Duality and Existence of Optimal Policies in Generalized Joint Replenishment

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Abstract

We establish a duality theory for a broad class of deterministic inventory control problems on continuous spaces that includes the classical joint replenishment problem and inventory routing. Using this theory, we establish the existence of an optimal policy, which has been an open question. We show how a primal-dual pair of infinite dimensional linear programs encode both cyclic and non-cyclic schedules, and provide various results regarding cyclic schedules including an example showing that they need not be optimal.

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1 Introduction

The existence question is, to our knowledge, open, even for the simplest of all joint cost structures. –Federgruen and Zheng (1992)

The joint replenishment problem is one of the oldest, most studied problems in inventory theory, yet until now there has not existed a duality theory for it. It is generally regarded as the most basic extension of the classical economic order quantity (EOQ) model, due to Harris (1915), from a single item to multiple items. As best as we can tell, it was first formally posed by Naddor and Saltzman (1958), but it was likely discussed at least informally much earlier than this. In the classical statement of the problem, items interact only through fixed ordering costs, which include both a major cost if *any* item is replenished, and a minor item-specific cost. Each item is consumed continuously at an item-specific constant, deterministic rate, and incurs a linear carrying cost per unit held in inventory. The problem is to coordinate joint replenishments so as to minimize the long-run time average operating costs, subject to no stockouts.

A substantial number of articles have been written on the problem and its variants since Naddor and Saltzman (1958). Goyal and Satir (1989) review some of this work. Research on the problem continues but has slowed in the last decade or so, primarily due to the fact that the "power-of-two" heuristic of Roundy (1985; 1986) performs provably within 2% or 6% of optimality, which is close enough for many researchers to consider the problem as "solved." However, in recent years, there has been a flurry of research on inventory routing problems, see Adelman (2003) and references therein, which entails traveling salesman costs instead of major/minor costs. Such problems possess other problem features as well, such as constraints on delivery quantities arising from vehicle capacities, so that previous work on the joint replenishment problem does not carry over naturally. Yet the underlying structure of the inventory routing problem, in terms of item interaction through shared fixed costs, is the same as in the joint replenishment problem. The problem we consider here, which we call the *generalized joint replenishment problem*, includes as special cases both the inventory routing problem and the classical joint replenishment problem.

Rather than considering the general multi-item problem on infinite sequences of replenishments, all authors, excluding Adelman (2003) and Sun (2004), restrict attention to a class of policies known as *cyclic schedules* (actually some subclass of these), which repeat a finite cycle of replenishments continuously through time. The fundamental question of whether there exists an optimal policy is rarely stated in the literature, with the notable exception of Federgruen and Zheng (1992) quoted above and Schwarz (1973). By concatenating an infinite series of finite horizon policies, Hassin and Megiddo (1991) show the existence of an optimal policy for a deterministic single-item inventory problem on continuous spaces. Recently, Sun (2004) extended this idea to the unconstrained multi-item, multi-stage setting. These are the only papers we are aware of that address existence questions in related settings. We resolve the existence question using the powerful and elegant machinery of infinite linear programming duality (Anderson and Nash, 1987). Using this approach, we can accomodate constraints on replenishment quantities, which are essential in real-world applications such as inventory routing.

Our duality results are important not only because they lead to a resolution of the existence question. They also provide, at least theoretically, a way to verify whether a given policy, or cyclic schedule, is optimal. Such a *certificate of optimality* has been missing in the inventory literature, and is essential if optimal control policies are ever to be identified. Whereas previous models in the literature yield bounds on optimal cost, our models are the first to provide the exact optimal cost. In the context of inventory routing without holding costs, Adelman (2003) reports significant progress in approximating an optimal policy using the infinite linear programs discussed herein, and future work will extend these methods to the more general setting discussed here. Having a complete duality theory will enable future researchers to not only better understand problems in this arena, but also to create brand new classes of math programming solution algorithms to solve them.

Specifically, we make three central contributions:

- We provide a new formulation of the generalized joint replenishment problem as a semi-Markov decision process on continuous spaces, which extends the model of Adelman (2003) to include holding costs.
- We provide a primal/dual pair of infinite linear programs for generalized joint replenishment and show that strong duality exists between them. We show how these primal/dual programs encode both cyclic and non-cyclic replenishment sequences.
- We prove the existence of an optimal stationary, deterministic policy.

Along the way, we provide the following new results:

- We provide an example showing that cyclic schedules need not be optimal.
- We show that cyclic schedules are ϵ-optimal, for every $\epsilon > 0$.
- We show that the generalized joint replenishment problem can be posed on compact spaces, without loss of optimality.
- We generalize the classical economic order quantity result stating that an optimal policy sets time-average holding cost equal to time-average fixed ordering cost.

This paper applies the general theory for stochastic semi-Markov decision processes developed in a companion paper, Klabjan and Adelman (2003). There we give a set of assumptions which, if satisfied, ensures strong duality and the existence of an optimal policy. This general approach to existence questions is not new, dating back to at least Fox (1966) (also see Hernández-Lerma and Lasserre (1996, 1999)). However, it has not been applied to the broad class of inventory problems we consider here because the established tradition has always been to consider only cyclic schedules having specially imposed structures, such as power-of-two policies. Consequently, such problems have never before been formulated in the framework of semi-Markov decision processes. Furthermore, once having formulated them in this way, it turns out that even the most general mathematical conditions currently available to make this approach work are violated. In Klabjan and Adelman (2003) we resolve this predicament by giving a new set of conditions, which we apply here.

In order to make this theory more easily applicable on other practical problems and to make our approach transparent, we first consider a general, deterministic semi-Markov decision process (SMDP). For this problem we provide a simpler set of assumptions and infinite linear programs than Klabjan and Adelman (2003). We then formulate the generalized joint replenishment problem as a deterministic SMDP and show that it satisfies these assumptions.

1.1 Duality and the Classical EOQ Problem

To illustrate how linear programming duality can resolve the existence question, we give a simple primal/dual pair of programs for the classical EOQ problem. These programs are special cases of the much more general programs that follow.

Suppose a single item of inventory is consumed at a constant, deterministic rate λ and incurs a per-unit per-time holding cost of h. It costs C to replenish, independently of the replenishment quantity. The problem is to find a replenishment policy that minimizes the long-run time average costs subject to no stockouts. Using simple calculus, it is easily seen that an optimal policy exists: replenish quantity $\sqrt{2\lambda C/h}$ whenever there is a stockout.

Let A denote a Borel space of permissible order quantities, for example $A = \mathbb{R}_+$ or $A = [0, \overline{A}]$ for some upper bound $\overline{A} < \infty$ on replenishment quantities. Consider now the following linear semi-infinite program with a single decision variable ρ, representing the long-run time average cost, but an uncountable number of constraints:

$$\begin{aligned} &\sup \rho \\ &\rho a/\lambda \leq C + (h/2\lambda)a^2 \qquad a \in A. \end{aligned}$$

This inequality says that if quantity a is replenished whenever there is a stockout, then the total cost incurred over a cycle of length a/λ, if it is accumulated at the long-run time average rate ρ, can be no larger than the actual total cost of any cycle. Rearranging terms, the optimal value is

$$\rho^* = \inf_{a \in A} \{C\lambda/a + ha/2\}$$

and the inequality is tight at $a^* = \sqrt{2\lambda C/h}$ when $A = \mathbb{R}_+$, which is the standard EOQ formula.

Now consider the dual program. The decision variable is a finite measure μ defined on the Borel space A. Letting $\mathcal{B}(A)$ be the Borel subsets of A, for any $\mathcal{A} \in \mathcal{B}(A)$, $\mu(\mathcal{A})$ represents the replenishment rate of quantities $\mathcal{A}$.

$$\begin{aligned} \inf \int_{a \in A} \left(C + (h/2\lambda)a^2\right) \mu(\mathrm{d}a) & \\ \int_{a \in A} (a/\lambda)\mu(\mathrm{d}a) &= 1 \\ \mu &\geq 0 \\ \mu(A) &< \infty \end{aligned}$$

Carrying λ onto the right-hand side of the equality constraint, we see that this constraint says replenishment must equal consumption. For the EOQ problem, this formulation is overly complex, but its generalization to multiple items is essential.

Consider the solution

$$\mu^*(\mathcal{A}) = \begin{cases} \lambda/a^* & \text{if } a^* \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \qquad \mathcal{A} \in \mathcal{B}(A),$$

which corresponds to the Dirac's measure concentrated on a^*. It is easy to see that this solution is feasible, yields objective value ρ^*, and satisfies complementary slackness when A is compact. Hence we see that the existence of an optimal primal/dual solution pair that satisfies strong duality yields a control policy and a proof that is optimal. The same idea holds for the generalized joint replenishment problem, although many additional complications arise which we address.

1.2 Outline

In Section 2 we formally define the generalized joint replenishment problem. In Section 3 we pose a general, deterministic SMDP, formulate infinite linear programs for it, and provide a set of assumptions under which there is strong duality and an optimal policy exists. Then in Section 4 we formulate the generalized joint replenishment problem as a deterministic SMDP, and verify that the assumptions are satisfied. Finally, in Section 5 we discuss how cyclic schedules are encoded by our infinite linear programs, and provide various related results.

2 Problem Description

A controller continuously monitors inventories for a finite set of items $\mathcal{I}$. An item may represent a product, a location, or a product-location pair. The inventory of each item $i \in \mathcal{I}$ is infinitely divisible, is consumed at a constant deterministic rate of $0 < \lambda_i < \infty$, and costs the firm $0 \leq h_i < \infty$ per unit per time to hold. It also cannot exceed a maximum allowable inventory level of $0 < \overline{X}_i \leq \infty$. For each i, to avoid degenerate cases, we assume that either $h_i > 0$ or $\overline{X}_i < \infty$ (or both). As inventories continuously deplete, the controller may at any time replenish a subset $I \subseteq \mathcal{I}$ of items, which incurs an ordering cost of $0 < C_I < \infty$ and is completed instantaneously. Without loss of generality, we assume $C_{I_1} \leq C_{I_2}$ if $I_1 \subseteq I_2$, since otherwise the controller can replenish I_1 by executing I_2 without replenishing items $I_2 \setminus I_1$. Although we can accommodate different item sizes, we assume for simplicity that all demands and inventories are measured in the same units, e.g. liters, and that no more than $0 < \overline{A} \leq \infty$ total units can be replenished across all items in a single replenishment. The controller's problem is to minimize the long-run time average cost, subject to allowing no stockouts.

It is useful at this point to indicate how this problem generalizes others in the literature. The literature is far too large to include everything, and so we select a representative subset and direct the reader to the literature reviews contained in these works. Zipkin (2000) is also an excellent resource. Table 1 is self-explanatory.

Table 1: Comparison of models in the literature as special cases.

	Roundy (1985) Roundy (1986)	Rosenblatt and Kaspi (1985) Queyranne (1987)	Federgruen and Zheng (1992)
$\overline{A}$	∞	∞	∞
$\overline{X}_i$	∞	∞	∞
h_i	> 0	> 0	> 0
C_I	major/minor	general	submodular
Heuristic	power-of-two	fixed partition	power-of-two

	Anily and Federgruen (1990) Bramel and Simchi-Levi (1995) Chan *et al.* (1998)	Adelman (2003)
$\overline{A}$	$< \infty$	$< \infty$
$\overline{X}_i$	∞	$< \infty$
h_i	> 0	$= 0$
C_I	traveling salesman	general/traveling salesman
Heuristic	partition	price-directed

Suppose quantity a_i is replenished of item i when its inventory level is x_i. We assign all future holding cost that results to the current replenishment. Consequently, as shown in Figure 1, the inventory holding cost associated with x_i is sunk, because it is assigned to the previous replenishments. The delivery of a_i moves the inventory level to $x_i + a_i$ and incurs additional holding cost. The area of the shaded region between the lines is $\frac{(a_i+x_i)^2}{2\lambda_i} - \frac{x_i^2}{2\lambda_i}$, or $(1/2\lambda_i)(a_i^2 + 2a_i x_i)$. Therefore, for every $(x, a) \in \mathbb{R}^{|\mathcal{I}|} \times \mathbb{R}^{|\mathcal{I}|}$ the cost of replenishment vector a is the sum of fixed ordering costs and holding costs, i.e.

$$c(x, a) = C_{\mathrm{supp}(a)} + \sum_{i \in \mathcal{I}} \frac{h_i}{2\lambda_i} (2a_i x_i + a_i^2) \;, \tag{1}$$

where we denote by $\mathrm{supp}(a)$ the support set of a.

The problem is to find an infinite sequence of replenishments $\{(x_n, a_n, t_n)\}_{n=0,1,\ldots}$, where x_n and a_n denote the vectors of item inventory levels and replenishment quantities respectively, at decision epoch n, and t_n represents the elapsed time between replenishments n and $n+1$. The notation $x_{i,n}$ and $a_{i,n}$ denotes the inventory level and replenishment quantity, respectively, of item i on replenishment n. Given a fixed initial inventory state $x_0 = x$, the

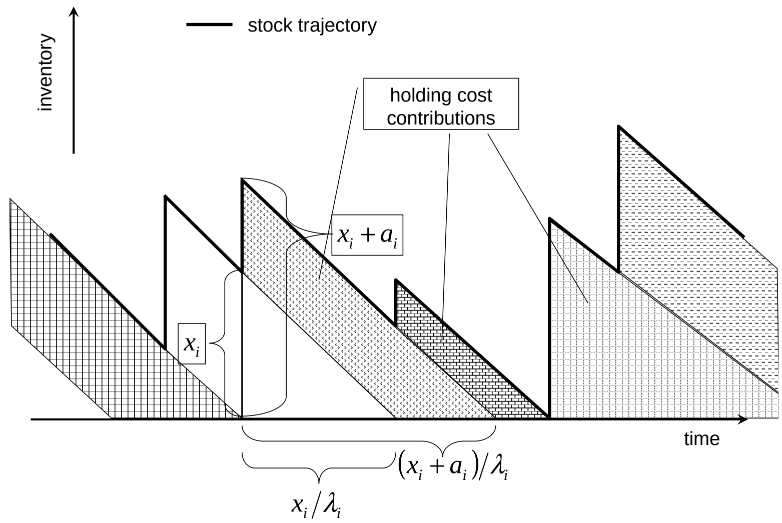

Figure 1: Accounting for inventory holding cost.

control problem can be formulated as

$$J^*(x) = \inf \limsup_{N\to\infty} \frac{\sum_{n=0}^{N} c(x_n, a_n)}{\sum_{n=0}^{N} t_n} \tag{2a}$$

$$x_{n+1} = x_n + a_n - \lambda t_n \quad n \in \mathbb{Z}_+ \tag{2b}$$

$$x_n + a_n \leq \overline{X} \quad n \in \mathbb{Z}_+ \tag{2c}$$

$$\sum_{i\in\mathcal{I}} a_{i,n} \leq \overline{A} \quad n \in \mathbb{Z}_+ \tag{2d}$$

$$x_0 = x \tag{2e}$$

$$x, a, t \geq 0, \tag{2f}$$

where $\mathbb{Z}_+ = \{0, 1, \dots\}$ and $\lambda = (\lambda_1, \dots, \lambda_{|\mathcal{I}|})$. Constraints (2b) maintain inventory flow balance, constraints (2c) ensure that the storage limits $\overline{X}_i$ are not violated, and constraints (2d) ensure that no replenishment delivers more than $\overline{A}$ in total across all items. The objective function minimizes the lim sup of the long-run time average cost.

The central existence result of this paper is the following theorem.

Theorem 1. *There exists a function $f(\cdot)$ and a constant J^* such that for all initial feasible inventory states $x_0 = x$, the infimum in (2a) equals $J^*(x) = J^*$ and an optimal sequence*

$\{(x_n^*, a_n^*, t_n^*)\}_{n=0,1,\ldots}$ *that attains* J^* *is given by*

$$\begin{aligned} a_n^* &= f(x_n^*), \\ t_n^* &= \min_{i \in \mathcal{I}} \left\{ \frac{x_{i,n}^* + a_{i,n}^*}{\lambda_i} \right\}, \text{ and} \\ x_{n+1}^* &= x_n^* + a_n^* - \lambda t_n^* \end{aligned} \tag{3}$$

for all $n \in \mathbb{Z}_+$.

3 A General Deterministic SMDP

3.1 Formulation

Consider a deterministic SMDP defined on a state space X and action space A, both assumed to be Borel spaces. For each $x \in X$, let $A(x) \subseteq A$ be a non-empty Borel subset that specifies the set of admissible actions from state x. We denote the collection of state-action pairs as $K = \{(x, a) : x \in X, a \in A(x)\}$, assumed to be a Borel subset of $X \times A$. Upon taking action a in state x, a cost $c(x, a)$ is incurred and then the system transitions to some state $s(x, a)$ after a time duration of length $\tau(x, a)$, all with probability one. We assume that $c : K \to \mathbb{R}$, $s : K \to X$, and $\tau : K \to [0, \infty)$ are measurable on K. Let $\{x_n, a_n, t_n\}_{n=0,1,\ldots} \in (K \times [0, \infty))^\infty$ denote any infinite sequence of state-action pairs and transition times. Suppose $f : X \to A$ is a measurable decision function that specifies for every $x \in X$ some action $a \in A(x)$. Define the long-run average cost of the system under control f, starting from an initial state $x_0 \in X$, as

$$J(f, x_0) = \limsup_{N \to \infty} \frac{\sum_{n=0}^N c(x_n, f(x_n))}{\sum_{n=0}^N t_n}.$$

The problem

$$J(x_0) = \inf_{f : X \to A} J(f, x_0) \tag{4}$$

finds an optimal decision rule f^* from starting state x_0. One of the main questions in Markov control processes is under what conditions does there exist an f^* such that $J^* = J(f^*, x_0) = J(x_0)$ for every $x_0 \in X$? Such a decision rule is said to be *long-run time average optimal*, in the class of stationary deterministic decision rules, from every starting state.

More generally, rather than restricting the class of policies to deterministic decision rules $f : X \to A$, we could pose the existence question over all admissible, non-anticipatory policies $\pi \in \Pi$, including randomized history-dependent ones. It follows from Klabjan and Adelman (2003) that all of our existence results hold even when the class of admissible policies is allowed to be all of Π, i.e. stationary deterministic policies still suffice for optimality. However, to develop the theory here in full generality would require notation that is unnecessarily complex, for instance we would need to carry expectations in an otherwise deterministic setting. So for ease of exposition we restrict discussion to policies based on stationary deterministic decision rules f.

Given a Borel space S let $\mathbb{B}(S)$ be the Banach space of bounded measurable functions u, i.e. having finite norm

$$\|u\| = \sup_S |u(s)| \,.$$

In addition, let $\mathbb{M}(S)$ be the Banach space of signed measures μ on the Borel space on S with finite *total variation norm*

$$\|\mu\|^{\mathrm{TV}} = \sup_{\|u\|\le 1} \left| \int_S u \,\mathrm{d}\mu \right| .$$

Let $\mathcal{B}(S)$ be the Borel σ-algebra on S and let $\mathbb{C}_b(S)$ be the set of all continuous, bounded functions on S.

The *average cost optimality equation* is

$$u(x) = \inf_{a\in A(x)} \{c(x,a) - \rho\tau(x,a) + u(s(x,a))\} \quad \text{for every } x \in X, \tag{5}$$

where $\rho \in \mathbb{R}$ and $u \in \mathbb{B}(X)$. The constant ρ is the optimal loss, whereas $u(x)$ is the bias function and reflects transient costs starting from state x. If (5) is solvable, then it shows the existence of an optimal policy.

3.2 Infinite Linear Programming Theory

Rather than work with the optimality equation (5) directly, we instead reformulate it as an infinite-dimensional linear program. We make the following assumptions.

Assumption B1. $c \in \mathbb{B}(K)$ *and nonnegative.*

Assumption B2. $\tau \in \mathbb{B}(K)$ *and nonnegative.*

Now consider the following primal/dual linear programs on the spaces $(\mathbb{M}(K), \mathbb{B}(K))$, $(\mathbb{R} \times \mathbb{M}(X), \mathbb{R} \times \mathbb{B}(X))$. The primal problem is

$$\inf \int_K c(x,a)\mu(\mathrm{d}(x,a)) \tag{6a}$$

$$\int_K \tau(x,a)\mu(\mathrm{d}(x,a)) = 1 \tag{6b}$$

$$\mu((B\times A)\cap K) - \mu(\{(x,a)\in K : s(x,a)\in B\}) = 0 \qquad \text{for every } B \in \mathcal{B}(X) \tag{6c}$$

$$\mu \ge 0, \mu \in \mathbb{M}(K) \tag{6d}$$

and the corresponding dual problem reads

$$\sup \rho \tag{7a}$$

$$\tau(x,a)\rho + u(x) - u(s(x,a)) \le c(x,a) \qquad \text{for every } (x,a) \in K \tag{7b}$$

$$\rho \in \mathbb{R}, u \in \mathbb{B}(X) \,. \tag{7c}$$

We denote by $\inf(P)$ and $\sup(D)$ the optimal values of the primal and dual programs, respectively.

Consider the following operators. Let $L_0 : \mathbb{M}(K) \longrightarrow \mathbb{M}(X)$ be defined as

$$(L_0\mu)(B) = \mu((B \times A) \cap K) - \mu(\{(x, a) \in K : s(x, a) \in B\}) \qquad \text{for every } B \in \mathcal{B}(X)$$

and let $L : \mathbb{M}(K) \to \mathbb{R} \times \mathbb{M}(X)$ be

$$L\mu = \left(\int_K \tau(x, a)\mu(d(x, a)), L_0\mu \right).$$

To see that the integral $\int_K \tau(x, a)\mu(d(x, a))$ is finite, let $\bar{\tau} = \sup_{(x,a)\in K} \tau(x, a) < \infty$ as $\tau \in \mathbb{B}(K)$ from Assumption B2. Since, $\mu \in \mathbb{M}(K)$, then $\int_K \frac{\tau(x,a)}{\bar{\tau}} \mu(d(x, a)) \leq \|\mu\|^{\mathrm{TV}} < \infty$. The adjoint operator $L^* : \mathbb{R} \times \mathbb{B}(X) \to \mathbb{B}(K)$ is given by

$$L^*(\rho, u)(x, a) = \tau(x, a)\rho + u(x) - u(s(x, a)).$$

Again by Assumption B2, $\tau(x, a)\rho \in \mathbb{B}(K)$, and the remaining terms trivially are also. It follows from Anderson and Nash (1987)[pp. 35-40] that (7) is a dual linear program to (6).

Next we provide a set of assumptions under which strong duality holds between these two programs.

Assumption B3. *$c(x, a) + \tau(x, a) \geq 1$ for every $(x, a) \in K$.*

Here the right-hand side can be changed to any $\epsilon > 0$, but we normalize to 1 for convenience.

Assumption B4. *$\tau \in \mathbb{C}_b(K)$.*

Assumption B5. *c is lower semi-continuous.*

Assumption B6. *$\{a \in A(x) : c(x, a) + \tau(x, a) \leq r\}$ is compact for every $x \in X, r \in \mathbb{R}$.*

Assumption B7. *There exists a decision rule $f : X \to A$ and initial state $x_0 \in X$ such that $J(f, x_0) < \infty$.*

Assumption B8. *s is continuous on K.*

Assumption B9. *K is compact.*

The following assumption says that all states communicate with bounded cost and time.

Assumption B10. *There exist constants $C < \infty$, $\Gamma < \infty$ such that for every measurable subset $S \subseteq X$ there is a decision rule $f : X \setminus S \to A$ with the property that for every $x' \in X \setminus S$ there exists a finite integer N and a set of states $x_0, x_1, \ldots, x_N$ with*

- *$x_0 = x'$,*
- *$a_n = f(x_n) \in A(x_n)$ for every $n = 0, \ldots, N - 1$,*

- $x_{n+1} = s(x_n, a_n)$ *for every* $n = 0, \dots, N-1$,
- $x_N \in S$,
- $\sum_{n=0}^{N-1} c(x_n, a_n) \le C$, *and*
- $\sum_{n=0}^{N-1} \tau(x_n, a_n) \le \Gamma$.

The infinite linear programs in Klabjan and Adelman (2003) are defined on spaces endowed with more general norms than here. Therefore, to apply their results we need to show that under the assumptions made here our spaces contain the same elements. Given a Borel space Z and a measurable weight function $f \ge 1$, let $\mathbb{B}_f(Z)$ be the Banach space of measurable functions u with finite *f-norm*

$$\|u\|_f = \sup_Z \frac{|u(s)|}{|f(s)|} .$$

In addition, let $\mathbb{M}_f(Z)$ be the Banach space of signed measures μ on the Borel space on Z with finite f *total variation norm*

$$\|\mu\|_f^{\mathrm{TV}} = \sup_{\|u\|_f \le 1} |\int_Z u \, \mathrm{d}\mu| .$$

The total variation norm of μ from above is $\|\mu\|^{\mathrm{TV}} = \|\mu\|_1^{\mathrm{TV}}$.

Lemma 1. *Given a Borel space* Z *and a measurable weight function* $f \in \mathbb{B}(Z)$ *such that* $f \ge 1$, *then* $\mathbb{B}_f(Z) = \mathbb{B}(Z)$ *and* $\mathbb{M}_f(Z) = \mathbb{M}(Z)$.

Proof. Suppose $u \in \mathbb{B}_f(Z)$. Then because there exists an $\bar{f} < \infty$ such that $|f(z)| < \bar{f}$,

$$\frac{\|u\|}{\bar{f}} = \sup_{z \in Z} \frac{|u(z)|}{\bar{f}} \le \sup_{z \in Z} \frac{|u(z)|}{|f(z)|} = \|u\|_f < \infty$$

and so $u \in \mathbb{B}(Z)$. On the other hand, for $u \in \mathbb{B}(Z)$, because $f \ge 1$,

$$\|u\|_f = \sup_{z \in Z} \frac{|u(z)|}{|f(z)|} \le \sup_{z \in Z} |u(z)| = \|u\| < \infty.$$

For all measurable $u : Z \to \mathbb{R}$, replacing the infinities above with 1 it is easy to see that if $\|u\| \le 1$, then $\|u\|_f \le 1$. Conversely, if $\|u\|_f \le 1$, then $\|u\| \le \bar{f}$. Thus if $\mu \in \mathbb{M}_f(Z)$ then

$$\|\mu\|^{\mathrm{TV}} = \sup_{\|u\| \le 1} |\int_Z u d\mu| \le \sup_{\|u\|_f \le 1} |\int_Z u d\mu| = \|\mu\|_f^{\mathrm{TV}} < \infty$$

and therefore $\mu \in \mathbb{M}(Z)$. On the other hand, if $\mu \in \mathbb{M}(Z)$ then

$$\|\mu\|_f^{\mathrm{TV}} = \sup_{\|u\|_f \le 1} |\int_Z u d\mu| \le \sup_{\|u\| \le \bar{f}} |\int_Z u d\mu| = \bar{f} \sup_{\|\bar{u}\| \le 1} |\int_Z \bar{u} d\mu| < \infty,$$

where the last equality follows from $\bar{u} = u/\bar{f}$. So $\mu \in \mathbb{M}_f(Z)$. $\square$

Let $w : K \longrightarrow \mathbb{R}, w_0(x) : X \longrightarrow \mathbb{R}$ be defined as

$$w(x,a) = \tau(x,a) + c(x,a) \tag{8}$$
$$w_0(x) = \inf_{a \in A(x)} w(x,a) \,. \tag{9}$$

From Assumptions B4, B5, and B6, w_0 is well-defined, the infimum can be replaced by the minimum, and, in addition, w_0 is measurable, Rieder (1978). It then follows from Assumptions B1, B2, B3, and Lemma 1 that $\mathbb{B}_w(K) = \mathbb{B}(K)$ and $\mathbb{B}_{w_0}(X) = \mathbb{B}(X)$, and also $\mathbb{M}_w(K) = \mathbb{M}(K)$ and $\mathbb{M}_{w_0}(X) = \mathbb{M}(X)$. Consequently, under our assumptions the primal/dual pair (6) and (7) is a special case of the one in Klabjan and Adelman (2003).

Lemma 2. *Assumptions B1–B10 imply Assumptions A1-A9 in Klabjan and Adelman (2003).*

Proof. Assumptions B4–B6 imply Assumption A1. Assumptions B1, B2, B4, and B5 imply A2. Assumption B3 is the same as Assumption A3. The condition in Assumption A4 reduces here to requiring the existence of some finite k such that $w_0(s(x,a)) \leq kw(x,a)$ for all $(x,a) \in K$. From Assumptions B1 and B2, we have

$$w_0(s(x,a)) = \inf_{a' \in A(s(x,a))} w(s(x,a), a') \leq \bar{\tau} + \bar{C} \leq (\bar{\tau} + \bar{C}) w(x,a)$$

from Assumption B3, which implies Assumption A4. Assumption B7 implies Assumption A5. Since the kernel is Dirac's, we have that Assumption A6, i.e. weak continuity of the kernel, is equivalent to s continuous, which is Assumption B8. Assumption B4 and B2 are the same as Assumption A7. Assumption B9 implies Assumption A8, because in the definition of strict unboundedness from Klabjan and Adelman (2003) with $S = K$, we can take $S_n = K$ for every n and the infimum over an empty set is infinity. Lastly, Assumption B10 is the same as Assumption A9. □

The next two theorems follow directly from Theorems 9 and 10 of Klabjan and Adelman (2003), and Lemmas 1 and 2 above.

Theorem 2. (Strong duality) *Under assumptions B1–B10, there exists an optimal primal/dual solution pair $(\mu^*, (\rho^*, u^*)) \in (\mathbb{M}(K), (\mathbb{R}, \mathbb{B}(X)))$ such that $\inf(P) = \sup(D)$ and complementary slackness holds, i.e. for μ^*-almost all $(x,a) \in K$ we have*

$$\tau(x,a)\rho^* + u^*(x) = c(x,a) + u^*(s(x,a)). \tag{10}$$

It is important to observe that (10) does not solve the optimality equation (5) everywhere, but Assumption B10 will give us a control policy for states elsewhere.

Theorem 3. (General existence result) *Under assumptions B1–B10, there exists a decision rule $f^* : X \to A$ such that*

$$J(x) = J(f^*, x) = J^* \quad \textit{for all } x \in X.$$

Proof. Here we sketch the proof. The detailed proof is given in Klabjan and Adelman (2003). Let $(\mu^*, (\rho^*, u^*))$ be defined as in Theorem 2. Let

$$L = \{x \in X : \text{ there exists a trajectory with } x_0 = x \text{ and all state-action pairs } (x_n, a_n) \text{ satisfy (10)}\}.$$

The key fact is to show that $L \neq \emptyset$.

Let the reduced cost of a state-action pair (x, a) be defined as $c(x, a) + u^*(s(x, a)) - \tau(x, a)\rho^* - u^*(x)$. By dual feasibility the reduced cost is nonnegative and by Theorem 2 it is 0 for μ^*-almost all $(x, a) \in K$. For any trajectory ω and $n \in \mathbb{N}$ we define $r_n(\omega)$ to be the sum of the reduced costs of the first n state-action pairs in ω and let $r(\omega)$ be the sum of the reduced costs of all the state-action pairs in ω. Thus by definition $r(\omega) = \lim_{n\to\infty} r_n(\omega)$. For any fixed n by using (6c) it is possible to show that $r_n(\omega)$ is 0 for almost every trajectory ω, under a (possibly randomized) policy derived from μ^*. The monotone convergence theorem implies that $r(\omega)$ is 0 for almost all trajectories. Since $r \geq 0$ and $\mu^* \neq 0$, it follows that there is a trajectory with 0 reduced cost, i.e. there is a trajectory satisfying (10). This clearly implies that $L \neq \emptyset$ and in turn that every $x \in L$ has an action $a \in A(x)$ such that (x, a) has 0 reduced cost.

An optimal policy chooses

$$f^*(x) \in \underset{\{a \in A(x): s(x,a) \in L\}}{\operatorname{argmin}} \{c(x, a) - \tau(x, a)\rho^* + u^*(s(x, a))\} \text{ for all } x \in L,$$

and otherwise lets f^* be as in Assumption B10 with $S = L$. □

4 Existence Proof for Generalized Joint Replenishment

We approach Theorem 1 by formulating (2) as a deterministic semi-Markov decision process that satisfies the assumptions above. We first need to show that without loss of optimality we can consider a discrete set of embedded decision epochs. These correspond with stockout times.

Definition 1. *A sequence $\{(x_n, a_n, t_n)\}_{n=0,1,\dots}$ is said to satisfy the* **just-in-time** *property if for every $n = 0, 1, \dots$ there exists an $i \in \mathcal{I}$ such that $x_{i,n} = 0$ and $i \in supp(a_n)$.*

An implication of this property is that $t_n = \min_{i\in\mathcal{I}} \left\{\frac{x_{i,n}+a_{i,n}}{\lambda_i}\right\}$ for all n.

Lemma 3. *Any feasible sequence $\{(x_n, a_n, t_n)\}_{n=0,1,\dots}$ for (2) that violates the just-in-time property can be transformed into one that satisfies it without increasing the objective value.*

Proof. Identical to Adelman (2003), except note that here c is given by (1) and includes holding costs. The costs $c(x_n, a_n)$ for all intervening dispatches before time τ are no greater than before because the inventories are not, and the costs for dispatches after time τ are the same. □

Note that the same proof works for any $c(x,a)$ that is non-decreasing in a.

Define the state space as the Borel space

$$X = \{x \in \mathbb{R}_+^{|\mathcal{I}|} : \text{there exists } j \in \mathcal{I} \text{ with } x_j = 0, x \le \overline{X}\}.$$

For every state $x \in X$ the action space is the non-empty Borel subset of $A = \{a \in \mathbb{R}_+^{|\mathcal{I}|} : \sum_{i\in\mathcal{I}} a_i \le \overline{A}\}$ defined by

$$A(x) = \{a \in \mathbb{R}_+^{|\mathcal{I}|} : \sum_{i\in\mathcal{I}} a_i \le \overline{A}, x + a \le \overline{X}\}.$$

For all $(x,a) \in K$, the cost of taking action a in state x is $c(x,a)$ given by (1). For every $(x,a) \in K$ define the transition time by

$$\tau(x,a) = \min_{i\in\mathcal{I}} \left\{ \frac{x_i + a_i}{\lambda_i} \right\}, \tag{11}$$

which may equal 0 if not all stocked out items are replenished. The next inventory state is then given by the function

$$s(x,a) = x + a - \lambda\tau(x,a).$$

We first show that if the inventories are bounded, i.e. $\overline{X}_i < \infty$, then Assumption B10 holds. Next we show that without loss of optimality the inventories can be bounded, so that Assumption B9 holds. Subsequently, we verify the remaining assumptions.

When inventories are bounded, Lemma 2 in Adelman (2003) shows that any two states indeed communicate, and shows that the required number of dispatches, amount of time, and amount of cost are finite. However, Assumption B10 requires that these quantities be bounded instead of merely finite. Furthermore, it requires the construction of a decision rule f for states outside of S that drives the system into S. The next lemma achieves both of these requirements.

Lemma 4. *Suppose $0 < \overline{X}_i < \infty$ for every $i \in \mathcal{I}$. Then Assumption B10 holds.*

Proof. Let S be a subset of X. Since $\overline{X}_i < \infty$ for every $i \in \mathcal{I}$, we can assume that $\overline{A} \le \sum_{i\in\mathcal{I}} \overline{X}_i < \infty$. Choose any $x' \in X \setminus S$ and $x'' \in S$. We demonstrate a sequence of replenishments with a bounded number of steps that moves the system from x' to 0 in the first stage, and then in the second stage moves the system from 0 to x''. Decision epochs are at stockout times, with the times between them given by (11).

First stage. Let $\epsilon' = \min\{\min_{i\in\mathcal{I}} \frac{\overline{X}_i}{\lambda_i}, \frac{\overline{A}}{2\sum_{j\in\mathcal{I}}\lambda_j}\} < \infty$. Whenever an item stocks out, for every i replenish quantity

$$a_i = (\lambda_i \epsilon' - x_i)^+, \tag{12}$$

where $(\cdot)^+$ denotes the positive part of the enclosed quantity and x_i is the current inventory level of item i. Our construction of ϵ' ensures that these replenishment quantities are feasible.

Each item stocks out at most once before reaching state 0, and so 0 is reached within a time duration no longer than $\max_{i\in\mathcal{I}} \frac{x_i'}{\lambda_i}$. Within this time frame there can be at most

$$\left\lceil (1/\epsilon') \max_{i\in\mathcal{I}} \frac{x_i'}{\lambda_i} \right\rceil + |\mathcal{I}| \leq \left\lceil (1/\epsilon') \max_{i\in\mathcal{I}} \frac{\overline{X}_i}{\lambda_i} \right\rceil + |\mathcal{I}|$$

dispatches, where $\lceil\cdot\rceil$ denotes the usual ceiling function.

Second stage. Define $\mathcal{I}_+ = \{i \in \mathcal{I} : x_i'' > 0\}$ and $\mathcal{I}_0 = \{i \in \mathcal{I} : x_i'' = 0\}$. Set $\epsilon'' = \min\{\epsilon', \min_{i\in\mathcal{I}_+} \frac{x_i''}{\lambda_i}\}$. Now replenish so that all items $i \in \mathcal{I}_0$ stock out every ϵ'' time units, and the inventories of all other items build until reaching their target. In particular, replenish quantity

$$a_i = \begin{cases} \lambda_i\epsilon'' & i \in \mathcal{I}_0 \\ \min\{\lambda_i\epsilon'' + (\overline{A}/2)\frac{x_i''}{\sum_{j\in\mathcal{I}} x_j''}, x_i'' - x_i\} & i \in \mathcal{I}_+. \end{cases} \tag{13}$$

The inventory level of each item $i \in \mathcal{I}_+$ increases by $(\overline{A}/2)\frac{x_i''}{\sum_{j\in\mathcal{I}} x_j''}$ on each dispatch. Therefore, the inventory state with $x_i = 0$ for $i \in \mathcal{I}_0$ and $x_i = x_i'' - \lambda_i\epsilon''$ for $i \in \mathcal{I}_+$ is reached in at most

$$\left\lceil 2\sum_{j\in\mathcal{I}} x_j''/\overline{A} \right\rceil \leq \left\lceil 2\sum_{j\in\mathcal{I}} \overline{X}_j/\overline{A} \right\rceil$$

dispatches. Then, as a final step, replenish quantity $a_i = \lambda_i\epsilon''$ to each item $i \in \mathcal{I}_+$, otherwise $a_i = 0$, which leads to state x''.

Hence, the number of dispatches N required to move the system from state x' to x'' is bounded. This bound does not depend on x' or x'', rather only on the problem data $\overline{A}$, $\overline{X}_i$, and λ_i. The assumption $0 < \overline{X}_i < \infty$ implies that $c(x, a) \leq \bar{c} < \infty$ and $\tau(x, a) \leq \bar{\tau} < \infty$, where $\bar{c} = \max_{I\subseteq\mathcal{I}} C_I + \sum_{i\in\mathcal{I}} \frac{h_i}{2\lambda_i}(3\overline{X}_i^2)$ and $\bar{\tau} = \max_{i\in\mathcal{I}} \frac{\overline{X}_i}{\lambda_i}$. It follows that $\sum_{n=0}^{N-1} c(x_n, a_n) \leq N\bar{c} < \infty$ and $\sum_{n=0}^{N-1} \tau(x_n, a_n) \leq N\bar{\tau} < \infty$.

Finally, we can construct the function $f : X \setminus S \to A$ as follows. Fix the trajectory in stage 2 above from 0 to x'', and let $\tilde{X}$ be the set of states visited on this trajectory. Then for all states x outside of $S \cup \tilde{X}$, let $f(x)$ be the replenishments given by (12). Otherwise, on states $x \in \tilde{X}$ set f to return the action specified by (13) on the fixed trajectory. □

Proposition 1. *Without loss of optimality, it suffices to consider only trajectories $\{(x_n, a_n, t_n)\}_{n=0,1,...}$ such that, for any item $i \in \mathcal{I}$ having $h_i > 0$ and $\overline{X}_i = \infty$ we have*

$$x_{i,n} + a_{i,n} \leq \max\left\{\sqrt{\frac{8\lambda_i C_{\{i\}}}{h_i}}, \frac{8\lambda_i C_{\{i\}}}{h_i\overline{A}}\right\} < \infty \tag{14}$$

for every n.

Proof. Given any trajectory, we can first modify it into one having the same long-run average cost as the original, but which visits state vector 0 between states x_0 and x_1. The algorithm that does this is described in the proof of Lemma 4, except choose any finite $\epsilon' > 0$ if the ϵ' specified there is infinite. In the rest of the proof, we assume that all trajectories have this property.

For ease of notation, we denote by $X_{\max}$ the right-hand side of (14). Let $\bar{n}$ be the decision epoch where (14) is violated for the first time for item i. We first form a trajectory that satisfies (14) for i and every $n \leq \bar{n}$, by only modifying replenishments for i. Let $N \leq \bar{n}$ be the last replenishment before $\bar{n}$ with $x_{i,N} = 0$. By the first paragraph above such an N exists.

We construct a new trajectory $\{(x'_n, a'_n)\}_{n=0,1,\ldots}$ with cost lower than or equal to the cost of the original trajectory and (14) holds for every $n \leq \bar{n}$. For $n < N$ the two trajectories coincide. We first show that there are $\{(x'_{i,j}, a'_{i,j})\}_{j=N,N+1,\ldots,\bar{n}}$ such that $x'_{i,j} \leq x_{i,j}, a'_{i,j} \leq a_{i,j}$ and $x'_{i,\bar{n}} = 0$. If $N = \bar{n}$, there is nothing to show, otherwise we construct this iteratively. Let $k = \operatorname{argmin}_{N<j\leq\bar{n}} x_{i,j}$, so that $x_{i,k} > 0$. We define new replenishments as

$$a'_{i,j} = \begin{cases} a_{i,j} - x_{i,k} & j = N \\ a_{i,j} & N < j \leq \bar{n}\,. \end{cases}$$

Observe that

$$\min_{N<j\leq\bar{n}} x_{i,j} \leq x_{i,N+1} = a_{i,N} - \lambda_i \tau(x_N, a_N) \leq a_{i,N}$$

and hence $a'_{i,N} \geq 0$. It is easy to see that $x'_{i,j} = x_{i,j} - x_{i,k}$ for $j = N+1, N+2, \ldots, \bar{n}$. Consider now the decision epochs $k, k+1, \ldots, \bar{n}$ with $N < k$ and the new trajectory. The stock out now occurs at k, instead of N. Therefore we repeat the argument until $x'_{i,\bar{n}} = 0$.

Let

$$\begin{aligned} M &= \max\left\{2, \left\lceil \frac{x_{i,\bar{n}} + a_{i,\bar{n}}}{\bar{A}} \right\rceil \right\} \\ Q &= \frac{x_{i,\bar{n}} + a_{i,\bar{n}}}{M}\,. \end{aligned} \tag{15}$$

Note that $Q \leq \bar{A}$. In the new trajectory at times $\tilde{t}_l = t_{\bar{n}} + \frac{lQ}{\lambda_i}, l = 0, 1, \ldots, M-1$ we make a new replenishment of item i only in the amount Q. Thus at time $\tilde{t}_0 = t_{\bar{n}}$ we make two replenishments, one to i in the amount of Q and the other one equal to the original one except we do not replenish i. At time $\tilde{t}_{M-1}$ both trajectories have the same amount of stock, because the same total quantity is replenished up to that point in time. Figure 2 shows the old and the new trajectory. We denote by S the set of all decision epochs between $\tilde{t}_{\bar{n}}$ and $\tilde{t}_{M-1}$ in the original trajectory. Observe that the new trajectory might not have the just-in-time property.

The total extra ordering cost of the new trajectory is no more than $M \cdot C_{\{i\}}$. Next we discuss the difference in the holding cost. Let $T = [\tilde{t}_0, \tilde{t}_{M-1}]$. The holding cost of the original trajectory during T equals to

$$h_i \frac{(x_{i,\bar{n}} + a_{i,\bar{n}})^2}{2\lambda_i} - h_i \frac{Q^2}{2\lambda_i} + \delta\,,$$

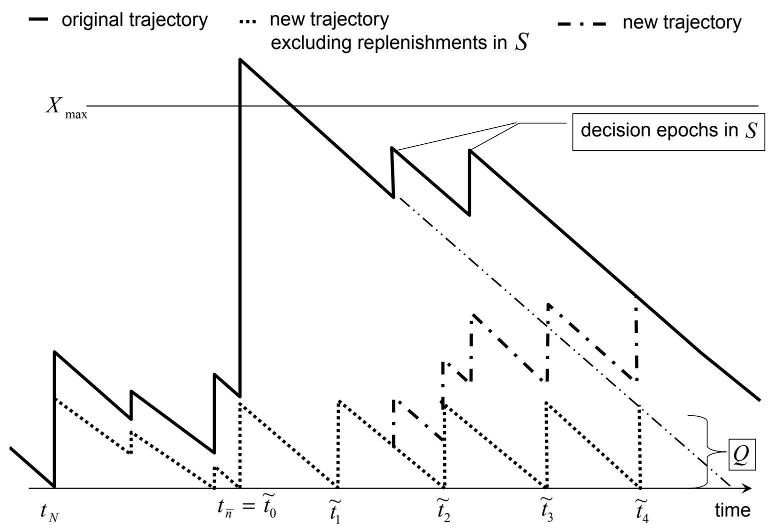

Figure 2: Holding cost for $M = 5$

where δ denotes the contribution in T corresponding to the replenishments in S. The first term corresponds to the holding cost associated with $x_{i,\bar{n}} + a_{i,\bar{n}}$ and incurred at time $\tilde{t}_0$ and the second term is the holding cost of the replenishment in the amount of Q at time $\tilde{t}_{M-1}$. (In Figure 2, δ corresponds to the area between the original trajectory and the hypotenuse of the big triangle.) On the other hand, the holding cost of the new trajectory in T is given by

$$h_i \frac{(M-1)Q^2}{2\lambda_i} + \delta \, .$$

(In Figure 2, δ is the area between the new trajectory and the new trajectory excluding replenishments in S.) Subtracting the two costs we obtain

$$h_i \frac{(x_{i,\bar{n}} + a_{i,\bar{n}})^2}{2\lambda_i} - h_i \frac{Q^2}{2\lambda_i} - h_i \frac{(M-1)Q^2}{2\lambda_i} = h_i \frac{M(M-1)Q^2}{2\lambda_i} \, , \tag{16}$$

where we use $x_{i,\bar{n}} + a_{i,\bar{n}} = QM$. To summarize, the holding cost of the new trajectory is less than or equal to the holding cost of the new trajectory during time $[0, \tilde{t}_0]$. After $\tilde{t}_0$ the two holding costs differ by the quantity given in (16). So the new trajectory is beneficial if

$$MC_{\{i\}} \leq \frac{h_i}{2\lambda_i} \cdot \frac{M-1}{M} \cdot (x_{i,\bar{n}} + a_{i,\bar{n}})^2 \, , \tag{17}$$

using the definition of Q.

Next we show that by the choice of $X_{\max}$ and M, (17) holds. By definition, $M \geq 2$, and therefore $\frac{M-1}{M} = 1 - \frac{1}{M} \geq 1/2$. Assume first that $2 < M$. We have $M \leq \frac{x_{i,\bar{n}} + a_{i,\bar{n}}}{\bar{A}} + 1 \leq$

$2\left(\frac{x_{i,\bar{n}}+a_{i,\bar{n}}}{\bar{A}}\right)$, which yields

$$\begin{aligned}\frac{h_i}{2\lambda_i}\cdot\frac{M-1}{M}(x_{i,\bar{n}}+a_{i,\bar{n}})^2 \geq \frac{h_i}{2\lambda_i}\cdot\frac{1}{2}(x_{i,\bar{n}}+a_{i,\bar{n}})^2 \geq \frac{h_i}{4\lambda_i}X_{\max}\cdot(x_{i,\bar{n}}+a_{i,\bar{n}})\\ \geq 2\left(\frac{x_{i,\bar{n}}+a_{i,\bar{n}}}{\bar{A}}\right)C_{\{i\}} \geq MC_{\{i\}}\,,\end{aligned} \tag{18}$$

where in (18) we have used that $X_{\max} \geq 8C_{\{i\}}\lambda_i/(\bar{A}h_i)$. If $M=2$, then we get

$$\frac{h_i}{2\lambda_i}\cdot\frac{1}{2}(x_{i,\bar{n}}+a_{i,\bar{n}})^2 \geq \frac{h_i}{4\lambda_i}X_{\max}^2 \geq 2C_{\{i\}} = MC_{\{i\}}\,,$$

where we use that $X_{\max} \geq \sqrt{8C_{\{i\}}\lambda_i/h_i}$. This shows (17) and therefore the new trajectory is beneficial.

Observe that for item i we have

$$x'_{i,\bar{n}}+a'_{i,\bar{n}} = Q = \frac{x_{i,\bar{n}}+a_{i,\bar{n}}}{M} \leq \frac{x_{i,\bar{n}}+a_{i,\bar{n}}}{2}\,. \tag{19}$$

If $Q > X_{\max}$, then we keep repeating the argument. Due to (19), in a finite number of iterations we produce a trajectory with $x'_{i,\bar{n}}+a'_{i,\bar{n}} \leq X_{\max}$. Since this trajectory might not satisfy the just-in-time property, by Lemma 3, it can be turned into one that has the just-in-time property without violating (14).

Since the described modification does not involve other items, we can repeat the process for any item violating (14) at the decision epoch $\bar{n}$. We conclude that there is a trajectory, where (14) holds for any $i \in \mathcal{I}$ and $n \leq \bar{n}$. Now we repeat the process for any decision epoch violating (14). □

As a result of this proposition, we may assume that $\overline{X}_i < \infty$ for every i (and $\overline{A} < \infty$). Assumption B9 follows directly from this and the definitions of X, $A(x)$, and K. Assumption B10 follows because the condition in Lemma 4 is satisifed. Also by Lemma 4, the functions c and τ are bounded on compact K, and so Assumptions B1 and B2 are satisfied. We also have $\tau \in \mathbb{C}_b(K)$, i.e. Assumption B4, from the definition (11) because the minimum of a finite set of continuous functions is a continuous function. From this it follows immediately that s is continuous, Assumption B8. We may assume that Assumption B3 is satisfied, because otherwise we can set $C_{\min} = \min_{I\subseteq\mathcal{I}} C_I > 0$ and rescale the cost data to $C_I \leftarrow C_I/C_{\min}$ for every I and $h_i \leftarrow h_i/C_{\min}$ for every i.

To show Assumption B5, note that a function is lower semi-continuous if and only if the level sets are closed. To make use of this fact, we show the following lemma.

Lemma 5. *For any $r \in \mathbb{R}$ we have*

$$\begin{aligned}&\left\{(x,a)\in K : C_{supp(a)} + \sum_{i\in\mathcal{I}}\frac{h_i}{2\lambda_i}(2a_ix_i+a_i^2) \leq r\right\}\\ &\quad= \bigcup_{I\subseteq\mathcal{I}}\{(x,a)\in K : a_i = 0 \text{ for every } i\in\mathcal{I}\setminus I, h_I(x,a) \leq r - C_I\}\,,\end{aligned}$$

where $h_I(x,a) = \sum_{i\in I}\frac{h_i}{2\lambda_i}(2a_ix_i+a_i^2)$.

Proof. Fix an r. First consider an (x,a) in the left-hand side set, i.e. $C_{\text{supp}(a)}+\sum_{i\in\mathcal{I}}\frac{h_i}{2\lambda_i}(2a_ix_i+a_i^2)\le r$. Then letting $I=\text{supp}(a)$, it is easy to see that (x,a) is included in the right-hand side set. On the other hand, for some I, if (x,a) is in the set $\{(x,a)\in K : a_i=0 \text{ for every } i\in\mathcal{I}\setminus I, h_I(x,a)\le r-C_I\}$, then $\text{supp}(a)\subseteq I$ implies

$$\sum_{i\in\mathcal{I}}\frac{h_i}{2\lambda_i}(2a_ix_i+a_i^2)=h_{\text{supp}(a)}(x,a)=h_I(x,a)\le r-C_I\le r-C_{\text{supp}(a)}$$

because $C_{\text{supp}(a)}\le C_I$ by our monotonicity assumption in the problem description. Hence, (x,a) is in the left-hand side set. □

Since h_I is continuous and K is compact under Assumption B9, the sets

$$\{(x,a)\in K : a_i=0 \text{ for all } i\in\mathcal{I}\setminus I, h_I(x,a)\le r-C_I\}$$

for each $I\subseteq\mathcal{I}$ are closed and therefore because a finite union of closed sets is closed, the level sets are closed. This shows Assumption B5. A similar argument shows Assumption B6.

Lastly, we construct a simple policy that satisfies Assumption B7. Let $x_0=0$ and choose any finite $\epsilon>0$ such that

$$\epsilon\le\min\left\{\min_{i\in\mathcal{I}}\frac{\overline{X}_i}{\lambda_i},\frac{\overline{A}}{\sum_{i\in\mathcal{I}}\lambda_i}\right\}.$$

Now set $f(0)$ so that $a_{i,n}=\lambda_i\epsilon$ for every i,n, and for $x\ne 0$ set $f(\cdot)$ arbitrarily. The long-run average cost of this policy equals $J(f,0)=C_{\mathcal{I}}/\epsilon<\infty$.

Proof of Theorem 1. All of the above arguments combine to show that Assumptions B1–B10 are satisfied. The just-in-time property in Lemma 3 implies (3). Therefore, the conclusion of Theorem 3 (and also Theorem 2) holds. □

5 Infinite Linear Programming and Cyclic Schedules

We now exploit the strong duality result in Theorem 2. First we discuss how the primal/dual linear programs (6) and (7) encode finite cyclic schedules for the general, deterministic SMDP. Then by exploiting the structure of the generalized joint replenishment problem, we say more about the structure of optimal cyclic schedules. We conclude with a problem instance for which there does not exist an optimal cyclic schedule, and by showing how the primal/dual linear programs encode such a solution.

5.1 Cyclic Schedules for the General, Deterministic SMDP

We assume throughout that Assumptions B1–B10 hold.

Definition 2. *A sequence $\{(x_n,a_n)\}_{n=0,\dots,N-1}$ of $N<\infty$ state-action pairs is called a* **cyclic schedule** *if*

$$x_n=\begin{cases} s(x_{N-1},a_{N-1}) & \text{for } n=0\\ s(x_{n-1},a_{n-1}) & \text{for } n=1,\dots,N-1.\end{cases}$$

A cyclic schedule is said to be optimal if its long-run average cost equals $J^* = \rho^*$. By Lemma 3 in Klabjan and Adelman (2003), in an optimal cyclic schedule $\sum_{n=0}^{N-1} \tau(x_n, a_n) > 0$. The next lemma shows that an arbitrary dual optimal solution encodes all optimal cyclic schedules, provided they exist.

Lemma 6. *Suppose there exists an optimal cyclic schedule* $\{(x_n, a_n)\}_{n=0,\dots,N-1}$. *Then for any dual optimal solution* (ρ^*, u^*) *we have*

$$u^*(x_n) = c(x_n, a_n) - \rho^* \tau(x_n, a_n) + u^*(s(x_n, a_n)) \quad n = 0, \dots, N-1.$$

Proof. Suppose there exists an $n' \in \{0, \dots, N-1\}$ such that (7b) for $(x_{n'}, a_{n'})$ is satisfied with strict inequality. Then, telescoping (7b) implies that

$$0 < \sum_{n=0}^{N-1} c(x_n, a_n) - \rho^* \sum_{n=0}^{N-1} \tau(x_n, a_n),$$

which rewritten becomes

$$\rho^* < \frac{\sum_{n=0}^{N-1} c(x_n, a_n)}{\sum_{n=0}^{N-1} \tau(x_n, a_n)} = \rho^*,$$

because the cyclic schedule is optimal. This is clearly a contradiction. □

We next discuss how the primal program (6) encodes cyclic schedules. Whereas the dual program simultaneously encodes all optimal cyclic schedules, provided they exist, each optimal cyclic schedule gives rise to a different primal optimal solution.

Lemma 7. *Suppose there exists an optimal cyclic schedule* $\{(x_n, a_n)\}_{n=0,\dots,N-1}$. *Then*

$$\mu^*(\mathcal{K}) = \frac{1}{T^*} \sum_{n=0}^{N-1} \mathbb{1}_{\mathcal{K}}\{(x_n, a_n)\} \quad \mathcal{K} \in \mathcal{B}(K)$$

is an optimal solution to the primal program (6), where $\mathbb{1}_{\mathcal{K}}\{(x_n, a_n)\}$ *is the indicator function that is 1 if* $(x_n, a_n) \in \mathcal{K}$ *and 0 otherwise, and* $T^* = \sum_{n=0}^{N-1} \tau(x_n, a_n)$ *is the time duration of the cycle.*

Proof. It is easy to see that this solution is feasible to (6) and has objective value equal to

$$\frac{1}{T^*} \sum_{n=0}^{N-1} c(x_n, a_n) = \rho^*.$$ □

So cyclic schedules correspond with feasible primal solutions for which all mass is concentrated on a finite number of singletons.

More generally, feasible primal solutions μ may have all mass concentrated on a countably infinite number of singletons, rather than just a finite number. Let μ be a measure on K, and

$$\begin{aligned} \mathcal{A}^\mu(x) &= \{a \in A(x) : \mu(x, a) > 0\} \quad \text{for every } x \in X, \text{ and} \\ X^\mu &= \{x \in X : \mathcal{A}^\mu(x) \neq \emptyset\}, \end{aligned}$$

so that X^μ is the set of states having singleton actions of positive mass, and $\mathcal{A}^\mu(x)$ is the set of such singleton actions from state x. From the following lemma, it follows that the sets $\mathcal{A}^\mu(x)$ and X^μ are countable.

Lemma 8. *Let μ be a positive measure on X such that $\mu(X) < \infty$. If $W = \{x \in X : \mu(\{x\}) > 0\}$, then W is countable.*

Proof. We show the claim by contradiction and thus assuming that W is uncountable. For every $n \in \mathbb{N}$, let $W_n = \{x \in W : \mu(\{x\}) > \frac{1}{n}\}$. Then clearly $W = \cup_{n=1}^\infty W_n$. Since W is uncountable, there exists $n_0 \in \mathbb{N}$ such that W_{n_0} is uncountable. But then

$$\infty > \mu(X) \geq \mu(W_{n_0}) \geq \sum_{i=1}^\infty \mu(\{x_i^{n_0}\}) \geq \infty \cdot \frac{1}{n_0} = \infty \,,$$

where $\{x_i^{n_0}\}_i$ is any infinite sequence of elements from W_{n_0}. Clearly this is a contradiction. □

The next lemma shows that if these countable sets contain a cyclic schedule, then it is optimal.

Lemma 9. *Let $(\mu^*, (\rho^*, u^*))$ be an optimal primal/dual solution to (6)-(7). If there exists a cyclic schedule $(x_n, a_n)_{n=0,1,\ldots,N-1}$ in which $x_n \in X^{\mu^*}$ and $a_n \in \mathcal{A}^{\mu^*}(x_n)$ for all $n = 0, \ldots, N-1$, then it is optimal.*

Proof. By complementary slackness from Theorem 2, and the assumption that $x_n \in X^{\mu^*}$, for all $n = 0, \ldots, N-1$ we have

$$u^*(x_n) = c(x_n, a_n) - \rho^* \tau(x_n, a_n) + u^*(x_{n+1}),$$

where we define $x_N = x_0$. Starting with $u^*(x_0)$ and telescoping we have

$$u^*(x_0) = \sum_{n=0}^{N-1} (c(x_n, a_n) - \rho^* \tau(x_n, a_n)) + u^*(x_0)$$

which implies

$$\rho^* = \frac{\sum_{n=0}^{N-1} c(x_n, a_n)}{\sum_{n=0}^{N-1} \tau(x_n, a_n)}.$$

□

Under the condition of this lemma, according to Lemma 7, the solution μ^* can be converted into an alternative optimal solution in which μ^* consists only of N point masses that each correspond with a step on the cyclic schedule.

Let $K^\mu = \{(x, a) \in K : x \in X^\mu, a \in \mathcal{A}^\mu(x)\}$ denote the set of singleton state-action pairs with positive mass under μ. This set is countable by Lemma 8.

Definition 3. *A feasible primal solution μ is said to be* **purely atomic** *if*

$$\mu(\mathcal{K}) = \sum_{(x,a) \in \mathcal{K} \bigcap K^\mu} \mu(x, a) \quad \mathcal{K} \in \mathcal{B}(K).$$

This means that all of the mass is concentrated on a countable subset of singletons. Our next result shows that in this case, there must exist an embedded cyclic schedule. So in particular, if there does not exist an optimal cyclic schedule, i.e. an optimal trajectory is an infinitely long non-cyclic sequence, then one cannot construct an optimal primal solution by giving each step on this trajectory positive mass because then it would be countable. In the next section, we show with an example how non-cyclic sequences are encoded by primal solutions.

For convenience, let $\phi^+(B)$ and $\phi^-(B)$ represent the total (countable) flow out and in, respectively, of states $B \subseteq X^\mu$. Also let $\phi(B_1, B_2)$ represent the total (countable) flow from states B_1 to states B_2. Formally,

$$\begin{aligned} \phi^+(B) &= \mu\left\{(x,a) \in \mathcal{K}^\mu : x \in B\right\} && B \in X^\mu \\ \phi^-(B) &= \mu\left\{(x,a) \in \mathcal{K}^\mu : s(x,a) \in B\right\} && B \in X^\mu \\ \phi(B_1, B_2) &= \mu\left\{(x,a) \in \mathcal{K}^\mu : x \in B_1, s(x,a) \in B_2\right\} && B_1, B_2 \in X^\mu. \end{aligned}$$

Using this notation, if μ is purely atomic then we can rewrite the primal constraints (6c) as

$$\phi^+(B) = \phi^-(B) \qquad B \in \mathcal{B}(X), \tag{20}$$

i.e. the flow rate out of B equals the flow rate into B.

Proposition 2. *There exists an optimal primal solution μ^* that is purely atomic if and only if there exists an optimal cyclic schedule.*

Proof. Lemma 7 shows that an optimal cyclic schedule corresponds with an optimal solution μ^* having countable (in fact finitely countable) mass.

Suppose there exists an optimal primal solution μ^* that is purely atomic. Then we can construct a set-to-set function F defined as the set of states reachable in one step from a set of originating states, i.e.

$$F(B) = \left\{x' \in X^{\mu^*} : s(x,a) = x' \text{ for some } x \in B, a \in \mathcal{A}^{\mu^*}(x)\right\} \qquad B \subseteq X^{\mu^*}.$$

Note that because of (6c), all states in X^{μ^*} lead to an infinite sequence of subsequent states under F. Denote the set of states reachable after n steps by $F^n(B)$, where $F^0(B) = B$, $F^1(B) = F(B)$, $F^2(B) = F(F^1(B))$, etc.

Suppose there does not exist an optimal cyclic schedule. Let $\tilde{x}$ be an arbitrary state in X^{μ^*}. Then we can partition the countable set X^{μ^*} into $\tilde{x}$ and three sets B_+,B_-, and B_0 defined as follows. The set B_+ denotes all states reachable in a finite number of steps starting from state $\tilde{x}$. The set B_- denotes all states from which $\tilde{x}$ is reachable in a finite number of steps. The set B_0 denotes all other states in X^{μ^*}. Formally,

$$\begin{aligned} B_+ &= \{x \in X^{\mu^*} : x \in F^n(\tilde{x}) \text{ for some } 1 \le n < \infty\} \\ B_- &= \{x \in X^{\mu^*} : \tilde{x} \in F^n(x) \text{ for some } 1 \le n < \infty\} \\ B_0 &= X^{\mu^*} \setminus \{B_+ \cup B_- \cup \{\tilde{x}\}\}. \end{aligned}$$

If $B_+ \cap B_- \neq \emptyset$, or if either $\tilde{x} \in B_+$ or $\tilde{x} \in B_-$, then we can construct an optimal cyclic schedule. Therefore, all pairwise intersections among $\{\tilde{x}\}$, B_+, B_-, and B_0 must be empty. Because $\phi(\{\tilde{x}\}, B_+) > 0$, by (6c) the set B_- must be nonempty and in particular $\phi(B_-, \{\tilde{x}\}) > 0$.

From flow rate feasibility (20) we have $\phi^+(B_-) = \phi^-(B_-)$. We can decompose $\phi^+(B_-)$ into

$$\phi^+(B_-) = \phi(B_-, \tilde{x}) + \phi(B_-, B_+) + \phi(B_-, B_0) + \phi(B_-, B_-).$$

Similarly, we can decompose $\phi^-(B_-)$ into

$$\phi^-(B_-) = \phi(\tilde{x}, B_-) + \phi(B_+, B_-) + \phi(B_0, B_-) + \phi(B_-, B_-).$$

By construction, and because there does not exist a cyclic schedule, there is no flow into B_- from the outside, meaning that $\phi(B_+, B_-) = \phi(B_0, B_-) = \phi(\tilde{x}, B_-) = 0$. Hence, we have

$$\phi(B_-, \tilde{x}) + \phi(B_-, B_+) + \phi(B_-, B_0) + \phi(B_-, B_-) \quad = \quad \phi(B_-, B_-),$$

which implies

$$\phi(B_-, \tilde{x}) + \phi(B_-, B_+) + \phi(B_-, B_0) \quad = \quad 0.$$

This is a contradition because $\phi(B_-, \tilde{x}) > 0$. Therefore, there must exist an optimal cyclic schedule and it satisfies the conditions of Lemma 9. □

Our example in the next section shows that cyclic schedules need not be optimal. Cyclic schedules are said to be *ϵ-optimal* if for every $\epsilon > 0$ there exists a cyclic schedule $\{(x_n, a_n)\}_{n=0,\ldots,N-1}$ such that

$$\frac{\sum_{n=0}^{N-1} c(x_n, a_n)}{\sum_{n=0}^{N-1} \tau(x_n, a_n)} - J^* \leq \epsilon,$$

i.e. they can get ϵ close to any optimal policy.

Theorem 4. *Cyclic schedules are ϵ-optimal.*

Proof. Suppose not. Then for some $\epsilon > 0$,

$$J^* < \frac{\sum_{n=0}^{N} c(\bar{x}_n, \bar{a}_n)}{\sum_{n=0}^{N} \tau(\bar{x}_n, \bar{a}_n)} - \epsilon$$

for every cyclic schedule $\{(\bar{x}_n, \bar{a}_n)\}_{n=0,\ldots,N-1}$. Suppose $\{(x_n, a_n)\}_{n=0,1,\ldots}$ is an optimal trajectory that attains J^*, which exists by Theorem 1. Then in particular, take just the first $\tilde{N}$ replenishments, where $\tilde{N}$ is an arbitrary positive integer, and construct a cyclic schedule by appending to it the finite sequence of $M < \infty$ steps $\{(\tilde{x}_n, \tilde{a}_n)\}_{n=\tilde{N},\ldots,\tilde{N}+M-1}$ leading from $x_{\tilde{N}-1}$ to x_0, given by Assumption B10. For all $\tilde{N}$ we have

$$\begin{aligned} J^* + \epsilon \quad &< \quad \frac{\sum_{n=0}^{\tilde{N}-1} c(x_n, a_n) + \sum_{n=\tilde{N}}^{\tilde{N}+M-1} c(\tilde{x}_n, \tilde{a}_n)}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n) + \sum_{n=\tilde{N}}^{\tilde{N}+M-1} \tau(\tilde{x}_n, \tilde{a}_n)} \\ &< \quad \frac{\sum_{n=0}^{\tilde{N}-1} c(x_n, a_n) + C}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)}, \end{aligned}$$

where C is the constant from Assumption B10. Therefore,

$$\begin{aligned}
J^* + \epsilon &< \limsup_{\tilde{N}\to\infty} \left(\frac{\sum_{n=0}^{\tilde{N}-1} c(x_n, a_n)}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)} + \frac{C}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)} \right) \\
&\leq \limsup_{\tilde{N}\to\infty} \frac{\sum_{n=0}^{\tilde{N}-1} c(x_n, a_n)}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)} + \limsup_{\tilde{N}\to\infty} \frac{C}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)} \\
&= \limsup_{\tilde{N}\to\infty} \frac{\sum_{n=0}^{\tilde{N}-1} c(x_n, a_n)}{\sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n)} \\
&= J^*,
\end{aligned}$$

which implies $\epsilon < 0$, contradiction. Note that in the above we use that $\lim_{\tilde{N}\to\infty} \sum_{n=0}^{\tilde{N}-1} \tau(x_n, a_n) = \infty$ using Lemma 3 of Klabjan and Adelman (2003) and the fact that $J^* < \infty$. □

An implication is that if there exists a cyclic schedule that is optimal among all cyclic schedules, then it is optimal among all possible trajectories in that it achieves J^*.

5.2 Cyclic Schedules for Generalized Joint Replenishment

In the classical EOQ problem introduced by Harris (1915), it is well known that under the optimal order quantity the long-run average holding cost equals the long-run average fixed ordering cost. Our next result generalizes this property. For any cyclic schedule $(x, a) = \{(x_n, a_n)\}_{n=0,\dots,N-1}$, let $\mathcal{H}(x, a) = \sum_{n=0}^{N-1} \sum_{i\in\mathcal{I}} \frac{h_i}{2\lambda_i} \left(2a_{i,n}x_{i,n} + a_{i,n}^2\right)$ be the total holding cost over the cycle, and let $\mathcal{C}(x, a) = \sum_{n=0}^{N-1} C_{\text{supp}(a_n)}$ be the total fixed ordering cost over the cycle. Also define

$$\bar{\alpha}(x, a) = \min \left\{ \min_{\substack{n\in\{0,\dots,N-1\} \\ i\in\mathcal{I}}} (\overline{X}_i / x_{i,n}), \min_{n\in\{0,\dots,N-1\}} (\overline{A} / \sum_{i\in\mathcal{I}} a_{i,n}) \right\}.$$

So $\bar{\alpha}(x, a) \geq 1$, with $\bar{\alpha}(x, a) = 1$ if at least one of the upper bounds $\overline{X}_i$ for some i or $\overline{A}$ is tight.

Theorem 5. *Without loss of optimality among cyclic schedules, it suffices to consider those (x, a) for which either*

- $\mathcal{H}(x, a) = \mathcal{C}(x, a)$, *or*
- $\bar{\alpha}(x, a) = 1$ *and* $\mathcal{H}(x, a) \leq \mathcal{C}(x, a)$.

Proof. For any scaling factor α where $0 < \alpha \leq \bar{\alpha}(x, a)$, consider a modified cyclic schedule $\{(x_n', a_n')\}_{n=0,\dots,N-1}$ in which $x_{i,n}' = \alpha x_{i,n}$ and $a_{i,n}' = \alpha a_{i,n}$ for each i and n. To simplify notation, we drop the dependence of $\mathcal{H}$, $\mathcal{C}$, and $\bar{\alpha}$ on (x, a), and use primes when referring to these quantities for (x', a'). Letting T and T' denote the time duration of each of the cyclic schedules, observe that

$$\tau(x', a') = \min_{i \in \mathcal{I}} \left\{ \frac{x'_i + a'_i}{\lambda_i} \right\} = \alpha \tau(x, a),$$

and so $T' = \sum_{n=0}^{N-1} \tau(x'_n, a'_n) = \alpha \sum_{n=0}^{N-1} \tau(x_n, a_n) = \alpha T$. Thus, multiplying

$$x_{i,n+1} = x_{i,n} + a_{i,n} - \lambda_i \tau(x_n, a_n)$$

by α yields

$$\alpha x_{i,n+1} = \alpha x_{i,n} + \alpha a_{i,n} - \lambda_i \tau(\alpha x_n, \alpha a_n)$$

so that the new cyclic schedule satisfies flow balance (2b). Furthermore, by definition of $\bar{\alpha}$, $x'_{i,n} = \alpha x_{i,n} \leq \bar{\alpha} x_{i,n} \leq \overline{X}_i$ for every i and n, and for every n we have $\sum_{i \in \mathcal{I}} a'_{i,n} = \alpha \sum_{i \in \mathcal{I}} a_{i,n} \leq \bar{\alpha} \sum_{i \in \mathcal{I}} a_{i,n} \leq \overline{A}$. Hence, constraints (2c) and (2d) are satisfied.

The total holding cost of the new cyclic schedule equals

$$\mathcal{H}' = \sum_{n=0}^{N-1} \sum_{i \in \mathcal{I}} \frac{h_i}{2\lambda_i} \left(2a'_{i,n} x'_{i,n} + (a'_{i,n})^2 \right) = \alpha^2 \mathcal{H},$$

and the new total fixed ordering cost is the same as the old one, i.e. $\mathcal{C}' = \mathcal{C}$. Therefore, the total long-run average cost of the new cyclic schedule equals

$$\frac{\mathcal{C} + \alpha^2 \mathcal{H}}{\alpha T} = \frac{1}{T} \left(\frac{\mathcal{C}}{\alpha} + \alpha \mathcal{H} \right).$$

Now we find the best such cyclic schedule, i.e. we solve

$$\min_{0 < \alpha \leq \bar{\alpha}} \left\{ \mathcal{C}/\alpha + \mathcal{H}\alpha \right\},$$

which yields $\alpha^* = \min\{\bar{\alpha}, \sqrt{\mathcal{C}/\mathcal{H}}\}$. If this minimum equals $\sqrt{\mathcal{C}/\mathcal{H}}$, then $\mathcal{H}' = (\alpha^*)^2 \mathcal{H} = (\mathcal{C}/\mathcal{H})\mathcal{H} = \mathcal{C} = \mathcal{C}'$, which is the first conclusion. If this minimum equals $\bar{\alpha}$, that is $\bar{\alpha} \leq \sqrt{\mathcal{C}/\mathcal{H}}$, then

$$\mathcal{H}' = \bar{\alpha}^2 \mathcal{H} \leq (\mathcal{C}/\mathcal{H})\mathcal{H} = \mathcal{C} = \mathcal{C}'.$$

Furthermore, $\bar{\alpha}' = \bar{\alpha}(x', a') = 1$ for the new cycle. This yields the second conclusion. □

Although Theorem 4 shows that cyclic schedules are ϵ-optimal, they are not necessarily optimal, i.e. there may not exist a cyclic schedule that attains J^*.

Proposition 3. *All cyclic schedules are suboptimal for the following instances:*
$\mathcal{I} = \{1, 2\}$, $\lambda_1 = \lambda_2 = 1$, $C_{\{1\}} = C_{\{2\}} = 1$, $C_{\{12\}} = 2$, $h_1 = h_2 = 0$, *one of* $\overline{X}_1$ *and* $\overline{X}_2$ *is rational and the other is irrational,* $\overline{A} = \infty$.

Proof. An optimal policy manages the two items independently because $C_{\{1\}} + C_{\{2\}} = 2 \leq C_{\{12\}}$, i.e. there is no economic incentive to replenish items together. Hence, the optimal policy replenishes quantity $\overline{X}_i$ of item i whenever it stocks out. Now suppose there exists an optimal cyclic schedule, and $(x, 0)$ (or $(0, x)$) is some state on it. Then there exists a cycle length $T < \infty$ such that state $(x, 0)$ (or $(0, x)$) is revisited. Hence, by flow balance (2b), there must exist $n_1, n_2 \in \mathbb{N}$ such that $x + n_1\overline{X}_1 - T = x$, which implies $T = n_1\overline{X}_1$ and similarly $T = n_2\overline{X}_2$. However, if one of $\overline{X}_1$ and $\overline{X}_2$ is rational and the other irrational, then $n_1\overline{X}_1 = n_2\overline{X}_2$ equates an irrational number with a rational number, which is a contradiction. $\square$

The same scenerio can occur even when all of the input data are rational. For instance, consider $h_1 = 1$, $h_2 = 2$, and $\overline{X}_1 = \overline{X}_2 = \infty$. Then, as before, it is optimal to manage the items indepedently, but in this case each item follows the classical economic order quantity, which equals quantity $\sqrt{2}$ for item 1 and quantity 1 for item 2. Because the former is irrational and the later is rational, the same argument holds, so that there does not exist an optimal cyclic schedule.

These examples immediately decompose into two independent single-item problems. This remains true even if $C_{\{12\}} > 2$. In the general multi-item problem, an optimal policy may decompose into a partition of the items, where on each partition there is an independent cyclic schedule. It is still not known whether or not such a decomposition always exists. Note that the infinite linear program (6) does not require any *a priori* knowledge of this decomposition, if it does exist, to solve the problem. We need only to solve (6) once, rather than once for every subset of items.

Lemmas 6 and 7 show how the infinite linear programs (6) and (7) encode cyclic schedules, but it is not yet clear how non-cyclic solutions are encoded. When there does not exist an optimal cyclic schedule, we already know from Proposition 2 that the corresponding primal optimal solution is not purely atomic. For the example of Proposition 3, as we will see shortly the primal optimal solution involves Lebesgue measure.

First we provide a dual optimal solution.

Proposition 4. *A dual optimal solution for the example in Proposition 3 is*

$$\begin{aligned} u^*(x, 0) &= -x/\overline{X}_1 && \forall x \in [0, \overline{X}_1] \\ u^*(0, x) &= -x/\overline{X}_2 && \forall x \in [0, \overline{X}_2] \\ \rho^* &= 1/\overline{X}_1 + 1/\overline{X}_2. \end{aligned}$$

Proof. The objective value ρ^* equals the value J^* of the optimal policy from the proof of Proposition 3, so we just need to show that the dual solution is feasible and that the optimal policy's actions have zero reduced-cost. Without loss of generality, consider any state $(0, x) \in X$ as a symmetric argument holds for states $(x, 0)$ as well. Note that we can write $u^*(x) = -(\frac{x_1}{\overline{X}_1} + \frac{x_2}{\overline{X}_2})$ for all $x \in X$. Collecting terms to one side of (7b), for any

$a \in A((0,x))$ we have to show that

$$
\begin{aligned}
& C_{\mathrm{supp}(a)} - \rho^* \tau((0,x),a) + u^*(s((0,x),a)) - u^*((0,x)) \\
& = C_{\mathrm{supp}(a)} - \rho^* \tau((0,x),a) - \frac{a_1 - \tau((0,x),a)}{\overline{X}_1} - \frac{x + a_2 - \tau((0,x),a)}{\overline{X}_2} + \frac{x}{\overline{X}_2} \\
& = C_{\mathrm{supp}(a)} - \frac{a_1}{\overline{X}_1} - \frac{a_2}{\overline{X}_2} \geq 0. \qquad (21)
\end{aligned}
$$

If $a_1 > 0$ and $a_2 = 0$, then (21) reduces to $1 - a_1/\overline{X}_1 \geq 0$, with equality when $a_1 = \overline{X}_1$. Similarly, if $a_1 = 0$ and $a_2 > 0$, then (21) reduces to $1 - a_2/\overline{X}_2 \geq 0$, with equality when $a_2 = \overline{X}_2$, which can only happen when $x = 0$. If $a_1 > 0$ and $a_2 > 0$, then (21) reduces to $2 - a_1/\overline{X}_1 - a_2/\overline{X}_2 \geq 0$, with equality when $a_1 = \overline{X}_1$ and $a_2 = \overline{X}_2$, which can only happen if $x = 0$. If $a_1 = a_2 = 0$, then (21) reduces to $C_\emptyset \geq 0$. Finally, $\|u^*\| \leq 1$ and so $u^* \in \mathbb{B}(X)$. $\square$

Let

$$
K^* = \{(0,0), (\overline{X}_1, \overline{X}_2)\} \cup \{(0,x_2), (\overline{X}_1, 0) : x_2 > 0\} \cup \{(x_1, 0), (0, \overline{X}_2) : x_1 > 0\}
$$

be the set of state-action pairs with zero reduced cost under the dual optimal solution above, excluding the state-action pairs $((0,0),(0,\overline{X}_2))$ and $((0,0),(\overline{X}_1,0))$. For all Borel subsets $\mathcal{K} \in \mathcal{B}(K)$, from the definition of X we can decompose $\mathcal{K}$ into two sets $((\beta_1(\mathcal{K}) \times \{0\}) \times A) \cap \mathcal{K}$ and $((\{0\} \times \beta_2(\mathcal{K})) \times A) \cap \mathcal{K}$, where $\beta_i(\mathcal{K}) \in \mathcal{B}([0, \overline{X}_i])$. Formally,

$$
\beta_1(\mathcal{K}) = \left\{ x \in [0, \overline{X}_1] : \text{ there exists } a \in A((x,0)) \text{ with } ((x,0),a) \in \mathcal{K} \right\}
$$

and analogously for $\beta_2(\mathcal{K})$. Let m denote the Lebesgue measure in $\mathbb{R}$.

Proposition 5. *For the example in Proposition 3,*

$$
\mu^*(\mathcal{K}) = \frac{m(\beta_1(\mathcal{K} \cap K^*)) + m(\beta_2(\mathcal{K} \cap K^*))}{\overline{X}_1 \overline{X}_2} \qquad \mathcal{K} \in \mathcal{B}(K)
$$

is an optimal solution to the primal problem (6).

Proof. It is easy to see that μ^* is a measure. Note also that $\|\mu^*\|^{\mathrm{TV}} = \mu^*(K) = 1/\overline{X}_1 + 1/\overline{X}_2 < \infty$. Without loss of generality, we assume that $\overline{X}_1 \leq \overline{X}_2$. Since $\mu^*(\mathcal{K} \setminus K^*) \leq \mu^*(K \setminus K^*) = 0$, the constraint (6c) reduces to

$$
\mu^*(\{(x,a) \in K^* : x \in B\}) = \mu^*(\{(x,a) \in K^* : s(x,a) \in B\}) \qquad B \in \mathcal{B}(X), \qquad (22)
$$

which we now show holds. Fix a $B \in \mathcal{B}(X)$ and decompose it into two sets $(B_1 \times \{0\})$ and $(\{0\} \times B_2)$, where $B_i \in \mathcal{B}([0, \overline{X}_i])$ for $i = 1, 2$. Note that these two sets are disjoint unless $(0,0) \in B$, but that $\mu^*(((0,0),(\overline{X}_1, \overline{X}_2))) = 0$, since the Lebesgue measure of a singleton equals 0. Therefore, under our proposed solution,

$$
\begin{aligned}
& \mu^*(\{(x,a) \in K^* : x \in B\}) \\
& = \mu^*(\{(x,a) \in K^* : x \in (B_1 \times \{0\})\}) + \mu^*(\{(x,a) \in K^* : x \in (\{0\} \times B_2)\}) \\
& = \frac{m(B_1) + m(B_2)}{\overline{X}_1 \overline{X}_2},
\end{aligned}
$$

because every state $x \in X$ has an action $a \in A(x)$ such that $(x, a) \in K^*$. Furthermore, this action is unique and leads to the unique next state $s(x, a)$. Consequently, the state under K^* immediately preceeding $(0, x)$ under s for $x \in [0, \overline{X}_2]$ must be either $(0, x + \overline{X}_1)$ if $0 \leq x \leq \overline{X}_2 - \overline{X}_1$, or $(\overline{X}_2 - x, 0)$ if $\overline{X}_2 - \overline{X}_1 \leq x \leq \overline{X}_2$. The state under K^* immediately preceeding $(x, 0)$ for $x \in [0, \overline{X}_1]$ must be $(0, \overline{X}_1 - x)$. Therefore, we can decompose the set $K_0 = \{(x, a) \in K^* : s(x, a) \in B\}$ into three sets

$$
\begin{aligned}
K_1 &= \{(x, a) \in K^* : \text{ there exists } \tilde{x} \in B_2 \cap [0, \overline{X}_2 - \overline{X}_1] \text{ such that } x = (0, \tilde{x} + \overline{X}_1)\} \\
K_2 &= \{(x, a) \in K^* : \text{ there exists } \tilde{x} \in B_2 \cap [\overline{X}_2 - \overline{X}_1, \overline{X}_2] \text{ such that } x = (\overline{X}_2 - \tilde{x}, 0)\} \\
K_3 &= \{(x, a) \in K^* : \text{ there exists } \tilde{x} \in B_1 \cap [0, \overline{X}_1] \text{ such that } x = (0, \overline{X}_1 - \tilde{x})\}
\end{aligned}
$$

so that $K_0 = K_1 \cup K_2 \cup K_3$. Note that all pairwise intersections are empty except for possibly the state-action pairs $((0, 0), (\overline{X}_1, \overline{X}_2))$ and $((0, \overline{X}_1), (\overline{X}_1, 0))$. Likewise, the sets $\beta_1(K_j)$ and $\beta_2(K_j)$ for $j = 1, 2, 3, 4$ are pairwise disjoint except for possibly the states $(0, 0)$ and $(0, \overline{X}_1)$. Since $\mu^*((x, a)) = 0$ for all singletons $(x, a) \in K$, we have $\mu^*(K_0) = \mu^*(K_1) + \mu^*(K_2) + \mu^*(K_3)$. Note also that

$$
\begin{aligned}
m(\beta_1(K_0)) &= m(\beta_1(K_2)) = m(B_2 \cap [\overline{X}_2 - \overline{X}_1, \overline{X}_2])) \\
m(\beta_2(K_0)) &= m(\beta_2(K_1)) + m(\beta_2(K_3)) = m(B_2 \cap [0, \overline{X}_2 - \overline{X}_1]) + m(B_1 \cap [0, \overline{X}_1]).
\end{aligned}
$$

Therefore,

$$
\mu^*(K_0) = \frac{m(B_1) + m(B_2)}{\overline{X}_1 \overline{X}_2},
$$

which yields (22).

We now show that (6b) is also satisfied. Using the definition of τ in (11), we have

$$
\begin{aligned}
\int_K \tau(x, a) \mu^*(\mathrm{d}(x, a)) &= \int_{K^*} \tau(x, a) \mu^*(\mathrm{d}(x, a)) \\
&= \int_{0 \leq x_1 \leq \overline{X}_1} \frac{\min\{x_1, \overline{X}_2\}}{\overline{X}_1 \overline{X}_2} \mathrm{d}x_1 + \int_{0 \leq x_2 \leq \overline{X}_2} \frac{\min\{\overline{X}_1, x_2\}}{\overline{X}_1 \overline{X}_2} \mathrm{d}x_2 \\
&= \int_{0 \leq x_1 \leq \overline{X}_1} \frac{x_1}{\overline{X}_1 \overline{X}_2} \mathrm{d}x_1 + \int_{0 \leq x_2 \leq \overline{X}_1} \frac{x_2}{\overline{X}_1 \overline{X}_2} \mathrm{d}x_2 + \int_{\overline{X}_1 \leq x_2 \leq \overline{X}_2} \frac{\overline{X}_1}{\overline{X}_1 \overline{X}_2} \mathrm{d}x_2 \\
&= 1.
\end{aligned}
$$

The objective function (6a) equals

$$\begin{aligned}\int_K c(x,a)\mu(\mathrm{d}(x,a)) &= 1\mu^*(\{(x,a)\in K^* : a_1 = 0, a_2 = \overline{X}_2\}) \\ &\quad + 1\mu^*(\{(x,a)\in K^* : a_1 = \overline{X}_1, a_2 = 0\}) \\ &\quad + 2\mu^*(\{(x,a)\in K^* : a_1 = \overline{X}_1, a_2 = \overline{X}_2\}) \\ &\quad + C_\emptyset\mu^*(\{(x,a)\in K^* : a_1 = a_2 = 0\}) \\ &= \frac{1}{\overline{X}_1\overline{X}_2}\left[m((0,\overline{X}_1]) + m((0,\overline{X}_2]) + 2m(0) + C_\emptyset m(\emptyset)\right] \\ &= \frac{\overline{X}_1+\overline{X}_2}{\overline{X}_1\overline{X}_2} = 1/\overline{X}_1 + 1/\overline{X}_2,\end{aligned}$$

which is ρ^* from Proposition 4 and therefore there is no duality gap. □

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