mathcal{F}$. It then follows that $\alpha_i = t$, where t is a constant, for each $i \in C_F$. Similarly for a small enough $\epsilon > 0$ and for each $k \in T$ consider the vector $(y(z^k) + \epsilon e_l - \epsilon e_k, z^k) \in \mathcal{F}$, where $l = r$ if $|C_F| = 1$ and l is a fixed index from $C_F \setminus \{r\}$ otherwise. These vectors yield that $\alpha_k = \alpha_l = t$ for all $k \in T$.

Next we consider $(y(w), w)$ and $(y(u^k), u^k)$ for each $k \in C_F \setminus L$. We obtain that $\beta_k = 0$ for all $k \in C_F \setminus L$. Similarly by considering $(y(w), w)$ and $(y(z^k), z^k)$ for each $k \in T$, we get that $\beta_k = t \cdot i_k$.

Finally we consider the integer vectors $\tilde{u}^k = (v - e_k, 0, 0)$ for each $k \in L$. The vectors $(a \circ \tilde{u}^k, \tilde{u}^k)$ are in $\mathcal{F}$ and they yield $\beta_k = t(\lambda - a_k)$ for each $k \in L$. This completes the proof. □

Example (continued). There are two flow covers $\{1, 2\}, \{3\}$ and they yield facet-defining inequalities $y_1 + y_2 \leq 8 + 2x_1 + x_2, y_3 \leq 6 + 3x_3$, respectively. The corresponding simple lifted flow cover inequalities with $T = N \setminus C_F$ are in turn

$$\sum_{i=1}^{6} y_i \leq 8 + 2x_1 + x_2 + 4x_3 + 2x_4 + 4x_5 + 2x_6, \tag{21}$$

$$\sum_{i=1}^{6} y_i \leq 6 + 3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 2x_6. \tag{22}$$

Inequality (22) can also be obtained from the unbounded flow cover $\{5\}$ or $\{4, 5\}$ by using Theorem 4. By Theorem 6 (21) is facet-defining and additional facets are obtained by considering $T \subset N \setminus \{1, 2\}$. However from (22) only the inequalities with $T \subseteq \{1, 2, 4, 5\}$ are facet-defining. □

3.3 Sequence Independent Lifting

We first briefly review sequence independent lifting for flow cover inequalities, Gu, Nemhauser and Savelsbergh (1999, 2000). Even though the results from these two papers assume only binary variables, they can be rather easily extended to general integer variables. For simplicity of notation we define $v_i = \lceil b/a_i \rceil$ for all $i \in I$.

Assume that for a subset $C \subset N$ the inequality

$$\sum_{i \in C} \alpha_i y_i \leq \alpha_0 + \sum_{i \in C} \beta_i x_i \tag{23}$$

is valid for $P \cap \{(y,x) \in \mathbb{R}^{2n}_+ : y_i = x_i = 0 \text{ for all } i \in N \setminus C\}$. New valid inequalities can be derived from (23) by lifting. Lifting the variables in $N \setminus C$ means finding coefficients α_i, β_i for each $i \in N \setminus C$ such that the lifted inequality

$$\sum_{i \in N} \alpha_i y_i \leq \alpha_0 + \sum_{i \in N} \beta_i x_i \tag{24}$$

is also valid for P.

We lift pairs $(y_p, x_p), p \in N \setminus C$ simultaneously. For each $p \in N \setminus C$ and for each $z \in [0, a_p v_p]$ we define

$$\begin{aligned} h_p(z) = \max \alpha_p y_p - \beta_p x_p& \\ y_p &= z \\ 0 \leq y_p &\leq a_p x_p \\ 0 \leq x_p &\leq v_p \\ x &\text{ integer.} \end{aligned}$$

In addition for each $z \in [0, b]$ let

$$\begin{aligned} f(z) = \min \alpha_0 - \sum_{i \in C} \alpha_i y_i + \sum_{i \in C} \beta_i x_i& \\ \sum_{i \in C} y_i &\leq b - z \\ 0 \leq y_i &\leq a_i x_i \qquad \text{for all } i \in C \\ 0 \leq x_i &\leq v_i \qquad \text{for all } i \in C \\ x &\text{ integer.} \end{aligned}$$

Definition 1. A function f is superadditive on $Z \subseteq \mathbb{R}$ if $f(z_1) + f(z_2) \leq f(z_1 + z_2)$ for each $z_1 \in Z, z_2 \in Z$ such that $z_1 + z_2 \in Z$.

The following theorem from Gu, Nemhauser and Savelsbergh (1999) is crucial in deriving lifted valid inequalities.

Theorem 7. *If f is superadditive on $[0,b]$ and if for each $p \in N \setminus C$ the lifting coefficients α_p, β_p are selected in such a way that $h_p(z) \leq f(z)$ for each $z \in [0, \min\{b, a_p v_p\}]$, then (24) is a valid inequality for P. If in addition, (23) is facet-defining for the projection of P to C and for each $p \in N \setminus C$ the equation $h_p(z) = f(z)$ has two linearly independent vectors $(y_p^1, x_p^1), (y_p^2, x_p^2)$, then (24) is facet-defining for P.*

It is easy to see that

$$h_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ \alpha_p z - j\beta_p & (j-1)a_p < z \leq j a_p \qquad j = 1, \ldots, v_p. \end{cases}$$

The facet inducing property now reads that there must exist $z_1, z_2 \in [0, \min\{b, a_p v_p\}]$ such that $h_p(z_1) = f(z_1), h_p(z_2) = f(z_2)$ and if $(j_1 - 1)a_p < z_1 \leq j_1 a_p, (j_2 - 1)a_p < z_2 \leq j_2 a_p$ for $0 \leq j_1, j_2 \leq v_p$, then

$j_2z_1 - j_1z_2 \neq 0$. Note that since we lift pairs of variables simultaneously for a given p there can be more than one lifting pair α_p, β_p.

Gu, Nemhauser and Savelsbergh (1999) also study a class of superadditive functions. Given a positive number l, nonincreasing and nonnegative sequences $\bar{u}$ and $\bar{v}$ such that $\bar{u}_i + \bar{v}_i > 0$ for all $i = 1, 2, \ldots, \infty$, we define $w_i = \bar{u}_i + \bar{v}_i$ for all $i = 1, 2, \ldots, \infty$, and $W_h = \sum_{i=1}^h w_i$ for all $h = 1, 2, \ldots, \infty$. In addition let

$$g_1(z) = \begin{cases} hl & \text{if } hw_1 \leq z \leq hw_1 + \bar{u}_1, \quad h = 0, 1, \ldots, \infty, \\ hl + l(z - hw_1 - \bar{u}_1)/\bar{v}_1 & hw_1 + \bar{u}_1 < z < (h+1)w_1, \quad h = 0, 1, \ldots, \infty, \end{cases}$$

$$g_2(z) = \begin{cases} hl & \text{if } W_h \leq z \leq W_h + \bar{u}_{h+1}, \quad h = 0, 1, \ldots, \infty, \\ hl + l\big(1 - (W_{h+1} - z)/\bar{v}_1\big) & W_h + \bar{u}_{h+1} < z < W_{h+1}, \quad h = 0, 1, \ldots, \infty. \end{cases}$$

They show that g_1 and g_2 are superadditive functions on $[0, \infty]$.

3.3.1 Lifted Unbounded Flow Cover Inequalities

We first assume that (23) is an unbounded flow cover inequality (8). In this case

$$\begin{aligned} f(z) = \min\ & (k-1)\lambda - \sum_{i \in C_I} y_i + (\bar{a} - \lambda) \sum_{i \in C_I} x_i \\ & \sum_{i \in C_I} y_i \leq b - z \\ & 0 \leq y_i \leq a_i x_i \qquad \text{for each } i \in C_I \\ & 0 \leq x_i \qquad \text{for each } i \in C_I \\ & \quad x \ \text{ integer.} \end{aligned}$$

Proposition 3. *If $C_I \subseteq I$ is an unbounded flow cover, then*

$$f(z) = \begin{cases} j\lambda & \text{if } j\bar{a} \leq z \leq (j+1)\bar{a} - \lambda \qquad j = 0, \ldots, k-1, \\ z - (\bar{a} - \lambda)j & j\bar{a} - \lambda < z < j\bar{a} \qquad j = 1, \ldots, k, \end{cases}$$

and f is superadditive on $[0, b]$.

Proof. Let $a_r = \bar{a}$, where $r \in C_I$. If (y, x) is a feasible solution to $f(z)$ and $l \in C_I$, then $(y + y_l e_r - y_l e_l, x + x_l e_r - x_l e_l)$ is a feasible vector with the same objective value. We conclude that there exists an optimal solution (y, x) to $f(z)$ with $y_i = x_i = 0$ for all $i \in C_I \setminus \{r\}$. The claim is now easy to check.

$f = g_1$ by taking $\bar{u}_1 = \bar{a} - \lambda, \bar{v}_1 = \lambda, l = \lambda$ and is therefore superadditive. □

The following theorem gives lifting coefficients that are computationally easy to obtain.

Theorem 8. *Let $C_I \subseteq I$ be an unbounded flow cover. If $T \subseteq N \setminus C_I$ and for each $p \in T$ the lifting coefficients are defined as*

1. *$\alpha_p = 1, \beta_p = (\bar{a} - \lambda)s$, where $s\bar{a} - \lambda < a_p \leq s\bar{a}$ for an integer $s, 1 \leq s \leq k-1$, or*
2. *$\alpha_p = \lambda/\bar{a}, \beta_p = \frac{\lambda}{\bar{a}}(\bar{a} - \lambda)$ and $a_p \geq 2\bar{a} - \lambda$, or*
3. $$\begin{aligned} \alpha_p &= \max\{\frac{\lambda}{\lambda + a_p - s\bar{a}}, \frac{s\lambda}{\lambda + (q+1)a_p - (sq+1)\bar{a}}\} \qquad (25) \\ \beta_p &= \alpha_p a_p - \lambda s, \end{aligned}$$

   where $s\bar{a} + \frac{\bar{a}-\lambda}{q+1} \leq a_p \leq s\bar{a} + \frac{\bar{a}-\lambda}{q}$ for an integer s such that $1 \leq s \leq k-1$ and an integer $q, 1 \leq q \leq v_p - 1$, and the second term in (25) is not present if $qs \geq k$, or

4.

$$\alpha_p = \frac{\lambda}{\lambda + a_p - s\bar{a}}, \qquad \beta_p = \alpha_p a_p - \lambda s,$$

where $s\bar{a} < a_p \le s\bar{a} + \frac{\bar{a}-\lambda}{v_p}$ *for an integer* $s, 1 \le s \le k-1$,

then the lifted unbounded flow cover inequality

$$\sum_{p \in T} \alpha_p y_p + \sum_{i \in C_I} y_i \le (k-1)\lambda + (\bar{a} - \lambda) \sum_{i \in C_I} x_i + \sum_{p \in T} \beta_p x_p \tag{26}$$

is facet-defining for P.

Proof. It is easy to see that the functions h_p and f are piecewise linear and f is concave on every interval $[j\bar{a} - \lambda, (j+1)\bar{a} - \lambda]$, $j = 1, \dots, k-1$. Therefore if $h_p(z) \le f(z)$ for all points $z = ja_p, j = 1, \dots, v_p$ and $z = j\bar{a} - \lambda, j = 1, \dots, k$, then $h_p(z) \le f(z)$ for all $z \in [0, \min\{b, a_p v_p\}]$. For any $p \in N \setminus C_I$ note that $\alpha_p = \beta_p = 0$ are always a valid facet inducing lifting coefficients and therefore we set $\alpha_p = \beta_p = 0$ for all $p \in N \setminus (C_I \cup T)$. Based on Theorem 7 we have to show that $h_p(z) \le f(z)$ for all α_p, β_p given in the theorem.

1. Let $\alpha_p = 1, \beta_p = (\bar{a} - \lambda)s$ and s be such that $s\bar{a} - \lambda < a_p \le s\bar{a}$. First note that since $s \ge 1$, $\bar{a} - \lambda < a_p$ and therefore $h(a_p) = f(a_p)$. By the definition of h and since f is superadditive by Proposition 3, it follows that $h(ja_p) = jh(a_p) = jf(a_p) \le f(ja_p)$ for all $j, 1 \le j \le v_p$.

Next we consider points $j\bar{a} - \lambda$. Let i be such that $(i-1)a_p < j\bar{a} - \lambda \le ia_p$. Then $j\bar{a} - \lambda \le ia_p \le is\bar{a}$. Therefore since $\lambda < \bar{a}$ it follows that $j - is \le \lfloor \frac{\lambda}{\bar{a}} \rfloor = 0$. The inequality $j\bar{a} - \lambda - is(\bar{a} - \lambda) = h_p(j\bar{a} - \lambda) \le f(j\bar{a} - \lambda) = (j-1)\lambda$ is equivalent to $(\bar{a} - \lambda)(j - is) \le 0$ which clearly holds. Therefore we have shown that $h_p(z) \le f(z)$, which establishes feasibility.

To see that it yields a facet observe that $h_p(z) = f(z)$ for $z = a_p$ and $z = s\bar{a} - \lambda$ and that $a_p - (s\bar{a} - \lambda) \ne 0$.

2. Now let $\alpha_p = \lambda/\bar{a}, \beta_p = \frac{\lambda}{\bar{a}}(\bar{a} - \lambda)$ and $a_p \ge 2\bar{a} - \lambda$. We first show that $h_p(ja_p) \le f(ja_p)$ for $j = 1, \dots, v_p$. If $s\bar{a} - \lambda \le ja_p \le s\bar{a}$ for an integer $s, 1 \le s \le k$, then $h(ja_p) \le f(ja_p)$ is equivalent to $(\bar{a} - \lambda)(j\lambda - \bar{a}s + ja_p) \ge 0$ which holds by the definition of s. If $s\bar{a} \le ja_p \le (s+1)\bar{a} - \lambda$ for an integer $s, 0 \le s \le k-1$, then $h(ja_p) \le f(ja_p)$ is equivalent to $(\bar{a} - \lambda)(j-1) \ge 0$ which clearly holds.

Now let $j = 1, \dots, k$ and we consider $z = j\bar{a} - \lambda$. If $(s-1)a_p < j\bar{a} - \lambda \le sa_p$ for an $s, 1 \le s \le v_p$, then the inequality $h_p(z) \le f(z)$ is equivalent to $(\bar{a} - \lambda)(s-1) \ge 0$ which clearly holds. In addition, since $2\bar{a} - \lambda \le a_p$ we have $h_p(z) = f(z)$ for $z = \bar{a} - \lambda$ and $z = 2\bar{a} - \lambda$ and therefore we get a facet inducing lifting pair.

3. Now let q and s be as in statement 3 in the theorem. First observe that $s\bar{a} < a_p < (s+1)\bar{a} - \lambda$ and therefore $h_p(a_p) = \alpha_p a_p - \beta_p = \lambda s = f(a_p)$. Since f is superadditive it follows that $h_p(ja_p) \le f(ja_p)$ for all $j, 1 \le j \le v_p$.

Next we show the inequality for $z = j\bar{a} - \lambda, 1 \le j \le k$. If $(i-1)a_p < j\bar{a} - \lambda \le ia_p$, then $h_p(j\bar{a} - \lambda) = \alpha_p(j\bar{a} - \lambda) - i\beta_p$ and $f(j\bar{a} - \lambda) = (j-1)\lambda$. Therefore we have to show that

$$\alpha_p \ge \frac{\lambda(si - j + 1)}{ia_p - j\bar{a} + \lambda}. \tag{27}$$

For simplicity of notation we define $\bar{\alpha}(i,j)$ to be the fraction in (27), i.e. we need to show that $\bar{\alpha}(i,j) \le \alpha_p$ for all j. Even though it would be enough to consider i as a function of j and therefore defining $\bar{\alpha}$ as a function of a single argument, we define $\bar{\alpha}$ in this more general way since we will prove that (27) holds for a larger range of i and j. We write $j = us + l$, where $1 \le l \le s$.

It is easy to see by the definition of q that if $0 \le u \le q$, then $ua_p < j\bar{a} - \lambda \le (u+1)a_p$. For any j it follows that $\bar{\alpha}(u,j) \le \bar{\alpha}(u,j+1)$ if $u \le \frac{\bar{a}-\lambda}{a_p - \bar{a}s}$, which is equivalent to $u \le q$, and $\bar{\alpha}(u,j) \ge \bar{\alpha}(u,j+1)$

otherwise. Therefore

$$\begin{aligned}\max_{j=1,\ldots,(q+1)s} \bar{\alpha}(i,j) &= \max_{u=0,\ldots,q} \bar{\alpha}(u+1,(u+1)s) \\ &= \max_{u=0,\ldots,q} \frac{\lambda}{u(a_p-\bar{a}s)+\lambda+a_p-\bar{a}s} = \frac{\lambda}{\lambda+a_p-\bar{a}s},\end{aligned}$$

where the first equality follows from the monotonicity property and from $ua_p < j\bar{a}-\lambda \leq (u+1)a_p$, and the second equality follows by the definition of $\bar{\alpha}$. We have proven that $\bar{\alpha}(i,j) \leq \alpha_p$ for all $j, 1 \leq j \leq (q+1)s$. If $qs > k$, then this completes this part of the proof.

Next we claim that $\bar{\alpha}(i,j) \leq \bar{\alpha}(i+1,j)$ for all $j \geq (q+1)s+2$. This inequality is equivalent to $j \geq (a_p - s\lambda)/(a_p - \bar{a}s)$. But

$$\frac{a_p - s\lambda}{a_p - \bar{a}s} \leq \frac{\bar{a}s + \frac{\bar{a}-\lambda}{q} - s\lambda}{\bar{a}s + \frac{\bar{a}-\lambda}{q+1} - \bar{a}s} = s(q+1) + \frac{q+1}{q} \leq s(q+1)+2,$$

implying that if $j \geq s(q+1)+2$, then $\bar{\alpha}(i,j) \leq \bar{\alpha}(i+1,j)$.

It follows that for each $\bar{i} \geq 2$

$$\begin{aligned}\max_{l=1,\ldots,s} \bar{\alpha}(i,(q+\bar{i})s+l) &\leq \max_{l=1,\ldots,s} \bar{\alpha}(q+\bar{i}+1,(q+\bar{i})s+l) \leq \bar{\alpha}(q+\bar{i}+1,(q+\bar{i})s+1) \\ &= \frac{\lambda s}{(q+\bar{i})(a_p-\bar{a}s)+a_p-\bar{a}+\lambda} \leq \frac{\lambda s}{\lambda+(q+1)a_p-(sq+1)\bar{a}},\end{aligned}$$

where the first inequality follows from the monotonicity in i and $\bar{a}((q+\bar{i})s+l)-\lambda \leq (q+\bar{i}+1)a_p$, the second inequality from the monotonicity property with respect to the second argument of $\bar{\alpha}$, and the last one from $\bar{i} \geq 2$. Therefore we have shown that $\bar{\alpha}(i,j) \leq \alpha_p$ for all $j \geq (q+2)s+1$.

It remains to establish the inequality for $j = (q+1)s+1, \ldots, (q+2)s$. For $j = (q+1)s+1$ it follows by the definition of q that $(q+1)a_p < j\bar{a}-\lambda \leq (q+2)a_p$ and therefore it is easy to see that $\bar{\alpha}(q+2,(q+1)s+1) \leq \alpha_p$. For $j, (q+2)s \geq j \geq (q+1)s+2$ it follows that

$$\bar{\alpha}(i,j) \leq \bar{\alpha}(q+2,j) \leq \bar{\alpha}(q+2,(q+1)s+2) \leq \frac{\lambda s}{\lambda+(q+1)a_p-(sq+1)\bar{a}},$$

where the first and second inequality follow as above from monotonicity and the last inequality can be checked with a trivial but long calculation. This establishes that α_p, β_p are a valid lifting pair.

To show that they yield a facet, note that $h_p(z) = f(z)$ for $z = a_p$ and for either $z = \bar{a}s - \lambda$ or $z = (qs+1)\bar{a} - \lambda$, depending where the maximum in α_p is attained.

4. This case is similar to the previous one and its proof is therefore omitted. □

Note that statement 1 in Theorem 8 gives sufficient conditions for facet-definining simple lifted unbounded flow cover inequalities.

Example (continued). The facet-defining inequality (9) yields (by using statement 1 in Theorem 8 for variable 1 and statement 3 for variables 3 and 5) the facet-defining inequality $y_1 + \frac{1}{3}y_3 + y_4 + \frac{1}{3}y_5 \leq 3 + 3x_1 + x_3 + 3x_4 + x_5$, and facet-defining inequalities (10) and (12) produce the facet-defining inequality $y_1 + y_3 + y_4 + y_5 \leq 6 + 3x_1 + 3x_3 + 3x_4 + 3x_5$ by applying statement 1. The last facet can also be obtained by using Theorem 4 or Theorem 6. The facet-defining inequality (11) yields multiple lifting coefficients by using statements 1 and, either 2 or 3 or 4: $(\alpha_1, \beta_1) = (\alpha_4, \beta_4) \in \{(1,2), (\frac{1}{2}, \frac{1}{2})\}$ and $(\alpha_3, \beta_3) = (\alpha_5, \beta_5) \in \{(1,3), (\frac{1}{2}, \frac{1}{2})\}$. For example, $y_1 + \frac{1}{2}y_3 + \frac{1}{2}y_4 + y_5 + y_6 \leq 7 + 2x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + 3x_5 + x_6$ is facet-defining. □

3.3.2 Lifted Flow Cover Inequalities

Here we consider flow covers $C_F \subseteq F$. In this case the lifting function is

$$\begin{aligned} f(z) = \min b - \sum_{i\in C_F} y_i - \sum_{i\in C_F} (a_i-\lambda)^+(v_i-x_i) & \\ \sum_{i\in C_F} y_i \le b - z & \\ 0 \le y_i \le a_i x_i & \quad \text{for each } i \in C_F \\ 0 \le x_i \le v_i & \quad \text{for each } i \in C_F \\ x \;\text{ integer.} & \end{aligned}$$

For simplicity of notation we assume that $C_F = \{1,2,\dots,q\}, a_1 \ge a_2 \ge \cdots \ge a_q$ and that $a_i > \lambda$ for $i = 1,2,\dots,r$. We define the following two sequences.

$$\begin{aligned} M_{1,0} &= 0 \\ M_{s,l} &= \sum_{i=1}^{s-1} a_i v_i + l a_s & s &= 1,2,\dots,r & l &= 1,\dots,v_s \\ M_{s,v_s+1} &= M_{s+1,1} & s &= 1,2,\dots,r-1 \\ M_{r,v_r+1} &= -\infty \\ A_{s,l} &= \sum_{i=1}^{s-1} v_i + l & s &= 1,2,\dots,r & l &= 1,\dots,v_s \end{aligned}$$

We will repeatedly make use of the property

$$M_{s,l} - \lambda A_{s,l} = \sum_{i=1}^{j} (a_i - \lambda) > 0,$$

where $j = \sum_{i=1}^{s-1} v_i + l$.

Example (continued). For $C_F = \{1,2\}$ we have

$$\begin{aligned} (M_{1,0}, M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2}, M_{2,3}) &= (0,4,8,11,14,17), \\ (A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}, A_{2,3}) &= (1,2,3,4,5). \end{aligned}$$

□

Proposition 4. *For each $z \in [0,b]$*

$$f(z) = \begin{cases} z - M_{s,l} + \lambda A_{s,l} & \text{if } M_{s,l} - \lambda \le z \le M_{s,l} \quad s = 1,\dots,r,\ l = 1,\dots,v_s, \\ \lambda A_{s,l} & \text{if } M_{s,l} \le z \le M_{s,l+1} - \lambda \quad s = 1,\dots,r,\ l = 1,\dots,v_s, \\ z - M_{r,v_r} + \lambda A_{r,v_r} & M_{r,v_r} \le z \le b, \end{cases}$$

and f is superadditive on $z \in [0,b]$.

Proof. The claim has been proven by Gu, Nemhauser and Savelsbergh (1999) for the binary case, i.e. $v_i = 1$ for all $i \in C_F$. For the general case we can replace the variables y_i, x_i with new variables y_i^j, x_i^j, $j = 1,\dots,v_i$ and we can replace each inequality $y_i \le a_i x_i$ with $y_i^j \le a_i x_i^j$ for each $i \in C_F, j = 1,\dots,v_i$. In addition, we require that the x_i^j variables are binary. It is now easy to check the claim from the corresponding function listed in Gu, Nemhauser and Savelsbergh (1999).

To prove that f is superadditive we use the g_2 function. We define $\bar{u}_i = a_j - \lambda$ for $i = \sum_{s=1}^{j-1} v_s, \dots, \sum_{s=1}^{j} v_s$ and for $j = 1,\dots,r$, and $\bar{v}_i = \lambda$ for all i and $l = \lambda$. Then f is precisely g_2 and therefore is superadditive. □

The following theorem gives the lifting coefficients for the flow cover inequalities.

Theorem 9. *If $T \subseteq N \setminus C_F$ and for each $p \in T$ the lifting coefficients are defined as either*

1.

$$\alpha_p = \frac{1}{a_s}, \qquad \beta_p = \frac{\lambda}{a_s}(a_s - \lambda + \sum_{i=1}^{s-1}(v_i(a_i - a_s))),$$

where $1 \leq s \leq r$ and $a_p \geq 2\bar{a} - \lambda$ if $s = 1$ and $v_1 \geq 2$, and $a_p \geq \sum_{i=1}^{s-1} a_i v_i + a_s - \lambda$ otherwise, or

2. $\alpha_p = 1, \beta_p = M_{st} - \lambda A_{st}$ *and* $M_{st} - \lambda < a_p \leq M_{st}$,

then the lifted flow cover inequality

$$\sum_{p \in T} \alpha_p y_p + \sum_{i \in C_F} y_i \leq b - \sum_{i \in C_F} (a_i - \lambda)^+ (v_i - x_i) + \sum_{p \in T} \beta_p x_p \tag{28}$$

is facet-defining for P.

Proof. Note again that f is piecewise linear and concave in each interval $[M_{ij} - \lambda, M_{i,j+1} - \lambda]$ and therefore it is enough to show that $h_p(z) \leq f(z)$ for $z = ja_p$ and $z = M_{ij} - \lambda$.

1. We can rewrite the coefficients and the conditions as $\alpha_p = 1/\tilde{a}_s, \beta_p = \lambda(\frac{M_{s,t+1} - \lambda}{\tilde{a}_s} - A_{s,t})$, where $M_{12} - \lambda \leq M_{s,t+1} - \lambda \leq a_p, 0 \leq s \leq r, 1 \leq t \leq v_s$ and $\tilde{a}_s = a_s$ if $t < v_s$ and $\tilde{a}_s = a_{s+1}$ otherwise. Note that by the definition of $\tilde{a}_s$ it follows that $M_{s,t+1} - M_{s,t} = \tilde{a}_s$.

For each $z \in [0, a_p v_p]$ we have $h_p(z) \leq \alpha_p z - \beta_p$ and we next show that $\alpha_p z - \beta_p \leq f(z)$. It is enough to prove this inequality for $M_{ij} - \lambda$ since $\alpha_p z - \beta_p$ is a linear function. But for these points the inequality is equivalent to $M_{ij} - M_{st} \leq \tilde{a}_s(A_{jt} - A_{st})$ which holds because of the order imposed in C_F. Therefore the inequality is valid.

It is easy to check that $h_p(z) = f(z)$ for $z = M_{s,t+1} - \lambda$ and $z = M_{st} - \lambda$, and therefore is facet inducing.

2. Now let $\alpha_p = 1, \beta_p = M_{st} - \lambda A_{st}$ and $M_{st} - \lambda < a_p \leq M_{st}$. It is easy to see that $h_p(a_p) = f(a_p)$ and therefore by superadditivity of f the inequality $h_p(z) \leq f(z)$ holds for all $z = ja_p$.

Now let $1 \leq j \leq v_p$ and consider the function $g(z) = f(z) - z + j\beta_p$ defined on $[(j-1)a_p, ja_p]$. We have already argued that g is nonnegative at the boundaries of the interval. The derivative g' exists at all points except those that are of the form $M_{ij} - \lambda$ for indices i and j. At points where it exists g' is either 0 or -1. Therefore g is nonincreasing which implies that $g(z) \geq 0$ for all $z \in [(j-1)a_p, ja_p]$.

The above arguments show that $h_p(z) \leq f(z)$ for all z. The equality is attained for $z = a_p$ and $z = M_{s,t}$ and therefore we get lifting coefficients that yield facet-inducing inequalities. □

Example (continued). The flow cover $C_F = \{1, 2\}$ yields the facet-defining inequality $y_1 + y_2 + y_4 + \frac{1}{4}y_5 \leq 8 + 2x_1 + x_2 + 2x_4 + x_5$ by using statement 1 in Theorem 9 with $s = 1$ for variable 5 and statement 2 for variable 4, whereas by applying statement 2 the flow cover $C_F = \{3\}$ gives the facet-defining inequality $y_1 + y_3 + y_4 + y_5 \leq 6 + 3x_1 + 3x_3 + 3x_4 + 3x_5$, which has already been obtained. Except by considering subsets of the lifted variables, we do not get any other facets by applying Theorem 9.

4 Extentions

In this section we study two related sets

$$S_= = \{(y, x) \in \mathbb{R}_+^{2n} : \sum_{i=1}^n y_i = b, y_i \leq a_i x_i, x_i \leq v_i, i = 1, \ldots, n, x \text{ integer}\},$$

$$S_\geq = \{(y, x) \in \mathbb{R}_+^{2n} : \sum_{i=1}^n y_i \geq b, y_i \leq a_i x_i, x_i \leq v_i, i = 1, \ldots, n, x \text{ integer}\},$$

and the associated convex hulls $P_= = conv(S_=)$ and $P_\geq = conv(S_\geq)$. For $S_\geq$ we assume that if $I = \emptyset$, then $\sum_{i\in N} a_i v_i \geq b + a_k v_k$ for all $k \in F$ as otherwise $y_k \geq 0$ is dominated by $y_k \geq b - \sum_{i\in N\setminus\{k\}} a_i v_i$. For $S_\geq$ we also assume that if $|I| = 1$, then $\sum_{i\in F} a_i v_i \geq b$. For $S_=$ we use all these assumptions and the assumptions made for S before. Using these assumptions it is easy to see that $dim(P_\geq) = 2n, dim(P_=) = 2n-1$ and that all the inequalities listed in the description of the sets $S_=, S_\geq$ are facet-defining, called the trivial facets, for $P_=, P_\geq$, respectively.

Similar propositions to Proposition 1 hold for $P_=$ and $P_\geq$ and are given next.

Proposition 5. *If $\widetilde{m}y \leq u_0 + ux$ defines a nontrivial facet of $P_=$, then*

1. *$0 \leq u$,*
2. *there exists a vector $m \geq 0$ with $m_j = 0$ for some $j \in N$ such that the inequality $my \leq u_0 + ux$, called a normalized facet-defining inequality, defines the same facet,*
3. *if $i \in I$ and $m_i > 0$, then $u_i > 0$.*

Proposition 6. *If $my + ux \geq u_0$ defines a nontrivial facet of $P_\geq$, then $u \geq 0, m \geq 0, u_0 > 0$.*

The following proposition shows how to get facets for P and $P_\geq$ from facets of $P_=$. Its proof follows closely the proof in Padberg, Van Roy and Wolsey (1985) and is therefore omitted.

Proposition 7. *If $my \leq u_0 + ux$ defines a nontrivial normalized facet of $P_=$, then*

1. *$(M\mathbf{1} - m)y + ux \geq Mb - u_0$ defines a nontrivial facet of $P_\geq$,*
2. *$(m + (-t)^+\mathbf{1})y \leq u_0 + (-t)^+ b + ux$ defines a nontrivial facet of P, where*

$$t = \min\{\frac{u_0 + \sum_{i\in N}(u_i - a_i m_i)x_i}{b - ax} : ax < b, 0 \leq x \leq v, x \textit{ integer}\},$$

and $t^+ = \max\{0, t\}$.

Example (continued). The inequality $y_1 + y_2 \leq 8 + 2x_1 + x_2$ is facet-defining for $P_=$ and therefore by Proposition 7 is also facet-defining for P, and $7 \leq y_3 + y_4 + y_5 + y_6 + 2x_1 + x_2$ is facet-defining for $P_\geq$. □

Every valid inequality for the set $K = conv\{ax \geq b, 0 \leq x \leq v, x \text{ integer}\}$ is a valid inequality for $P_=$ and $P_\geq$. The following proposition identifies which facets of K are facets of $P_\geq$ or $P_=$. The proof is omitted since it closely follows the related proof in Padberg, Van Roy and Wolsey (1985).

Proposition 8. *Let $ux \geq 1$ be facet-defining for K. Then $ux \geq 1$ defines a nontrivial facet of $P_\geq/P_=$ if and only if for each $i \in N$ there exists a vector $(y, x) \in P_\geq/(y, x) \in P_=$ such that $ux = 1$ and $y_i < a_i x_i$.*

As in Section 2 we define

$$\begin{aligned} V_\geq = \{x \in \mathbb{R}^n_+ : 1 - u_0 + ux &\leq \sum_{i\in N}(j - m_i)^+ a_i x_i - jb \text{ for all } j = 0, \ldots, M, \\ -1 + u_0 - ux &\leq \sum_{i\in N} m_i a_i x_i, \\ ax &\geq b, \\ x &\leq v, \quad x \text{ integer}\}, \end{aligned}$$

An analoguos theorem to Theorem 1 holds for $P_\geq$.

Theorem 10. *The inequality $u_0 \leq my + ux$ is valid for $P_\geq$ if and only if $V_\geq$ is empty.*

The proof uses the following lemma, which can be proved from Lemma 1 by making the substitution $z_i = A_i - y_i$.

Lemma 2. *Let $A_i \in \mathbb{Q}_+$ for all $i \in N$, $B \in \mathbb{Q}_+$, $b \in \mathbb{Q}_+$, and we assume that $b \leq \sum_{i=1}^n A_i$ and $B \leq \sum_{i \in N} m_i A_i$. Then $jb - \sum_{i=1}^n (j - m_i)^+ A_i \leq B$ for all $j = 0, \ldots, M$ if and only if*

$$\{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i \geq b, my = B, y_i \leq A_i \text{ for all } i \in N\}$$

is nonempty,

In order to show that the facets obtained for P have corresponding facets in $P_=$ and $P_\geq$, we need the following corollary to Theorem 2, which states that all facets of P of the projection to the variables in F or I are facets of $P_=$.

Corollary 2. *If $my \leq u_0 + ux$ defines a nontrivial facet $\mathcal{F}$ of $\{(y, x) \in P, x_i = y_i = 0 \text{ for all } i \in F\}$ and*

a) $\sum_{i \in F} a_i v_i > b - \min\{\sum_{i \in N} y_i : (y, x) \in \mathcal{F}\}$, *and*

b) $\min\{\sum_{i \in I} a_i x_i : \sum_{i \in I} a_i x_i \leq b - 1, \sum_{i \in I}(m_i a_i - u_i) x_i = u_0, x \geq 0, x \text{ integer}\} \geq b - \sum_{i \in F} a_i v_i$,

then it also defines a facet of $P_=$.

If $my \leq u_0 + ux$ defines a nontrivial facet of $\{(y, x) \in P, x_i = y_i = 0 \text{ for all } i \in I\}$, then it also defines a facet of $P_=$.

Proof. We first prove the claim when $\mathcal{F}$ is a facet of the projection onto the variables in I, i.e. $x_i = y_i = 0$ for all $i \in F$.

Let $\sum_{i \in N} \alpha_i y_i + \sum_{i \in N} \beta_i x_i = \Pi_0$ be satisfied by all vectors in $\bar{\mathcal{F}} = \{(y, x) \in \mathbb{R}^{2n}_+ : (y, x) \in P_=, my = u_0 + ux\}$. We show that $\beta = 0$ and that $\alpha = K\mathbf{1}$ for a constant K. We write vectors in $\mathbb{R}^{2n}$ as $(y, x, \bar{y}, \bar{x})$ where y and x correspond to indices in I and $\bar{y}, \bar{x}$ correspond to indices in F.

By Corollary 1, there is a vector $(y, x) \in \mathbb{R}^{2 \cdot |I|}$ on the projection of $\mathcal{F}$ to I such that $\sum_{i \in I} y_i = b$. Let e_i be the ith unit vector and consider the vectors $(y, x, 0, 0) \in \bar{\mathcal{F}}$ and $(y, x, 0, e_i) \in \bar{\mathcal{F}}$ for all $i \in F$. It follows that $\beta_i = 0$ for all $i \in F$.

Since $\mathcal{F}$ is a nontrivial facet, there exists $(y, x) \in \mathcal{F}$ such that $\sum_{i \in I} y_i < b$. From assumption a) it follows that $\sum_{i \in F} a_i v_i > b - \sum_{i \in I} y_i$. Then there exist affinely independent vectors $y^1 \in \mathbb{R}^{|F|}_+, \ldots, y^{|F|} \in \mathbb{R}^{|F|}_+$ such that $y^j_i \leq a_i v_i$ for all $i \in F$, $\sum_{i \in F} y^j_i = b - \sum_{i \in I} y_i$ for all $j = 1, \ldots, |F|$. The vectors (y, x, y^j, v) are in $\bar{\mathcal{F}}$ and hence α_i are equal to a constant K for all $i \in F$.

Since $\mathcal{F}$ is a facet of the projection onto the variables in I, there are $2 \cdot |I| + 1$ affinely independent vectors $(y^j, x^j) \in \mathcal{F}$. We next show that by assumption b) these vectors yield vectors in $P_=$. Let U be the set of all vectors $(y^j, x^j) \in \mathcal{F}$ such that $\sum_{j \in I} y^j_i < b$. We know from Theorem 2 that every vector in U has to satisfy the property that $y^j_i = a_j x^j_i$ for all $i \in I$ such that $m_i > 0$. From assumption b) it follows that for each $(y^j, x^j) \in U$ there is a vector $(\bar{y}^j, \bar{x}^j)$ such that $(y^j, x^j, \bar{y}^j, \bar{x}^j) \in \bar{\mathcal{F}}$. These vectors show that $\alpha = K\mathbf{1}$ and that $\beta_i = 0$ for all $i \in I$. Therefore $\bar{\mathcal{F}}$ is a facet of $P_=$.

The remaining case for $P_=$ can be proven in the same way except that we replace v by $b\mathbf{1}$ and $\bar{x}^j$ above by $b\mathbf{1}$. □

Using the same vectors as in the proof of Theorem 3 and Corollary 2 we can establish the following result. The second statement follows from Proposition 7 and the first statement.

Corollary 3. *Let $C_I \subseteq I$ be an unbounded flow cover. Assume that if $F = \emptyset$, then $C_I \neq I$ and that if $C_I = I$ and $F \neq \emptyset$, then $\sum_{i \in F} a_i v_i > b - \bar{a}(k - 1)$. Then*

$$\sum_{i \in C_I} y_i \leq (k - 1)\lambda + (\bar{a} - \lambda) \sum_{i \in C_I} x_i$$

is facet-defining for $P_=$, and

$$(\bar{a} - \lambda)k \leq \sum_{i \in N \setminus C_I} y_i + (\bar{a} - \lambda) \sum_{i \in C_I} x_i$$

is facet-defining for $P_\geq$.

By using Proposition 7, Corollary 2, and Theorem 6 we obtain facets for $P_=$ and $P_\geq$ that are derived from C_F.

Corollary 4. *Let C_F, T be defined as in Theorem 6 and let us also assume that $\bar{a} \leq \lambda + \sum_{i \in N \setminus (C_F \cup T)} a_i v_i$ and that there is a $j \in C_F$ such that $\lambda < a_j < \lambda + \sum_{i \in N \setminus (C_F \cup T)} a_i v_i$. Then*

1. *(20) is facet-defining for $P_=$, and*
2. *the inequality*

$$\sum_{i \in C_F} (a_i - \lambda)^+ v_i \leq \sum_{i \in N \setminus (C_F \cup T)} y_i + \sum_{i \in C_F} (a_i - \lambda)^+ x_i + (\bar{a} - \lambda) \sum_{i \in T} x_i$$

is facet-defining for $P_\geq$.

Note that if $(N \setminus (C_F \cup T)) \cap I \neq \emptyset$, then the extra conditions in Corollary 4 are automatically fulfilled.

5 The Case $a_i = a$ for all $i \in N$

In this section we give a complete polyhedral description of P if $a_i = a$ for all $i \in N$. For the case $I = N$, such a description was recently given by Atamtürk (2002) using a completely different argument from the one presented here. It is easy to see that if a divides b, then the trivial inequalities completely describe P. Therefore we assume that $\lambda = ak - b > 0$, where $k = \lceil b/a \rceil$.

Proposition 9. *For any $C \subseteq N$ such that $\sum_{i \in C} v_i \geq k$,*

$$\sum_{i \in C} y_i \leq b - (a - \lambda)(k - \sum_{i \in C} x_i) \tag{29}$$

is facet-defining for P .

Proof. Let $C \subseteq N$ be such that $\sum_{i \in C} v_i \geq k$. If $C \cap I \neq \emptyset$, then $C \cap I$ is an unbounded flow cover and Theorem 8, rule 1, yields that (29) is facet-defining.

Let $C \subseteq F$. If $\sum_{i \in C} v_i = k$, then C is a flow cover and therefore (29) is facet-defining by Theorem 6 with $T = \emptyset$. Now consider a set $C \subseteq F$ such that $\sum_{i \in C} v_i \geq k + 1$. Validity can be easily checked by using Theorem 1. Without loss of generality let $C = \{1, \ldots, q\}$ and let $j, 0 \leq j \leq q - 1$ be such an index that $\sum_{i=1}^{j} v_i \leq k$ and $\sum_{i=1}^{j+1} v_i \geq k + 1$. Let us define the vector x^0 as

$$x_i^0 = \begin{cases} v_i & \text{if } i \leq j, \\ k - \sum_{t=1}^{j} v_t & \text{if } i = j + 1, \\ 0 & i > j + 1. \end{cases}$$

In addition, for $i > j + 1$ we define $x^i = x^0 - e_1 + e_i$ and for $2 \leq i \leq j$ we define $x^i = x^0 - e_i + e_{j+2}$. We denote also $x^{j+1} = x^0 - e_{j+1} + e_{j+2}$. The vectors x^i all have the property that $\sum_{t \in C} x_t^i = k$ and $0 \leq x_t^i \leq v_t$ for all $t \in C$. Using these vectors, the vector $x^0 - e_1$, and the technique used in the proof of Theorem 3, it is easy to see that (29) is facet-defining. $\square$

The main goal of this section is to prove that the trivial facets and facets defined by (29) completely describe P. Note that the facets induced by (29) are not always special cases of privously given facets.

Theorem 11.

$$P = \{(y, x) \in \mathbb{R}_+^{2n} : \sum_{i \in N} y_i \leq b \tag{30a}$$

$$y_i \leq a x_i \qquad i \in N \tag{30b}$$

$$x_i \leq v_i \qquad i \in N \tag{30c}$$

$$\sum_{i \in C} y_i \leq b - (a - \lambda)(k - \sum_{i \in C} x_i) \qquad \textit{for all } C \subseteq N \textit{ such that } \sum_{i \in C} v_i \geq k\} \tag{30d}$$

The theorem has been proven by Padberg, Van Roy and Wolsey (1985) if $v_i = 1$ for all $i \in N$ and for $P_{=}$. We generalize their approach to general upper bounds and to P.

Lemma 3. *Let m, u be two nonnegative n-dimensional vectors. Then the solution to* $\max\{my - ux : (y, x) \in P\}$ *is given by the LP*

$$\max \sum_{i \in N} (am_i - u_i)w_i + \sum_{i \in N} ((a - \lambda)m_i - u_i)z_i \tag{31a}$$

$$\sum_{i \in N} w_i \le k - 1 \tag{31b}$$

$$\sum_{i \in N} z_i \le 1 \tag{31c}$$

$$w_i + z_i \le v_i \qquad i \in N \tag{31d}$$

$$w \ge 0, z \ge 0 \tag{31e}$$

Proof. Let (y, x) be an optimal solution to $\max\{my - ux : (y, x) \in P\}$.

Case 1.) Suppose that $\sum_{i \in N} y_i = b$. If there exist i, j such that $0 < y_i < ax_i, 0 < y_j < ax_j$ and $m_i \ge m_j$, then there is an $\epsilon > 0$ such that $(y + \epsilon e_i - \epsilon e_j, x)$ is again a feasible vector. Note also that since $u \ge 0$, if $y_i = 0$, then $x_i = 0$. Therefore there exists an optimal solution (y, x) and an index $j \in N$ such that $y_i = ax_i$ for all $i \in N \setminus \{j\}$ and $y_j = b - a \sum_{i \in N \setminus \{j\}} x_i$. Let $l = k - 1 - \sum_{i \in N \setminus \{j\}} x_i$. Since $u_j \ge 0$, it follows that $x_j = l + 1$. It is easy to see that $z = e_j$ and $w_i = x_i$ for all $i \in N \setminus \{j\}$, $w_j = l$, is a feasible solution to (31) with the same objective value.

Case 2.) Now let $\sum_{i \in N} y_i < b$. In this case $y_i = ax_i$ for all $i \in N$ since otherwise either the solution is not optimal or we can obtain a solution that is covered by the first case. Now it is easy to see that $z = 0, w = x$ is a solution to $\max\{my - ux : (y, x) \in P\}$ with the same objective value.

Conversely, suppose that (w, z) is an integer solution to (31). An integer solution always exists since the constraint matrix is totally unimodular. Then we can define $(y, x) \in P$ as above with the same objective value. □

Lemma 4. *If (y, x) is a vertex of P, then $x_i \le k$ for all $i \in I$.*

Proof. Let (y', x') be a vertex of P. Then there exist $m \in \mathbb{R}^n, u \in \mathbb{R}^n$ such that (y', x') is the unique solution to $\max\{my - ux : (y, x) \in P\}$. Note that $u_i > 0$ for all $i \in I$ since otherwise the maximum is unbounded. Then there are sets $\tilde{S} \subseteq I, \bar{S} \subset I$ with the property $I = \tilde{S} \cup \bar{S}, |\bar{S}| \le 1$, and $y'_i = ax'_i$ for all $i \in \tilde{S}$, and $y'_l = b - a \sum_{i \in \tilde{S}} x_i, x'_l = \lceil y_l / a \rceil$, where $l \in \bar{S}$. If these conditions are not met, it is easy to see that (y', x') is not a unique optimal solution. For $i \in \tilde{S}$ it follows that $x'_i \le y'_i / a \le b/a \le k$. Similarly for $l \in \bar{S}$ it follows that $x'_l = \lceil y'_l / a \rceil \le \lceil b/a \rceil = k$. This proves the statement. □

Proof of Theorem 11. We prove that for any vectors m, u the objective values $\max\{my - ux : (y, x) \in P\}$ and $\max\{my - ux : (y, x)$ satisfies (30)$\}$ are equal. It is easy to see that it suffices to prove this property for all m, u that define a facet of P. Therefore by Proposition 1 we can assume that $m \ge 0, u \ge 0$ and by Corollary 1 for a given m, u we can consider only optimal solutions with $\sum_{i \in N} y_i = b$.

Let us define $\tilde{v}_i = v_i$ if $i \in F$ and $\tilde{v}_i = k$ if $i \in I$. By Lemma 4 and since for any $C \subseteq N$ $\sum_{i \in C} v_i \ge k$ if and only if $\sum_{i \in C} \tilde{v}_i \ge k$, it suffices to prove the theorem for $\tilde{v}$. We prove that the objective value of the dual of (31) is the same as the objective value of the dual of (30). By Lemma 3 this establishes the result.

The dual of (31) is

$$\begin{aligned} \min \sum_{i \in N} \tilde{v}_i \pi_i + (k - 1)\delta_1 + \delta_2 & \\ \delta_1 + \pi_i \ge am_i - u_i & \qquad i \in N \\ \delta_2 + \pi_i \ge (a - \lambda)m_i - u_i & \qquad i \in N \\ \delta_1 \ge 0, \delta_2 \ge 0, \pi \ge 0, & \end{aligned} \tag{32}$$

where δ_1 corresponds to (31b), δ_2 to (31c), and π to (31d), and the dual of (30) is

$$
\begin{aligned}
\min b\alpha + \sum_{i\in N} \tilde{v}_i\theta_i + (k-1)\lambda \sum_C \gamma_C & \\
\alpha + \beta_i + \sum_{C:i\in C} \gamma_C \geq m_i \qquad & i \in N \\
-a\beta_i + \theta_i - (a-\lambda) \sum_{C:i\in C} \gamma_C \geq -u_i \qquad & i \in N \\
\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \theta \geq 0, &
\end{aligned} \tag{33}
$$

where α corresponds to (30a), β to (30b), θ to (30c), and γ_C is defined only for subsets with $\sum_{i\in C} \tilde{v}_i \geq k$ and it corresponds to (30d).

Without loss of generality we assume that $m_1 \geq m_2 \geq \cdots \geq m_n$ and that if $m_i = m_j, i > j$, then $u_j \geq u_i$. An optimal solution (y,x) to $\max\{my - ux : (y,x) \in P\}$ with $\sum_{i\in N} y_i = b$ corresponds to an optimal solution (w,z) of (31) with $\sum_{i\in N} w_i = k-1$ and $\sum_{i\in N} z_i = 1$. Therefore there exists an index $j^* \in N$ such that $z_{j^*} = 1$. Among all such optimal solutions to (31) we select one that has the largest j^*. Observe that $(a-\lambda)m_{j^*} - u_{j^*} > 0$ since otherwise we get a better solution by setting $z_{j^*} = 0$. Let $(\delta_1, \delta_2, \pi)$ be an optimal solution to (32). Define

$$
\begin{aligned}
t^* &= \max\{i \in N : \delta_1 - \delta_2 \leq \lambda m_i\}, \\
s^* &= \begin{cases} n+1 & \text{if } m_i \geq \frac{\delta_2}{a-\lambda} \text{ for all } i \in N, \\ \min\{i \in N : m_i < \frac{\delta_2}{a-\lambda}\} & \text{otherwise.} \end{cases}
\end{aligned}
$$

Among all optimal solutions to (32) we select one that has the largest t^*. By the order imposed on m we have that $(\delta_1 - \delta_2)/\lambda \leq m_i$ for all i such that $i \leq t^*$ and $(\delta_1 - \delta_2)/\lambda > m_i$ for all i such that $i > t^*$. Similarly we have that $m_i < \delta_2/(a-\lambda)$ for all i such that $i \geq s^*$ and $m_i \geq \delta_2/(a-\lambda)$ for all i such that $i < s^*$.

Claim 1. $t^* \leq j^*$

Since $z_{j^*} = 1$, by complementary slackness $\delta_2 + \pi_{j^*} = (a-\lambda)m_{j^*} - u_{j^*}$ and therefore $m_{j^*} \leq (\delta_1 - \delta_2)/\lambda$. If $m_{j^*} < (\delta_1 - \delta_2)/\lambda$, then by the definition of t^* it follows that $t^* < j^*$. Suppose that $m_{j^*} = (\delta_1 - \delta_2)/\lambda$ and $j^* < t^*$. Then

$$\frac{\delta_1 - \delta_2}{\lambda} = m_{j^*} \geq m_{t^*} \geq \frac{\delta_1 - \delta_2}{\lambda}.$$

Therefore $m_{j^*} = m_{t^*}$ and since $u_{t^*} \leq u_{j^*}$ by the order imposed, it follows that $(a-\lambda)m_{j^*} - u_{j*} \leq (a-\lambda)m_{t^*} - u_{t^*}$. Suppose that $w_{t^*} < \tilde{v}_{t^*}$. Then consider $(w, \bar{z}) = (w, z - e_{j^*} + e_{t^*})$, which is a feasible solution to (31). If $u_{t^*} < u_{j^*}$, then (w,z) is not an optimal solution which is a contradiction. If $u_{t^*} = u_{j^*}$, then $(w, \bar{z})$ is an optimal solution to (31) with $\bar{z}_{t^*} = 1$, which contradicts the choice of j^*. Therefore we have shown that $w_{t^*} = \bar{v}_{t^*}$. Now consider $(\bar{w}, \bar{z}) = (w - e_{t^*} + e_{j^*}, \bar{z})$. It is easy to see that $(\bar{w}, \bar{z})$ is an optimal solution to (31), again contradicting the choice of $j*$. This proves Claim 1.

Claim 2. $j^* < s^*$

Since $z_{j^*} = 1$ and by complementary slackness it follows that $\delta_2 \leq \delta_2 + \pi_{j^*} = (a-\lambda)m_{j^*} - u_{j^*} \leq (a-\lambda)m_{j^*}$. Therefore by definition $j^* < s^*$, which shows Claim 2.

Claim 3. $\sum_{i=1}^{t^*} \tilde{v}_i \geq k$

By complementary slackness, if $w_i > 0$, then $\delta_1 + \pi_i = am_i - u_i$. Since $\delta_2 + \pi_i \geq (a-\lambda)m_i - u_i$, it follows that $\delta_1 - \delta_2 \leq \lambda m_i$, and therefore $i \leq t^*$. Note also that $\sum_{i:z_i>0} \tilde{v}_i \geq \sum_{i:z_i>0} z_i = k-1$. Therefore

$$\sum_{i=1}^{t^*} \tilde{v}_i \geq \sum_{i:z_i>0} \tilde{v}_i \geq k-1. \tag{34}$$

If $\sum_{i=1}^{t^*} \tilde{v}_i \geq k$, then the claim is true, so assume that $\sum_{i=1}^{t^*} \tilde{v}_i = k-1$. In this case clearly $t^* < n$. By analyzing lower bounds in (34), we get that $w_i = 0$ for all $i > t^*$ and $w_i = \tilde{v}_i$ for all $i \leq t^*$. If $\delta_1 + \pi_i = am_i - u_i$

and $i > t^*$, then this contradicts the choice of t^*. Therefore $\delta_1 + \pi_i > am_i - u_i$ for all $i > t^*$. Let us define $\epsilon = \min\{\min_{i=t^*+1,\dots,n}\{\delta_1 + \pi_i - am_i + u_i\}, \delta_1\}$. If $\delta_1 = 0$, then $t^* = n$, which is a contradiction and therefore $\epsilon > 0$. Consider the following optimal solution to (32): $\bar{\delta}_1 = \delta_1 - \epsilon, \bar{\delta}_2 = \delta_2$, $\bar{\pi}_i = \pi_i$ for all $i > t^*$, and $\bar{\pi}_i = \pi_i + \epsilon$ for $i = 1, \dots, t^*$. If $\bar{\delta}_1 = 0$, then t^* corresponding to this solution is n, which is a contradiction. Therefore $\bar{\delta}_1 > 0$ and there exists $l > t^*$ such that $\bar{\delta}_1 + \pi_l = am_l - u_l$, where l is the index attaining the minimum in the definition of ϵ. The solution $(\bar{\delta}_1, \bar{\delta}_2, \bar{\pi})$ is an optimal solution to (32) with a larger t^* than the fixed solution $(\delta_1, \delta_2, \pi)$. But this contradicts the choice of t^* since t^* is the maximum index over all optimal dual solutions to (32). This proves the claim.

For $i \in N$ let us denote $C_i = \{1, 2, \dots, i\}$. Consider the following dual solution to (33).

$$
\begin{aligned}
\alpha &= \frac{\delta_2}{a - \lambda} & & \\
\beta_i &= (m_i - \frac{\delta_1 - \delta_2}{\lambda})^+ & & i \in N, \\
\gamma_{C_{t^*}} &= \frac{\delta_1 - \delta_2}{\lambda} & & \\
\gamma_{C_i} &= m_i - m_{i+1} & & t^* < i < s^* - 1, \\
\gamma_{C_{s^*-1}} &= m_{s^*-1} - \frac{\delta_2}{a - \lambda} & & \\
\gamma_C &= 0 & & \text{otherwise.}
\end{aligned}
$$

This solution is well defined by Claims 1, 2, and Claim 3 and it is easy to see that it is dual feasible to (33) with the same objective value as the objective value of $(\delta_1, \delta_2, \pi)$ for (32). Therefore we have proven Theorem 11. $\square$

Since $P_=$ is a face of P, we get the following corollary.

Corollary 5.

$$
\begin{aligned}
P_= = \{(y, x) \in \mathbb{R}^{2n}_+ : \sum_{i \in N} y_i = b & & \\
y_i &\le ax_i & & i \in N \\
x_i &\le v_i & & i \in N \\
\sum_{i \in C} y_i &\le b - (a - \lambda)(k - \sum_{i \in C} x_i) & & \textit{for all } C \subseteq N \textit{ such that } \sum_{i \in C} v_i \ge k\}
\end{aligned}
\tag{35}
$$

Here we do not claim that this is a minimal representation of $P_=$, i.e. it is not necessarily true that all of the inequalities in (35) are facet-defining. We leave open the case $P_\ge$.

6 Computational Experiments

Here we present a branch-and-cut algorithm that uses our valid inequalities. We also compare the results obtained from CPLEX with and without our cuts. We first show how to use our cuts to separate fractional solutions from the LP relaxation.

6.1 Separation

For all the valid inequalities (13), (16), (26), and (28) the separation is done heuristically and we always try to find an inequality that is most violated. Let (y^*, x^*) be an LP solution at a given node of the branch-and-bound tree. Given either an unbounded flow cover or a flow cover C, for each index $p \in N \setminus C$ the lifting pair (α_p, β_p) is selected as the pair of coefficients that leads to the largest violation, i.e. among all possible lifting coefficients for p, which are given by the corresponding formulas in Theorem 4, Theorem 5, Theorem 8, and Theorem 9, we select the one that maximizes $\alpha_p y_p^* - \beta_p x_p^*$.

To detect a lifted unbounded flow cover inequality we proceed as follows. Before we start branch-and-cut, we sort the variables in N in nondecreasing order of $(k-1)\lambda$. In the separation routine, we scan the variables $i \in N$ and at each iteration we construct C_I on the fly such that $a_i = \max_{j \in C_I} a_j$. A variable pair $k \in N \setminus \{i\}$ is either lifted based on C_I, or we add k to C_I if $a_k \leq a_i$ and in this case the corresponding coefficients are computed based on (8). If both cases are possible, we choose the one that leads to the larger violation. If a cut is found, we add it to the LP relaxation of the node. This yields an algorithm with a worst case complexity $\mathcal{O}(n^2)$, which might be excessive, but in practice many variables have $y_p^* = x_p^* = 0$ and all such variables need not be considered for lifting. If the last cut found has $a_k = \max_{j \in C_I} a_j$, then the next time the separation routine is invoked it starts scanning the variables with index k. In other words, indices are considered in a wrap-around order and every time we start scanning from the index visited last.

Unfortunately finding a flow cover itself is an NP-complete problem. Note that in the case of binary variables finding a flow cover is solvable in polynomial time, however in the general case the constraint $\bar{a} > \lambda$ is not automatically fulfilled.

Proposition 10. *Given a, v and b, the problem of finding a flow cover is NP-complete.*

Proof. We give a reduction from the subset sum problem. Let $\widetilde{a} \in \mathbb{Z}_+^n$ and $\widetilde{b} \in \mathbb{Z}_+$ be the input to the subset sum problem, i.e. we want to find a subset $\widetilde{C} \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in \widetilde{C}} \widetilde{a}_i = \widetilde{b}$.

We define the input for the flow cover problem as follows. There are $n+1$ elements, where the first n elements correspond to the elements in the subset sum problem. Let $b = \widetilde{b} + 1, a = (\mathbf{1}, 2)$, and $v = (\widetilde{a}, 1)$ be the input data for the flow cover problem.

Let C be a flow cover for these input data. By definition C satisfies $\sum_{i \in C} a_i v_i > \widetilde{b} + 1$ and $\bar{a} > \sum_{i \in C} a_i v_i - \widetilde{b} - 1$. If $C \subseteq \{1, 2, \ldots, n\}$, then $\bar{a} = 1$ and it is easy to see that in this case $\widetilde{b} + 1 < \sum_{i \in C} \widetilde{a}_i < \widetilde{b} + 2$, which is a contradiction. We conclude that $\{n+1\} \in C$. We define $\widetilde{C} = C \setminus \{n+1\}$. Then the two flow cover conditions are equivalent to $\widetilde{b} - 1 < \sum_{i \in \widetilde{C}} \widetilde{a}_i < \widetilde{b} + 1$. Since the data are integral it follows that $\sum_{i \in \widetilde{C}} \widetilde{a}_i = \widetilde{b}$ and therefore $\widetilde{C}$ solves the subset set problem. □

At each node of the branch-and-cut tree we first sort the variables in nondecreasing order of

$$\frac{y_i^* + (a_i - 1)(v_i - x_i^*)}{v_i^2}.$$

The numerator guides the selection of the variables that give the largest violation, where we use the approximation $\lambda = 1$. On the other hand, a greedy approach for finding a flow cover is to select the variables in nondecreasing order of a_i / v_i, which is captured by our formula if $y_i^* = x_i^* = 0$ since the denominator is v_i^2. Next we construct a flow cover as follows. We start with an empty set C_F and we include variables in C_F based on the order described. If $\sum_{i \in C_F} a_i v_i \leq b$, then we expand C_F with the next variable. If $\sum_{i \in C_F} a_i v_i > b$ and $\bar{a} \leq \lambda$, we remove the last variable from C_F and keep scanning from the current position. If $\sum_{i \in C_F} a_i v_i > b$ and $\bar{a} > \lambda$, then we stop since C_F is a flow cover. Such a procedure typically leads to a flow cover but it often happens that the flow cover does not yield a violated lifted flow cover inequality. To generate additional covers, we randomly select a variable $i \in C_F$ and we start the greedy procedure all over from the beginning of the array, except that we assume that i is not a candidate for inclusion in C_F and that the initial set C_F consists of $C_F \setminus \{i\}$. We repeat the procedure at most 60 times, i.e. at most 60 covers can be obtained. Each flow cover obtained is lifted and if it yields a cut, it is added to the LP relaxation of the node.

6.2 Implementation

Computational experiments were carried out on an SGI Origin200 workstation with a RISC 12000 processor running at the clock speed of 270MHz. The operating system is IRIX, version 6.5, and the workstation is equipped with 512MB of main memory. The branch-and-cut algorithms were implemented on top of CPLEX,

CPLEX Optimization (2002), version 7.5. Except for cut generation, all other features and parameters of CPLEX are set to their default values.

Cuts are generated at every node of the branch-and-bound tree and at every node except the root node at most 3 rounds of cuts are added, i.e. we solve the LP relaxation at most 3 times. Since computational experiments have shown that lifted flow cover inequalities are the most useful cuts and the hardest to find, in each round at most 30 cuts resulting from the flow covers are added and at most 10 lifted unbounded flow cover cuts are added. We add only globally valid inequalities, i.e. those cuts that are valid at the root node, since CPLEX does not allow the addition of cuts that are valid only at a given node. Since at the root node the LP relaxation yields a global lower bound and it is important to find quickly a good primal solution, at the root node the number of rounds is unlimited but we limit the total number of cuts added at the root node to 300. The cut generation procedure is extremely fast since in all our computational experiments it never required more than a total of 30 seconds or 5% of the overall computational time.

Cuts are added to a pool and each time the cut generation procedure is invoked, the pool is scanned first for cuts. If the number of valid inequalities in the pool exceeds 1,500, the pool is cleaned by selecting at most 500 cuts that were most frequently violated. At every node after the LP relaxation is solved by applying at most 3 rounds of cuts, only the cuts that are in the basis are added to the formulation and the remaining cuts are put in the pool. This strategy prevents the formulation from growing too rapidly.

CPLEX does not allow modification of the formulation at a node and therefore the deletion of nonbasic cuts is not possible. To circumvent this restriction, at each node we copy the node LP into a separate LP and we solve the copied LP with the cut generation procedure. After 3 rounds of adding cuts, we add to the original formulation only those that are in the basis. The copied formulation is then freed. This strategy has a large overhead in copying the node LPs and therefore we chose to time and report only the time needed for solving the LP relaxations throughout the branch-and-bound tree. We observed that in the branch-and-bound algorithm this time represents on average 85% to 95% of the overall computation time, which in addition includes branching, node selection and the application of a primal heuristic, and therefore it is a reasonable estimate of the overall computational time. Since this extra time is proportional to the number of evaluated nodes and in general the number of evaluated nodes is lower for branch-and-cut algorithms, the overall time without the overhead of copying would benefit even further the branch-and-cut algorithm. In all of our computational experiments we set a time limit of 1800 seconds. When both CPLEX and our branch-and-cut algorithm reach the time limit, we compare the gap between the best upper and lower bounds.

6.3 Computational Results

We first generated some random instances of S and random objective functions which are to be maximized over S. The sequences b, a, v, c, d, where c, d are the objective function coefficients with respect to the y and x variables, are generated dependently in the following way. Let $X_i, i = 1, \ldots, 5$ be independent random variables from the uniform distribution on $[0, 1]$. We choose $b = 1000 \cdot X_1$ and $a_i = \lceil b \cdot X_2/10 \rceil, v_i = 1 + \lceil b \cdot X_3/a_i \rceil, c_i = X_4, d_i = -(v_i \cdot X_5 + 10)$ for all $i \in N$. We have generated 3 different types of instances and 5 instances of each type. The first instances all have $F = N$ and v is generated as above. For the second instances a v_i is generated as above with probability 0.75 and is infinity otherwise, so on average $|F| = 3|I|$. The third instances all have $I = N$. For $i \in I$ first v_i and d_i are generated based on the formulas and then v_i is set to infinity. In our experiments $n = 3000$ and $n = 4000$.

The computational results are shown in Table 1 and Table 2, where the CPLEX columns represent the time, the number of evaluated nodes, the value of the best integer solution obtained at the root, the objective value of the root node LP relaxation, the overall gap and the number of cuts generated by the default CPLEX implementation. The overall gap is defined as $100(z^{\mathrm{UB}} - z^{\mathrm{IP}})/z^{\mathrm{UB}}$, where z^{UB} is the best upper bound and z^{IP} is the best IP solution obtained at the end of the computation. The branch-and-cut columns have identical meaning except that they are obtained with our enhanced cut generation routine. The default CPLEX implementation generates various cuts at the root node (mostly Gomory and MIR inequalities), which we call the CPLEX cuts, and the number of generated cuts is reported in the 'no. cuts' column. The first number in 'no. cuts' under 'branch-and-cut' shows the number of CPLEX generated cuts and the second number is the number of cuts generated by our routine in our implementation of branch-and-cut. In the bottom row we show the average improvement of the branch-and-cut algorithm over CPLEX in terms of computational time, number of nodes, the value of the IP solution at the root node, and the value of the LP relaxation at the root node.

Easy instances, defined by the branch-and-bound execution time being less than a minute, are not included since the cuts do not help. For most instances we were able to generate many cuts and the cuts substantially reduce the number of evaluated nodes. The instances with finite upper bounds are harder than the instances with $I = N$. This is due to the fact that the former instances have more rows, i.e. those corresponding to the upper bounds on x, and therefore solutions of the LP relaxations tend to be more fractional. The largest improvement of branch-and-cut over CPLEX is seen in instances with $I = N$. In this case we have the richest set of cuts and they are easy to find and therefore this behavior is expected. On average the time improvement is 48.5% and the average reduction in the number of nodes is approximately 64%. For many instances CPLEX was able to find a better integer solution at the root node. Nevertheless, on average the integer solutions obtained at the root node by branch-and-cut are slightly better than those obtained by CPLEX. Our cuts do not improve the objective value of the root node LP relaxation substantially (on average by 0.08% and 0.04%), which indicates that not many cuts are found at the root node.

	CPLEX						branch-and-cut					
	time	no. nodes	root IP	root LP	gap %	no. cuts	time	no. nodes	root IP	root LP	gap %	no. cuts
$F = N$	39	800	862	871	0	4	5	33	864	870	0	3+87
	102	2340	730	737	0	10	84	1304	730	737	0	9+65
	† 231	2240	341	348	0	5	326	2306	340	348	0	6+1227
	61	679	521	523	0	4	46	644	520	523	0	4+244
	414	13294	299	303	0	9	72	440	298	302	0	11+207
$\|F\| \approx 3\|I\|$	114	898	264	266	0	4	90	489	260	265	0	2+869
	1082	14990	130	136	0	6	23	177	133	135	0	2+333
	† 1800	14473	772	783	.12	4	1800	4000	777	783	.16	2+4493
	† 393	7729	552	559	0	5	1550	1788	552	558	0	8+2643
	1800	32252	344	349	.37	8	90	731	345	348	0	4+1101
$I = N$	165	3775	452	458	0	6	28	262	452	457	0	4+437
	59	535	391	393	0	4	15	160	389	393	0	2+291
	36	291	348	352	0	3	10	17	351	352	0	1+140
	77	476	189	193	0	4	56	124	189	193	0	2+462
	154	718	418	423	0	5	40	211	420	423	0	2+422
Improvement							44%	69%	0.13%	0.08%		

Table 1: Computational results for P with $n = 3000$

For instances denoted by † CPLEX outperforms the branch-and-cut algorithm because either

- CPLEX found improved primal solutions quicker than branch-and-cut or
- branch-and-cut produced too many cuts.

Cuts reduce the number of nodes but the LP relaxations may be much more difficult and therefore the overall computational time can be greater. To understand the value of a good primal solution, we investigated running time under the assumption that the optimal solution value is known. We have taken the instances denoted by † and we have imposed the pruning upper bound of $z^{\mathrm{IP}} + \epsilon$, where ϵ is a small number, i.e. every node with the value of the LP relaxation above this value is pruned. Note that this substantially reduces the effect of a primal heuristic. The results are given in Table 3, where the last instance is another difficult instance not shown in the previous two tables. The column 'idx. opt. solution' shows the index of the

	CPLEX						branch-and-cut					
	time	no. nodes	root IP	root LP	gap %	no. cuts	time	no. nodes	root IP	root LP	gap %	no. cuts
	† 278	4719	281	282	0	2	615	3302	281	282	0	3+1126
	174	2499	446	451	0	5	40	558	446	451	0	6+215
	745	8985	386	392	0	6	524	4771	386	392	0	6+1187
$F = N$	† 1800	26343	317	320	0.12%	5	1800	12000	317	320	0.33%	5+4295
	1445	7869	655	669	0	6	401	4170	663	669	0	5+1055
	239	1962	233	237	0	8	66	272	233	237	0	3+526
	256	2458	822	828	0	5	11	56	825	828	0	2+147
$\|F\| \approx$	340	800	343	345	0	6	230	743	343	345	0	6+330
$3\|I\|$	231	656	263	277	0	9	167	565	263	277	0	9+177
	1261	3970	700	715	0	3	49	376	710	715	0	3+383
	1201	5934	646	650	0	3	76	692	646	650	0	4+878
	50	35	363	366	0	2	48	48	363	366	0	2+45
$I = N$	96	733	308	312	0	4	21	56	308	311	0	2+165
	269	1646	632	641	0	2	42	204	634	640	0	2+349
	53	76	643	652	0	5	6	8	647	652	0	2+41
Improvement							53%	59%	0.25%	0.04%		

Table 2: Computational results for P with $n = 4000$

branch-and-bound node where the optimal solution is found. We see that in proving the optimality branch-and-cut outperforms CPLEX in 4 instances but however there are still 2 instances where branch-and-cut is inferior. In branch-and-cut the optimal solution is always found towards the end of the execution, which is not the case for CPLEX in some of the instances. Once the optimal solution is found, branch-and-cut terminates quickly. This confirms that having a primal heuristic tailored for branch-and-cut would likely speed up the algorithm.

	CPLEX				branch-and-cut			
problem type	time	no. nodes	idx. opt. solution	no. cuts	time	no. nodes	idx. opt solution	no. cuts
3000/f	50	489	489	5	34	274	219	10+126
3000/m	243	5550	710	5	92	928	927	8+643
3000/m	602	18581	8700	4	534	5277	5276	2+2378
4000/f	† 129	2114	54	2	456	2976	2975	2+1120
4000/f	† 1271	22178	13176	5	1465	12551	8699	6+2105
3000/m	255	5277	5000	5	169	955	840	7+872

Table 3: Computational results with the optimality upper bound

In addition to these instances, we have tried the branch-and-cut algorithm on several capacitated facility location instances, where more than one facility is allowed to be opened, i.e. $x_i \leq v_i$, where the integer variable x_i represents the number of facilities opened at location i. Note that these are minimization problems. These random instances were generated by Aardal (1998) and are also used in the computational experiments performed by Gu, Nemhauser and Savelsbergh (1999). We have generated random upper bounds v_i using the formula given above and as above we generated three types of instances. The problem sizes are reported in Table 4. The first number in the name is the number of clients and the second number is the number of facilities. Problems with fewer than 75 facilities were easy to solve and are not considered in this study. These problems have a single row that together with the variable upper bound constraints correspond to $P_=$ and therefore cuts are generated from the P relaxation and from the $P_\geq$ relaxation based on Proposition 7. Namely, for every violated valid inequality for P the corresponding valid inequality for $P_\geq$ is violated as well and therefore both are added to the LP relaxation.

name	# vars	# int. vars	# rows	# nonzeros
10_200	2400	200	2411	8800
100_75	7650	75	7751	30300
75_75	5775	75	5851	22800
100_125	12750	125	12851	50500
125_100	12700	100	12826	50400

Table 4: Facility location problem sizes

The computational results are presented in Table 5, where the gaps are defined accordingly for minimization problems. Our implementation was able to find many cuts and for many problems the branch-and-cut implementation outperforms CPLEX. Due to a substantially lower number of integer variables the number of cuts found in these instances is lower. The problems where CPLEX performs better are denoted by †. The results for the facility location problems are not as good as those presented in Table 1 and Table 2. This is expected since the cuts do not necessarily yield a facet for the capacitated facility location problem and often the cut generation procedure did not find any cuts at the top of the tree, which is confirmed by an average gap of 3% at the root node. Similar to the $P_\leq$ computational experiments, the problems with finite upper bounds are harder.

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		CPLEX						branch-and-cut					
	name	time	no. nodes	root IP	root LP	gap %	no. cuts	time	no. nodes	root IP	root LP	gap %	no. cuts
$F = N$	10_200	1800	103068	32524	30767	.31	10	275	5691	32524	30767	0	9+300
	100_75	† 407	894	15035	14247	0	4	1800	1500	14754	14307	1.24	2+1295
	75_75	† 1088	6041	13671	12934	0	3	1299	5902	13671	12934	0	3+259
	75_75	1231	3373	13318	12764	0	43	1230	2985	13154	12764	0	45+212
	100_125	† 1363	2902	21989	21408	0	18	1800	2500	21989	21408	.50	17+202
$\|F\| \approx 3\|I\|$	100_75	1800	2708	15504	15144	.07	7	1190	2806	15801	15144	0	6+96
	100_125	† 1800	1877	21044	20676	.55	10	1800	1850	21044	20676	1.34	10+10
	125_100	1460	1949	19101	18178	0	5	1144	1020	19354	18183	0	4+122
	125_100	† 321	301	18385	18296	0	3	371	403	18385	18294	0	3+117
	75_75	1800	5459	14998	14024	.69	6	1800	4000	14966	14025	.48	6+873
$I = N$	100_75	347	432	14600	14101	0	5	347	412	14603	14102	0	5+9
	100_75	† 1122	1575	15607	14681	0	5	1507	1983	15608	14681	0	5+50
	75_75	645	2684	13625	13332	0	21	431	1717	13532	13336	0	12+760
	75_75	53	68	14647	13931	0	9	50	67	14647	13931	0	9+9
	75_75	296	807	14475	13775	0	3	290	791	14475	13775	0	3+74
Improvement								2%	10%	.04%	.03%		

Table 5: Computational results for facility location instances