Inventory Control in Serial Systems under Radio Frequency Identification

Jinxiang Pei (peipauj@northwestern.edu)
Diego Klabjan (d-klabjan@northwestern.edu)
Department of Industrial Engineering and Management Sciences
Northwestern University, Evanston, IL

Abstract: The widely adopted slap-and-ship radio frequency identification strategy provides valuable information to retailers. On the other hand, suppliers struggle to find benefits even though they are submerged with new data. Radio frequency identification provides complete visibility of their shipments, including the time and location of every pallet, case or even item. We provide a novel model relying on such data that is capable of producing better inventory and shipping control policies. We first propose a comprehensive inventory model for serial systems that captures both the supply and distribution information. We show that the underlying cost-to-go can be decomposed into two lower dimensional functions. In a special case, the optimal replenishment and shipping policies are base stock with respect to the underlying positions. In addition, we also analytically study the value of radio frequency identification in terms of the expected total minimum cost over a finite time horizon by introducing partial radio frequency deployment scenarios. Results indicate that additional cost reductions are possible with broader deployments.

Key Words: inventory control, radio frequency identification

1. Introduction

A basic radio frequency identification (RFID) system includes two components: a transponder and an interrogator. A transponder is a tiny microcomputer consisting of a microchip, antenna and memory connecting these two. In the simplest form, the so-called passive RFID transponders, transponders are idle until woken up by an interrogator via radio wave signals. Interrogators are, therefore, constantly emitting signals to provide power to the transponders within their antenna's field of work. When a transponder receives a signal from an interrogator, it absorbs some of the radio energy to power itself and sends back a response, which among other data stored in its memory, includes the transponder's unique identification number. The interrogator decodes this information and passes it to information systems. In a typical RFID deployment within a supply chain, interrogators are mounted at critical locations. Every time goods affixed with transponders come within the read range of an interrogator, the location, time, and identification are recorded. Under normal conditions a typical interrogator may interrogate hundreds of transponders per second.

One of the recent information technology advances is the adoption of RFID technology in inventory control systems. Early benefits earned from RFID deployments are inventory asset tracking, advance shipping notice, real-time order progress information for retailers, and real-time shipping visibility for suppliers. The additional RFID generated information could possibly result in improved inventory control policies and potential new business applications. One of the biggest setbacks to a wider RFID adoption is the lack of return of investment. Many entities in supply chains are overwhelmed with data generated from RFID deployment, yet this data is seldom used to enhance business intelligence.

As an emerging auto data-capture enabler, RFID technology intrigues supply chain researchers and practitioners. Companies have rushed to develop RFID solutions without having a clear idea about the potential value of RFID to their business. One of the values of RFID in supply chains is derived from better supply chain visibility. An RFID deployment improves supply chain visibility; however, many benefits in inventory control are still untapped. The value of RFID obtained from labor cost reductions and similar basic benefits can be satisfactorily assessed by case studies. Empirical studies and proofs of concepts are of limited scope since they have to rely on existing processes and data. It is not clear how RFID can further reduce supply chain costs via improved visibility. Educated guesses are currently driving such estimates. Analytically modeling inventory control systems with an RFID deployment is critical to enhance our understanding of the value of RFID.

Beyond replenishment inventory control on the supply side, the distribution side deals with shipping decisions. As is the case with the supply, RFID deployments yield real time visibility of shipments. To achieve such capability, it is typical to install portals with readers at important locations (e.g., in and/or outbound docks) and tag the corresponding goods. Consider an RFID mandate imposed by a retailer. A supplier places transponders on the products and ships them to the retailer's distribution centers, thus complying with the corresponding RFID mandate. Since typically transponders are affixed just before leaving the final facility of the supplier, this strategy is known as slap-and-ship. Clearly, to obtain further benefits from RFID, it is advised to push the tagging process further upstream in the supplier's own supply chain. The main drawback of slap-and-ship is the inability to produce return on investment. As already discussed, the retailer could benefit from continuously monitoring the inventory levels and outstanding order progress in its own chain. However, it is not clear how the supplier can benefit from the mandate even if real-time distribution information is provided by the retailer. This is a typical quote about such suppliers: "They (an apple supplier to Wal-Mart) know exactly what day and time the container was scanned through its portal, when it entered a distribution center and what day and time it went to the store. The company has yet to determine how best to use this data.", Inbound Logistics, June 2006. The main objective of this research is to show how suppliers can benefit under such circumstances even under slap-and-ship. We assume a decentralized system where each entity in the chain acts independently. The main entity is an installation somewhere in the middle of the entire chain. The firm makes two decisions: (1) the replenishment decision from its own supplier, and (2) the shipment decision how much to ship downstream.

We study a single-product, multi-echelon serial supply chain system, in which the supplier streamlines both its replenishment and distribution processes by using the RFID data. We propose a dynamic programming model to capture the real time inventory information generated from an RFID deployment across the entire supply chain. The paper is organized as follows. A comprehensive inventory model is first presented in Section 2. In addition, we provide a special case with an instantaneous replenishment process. We apply multi-echelon techniques to decompose the proposed model into two sub-problems. The optimal control policy under certain conditions is characterized as the echelon base stock policy. The value of RFID in a serial distribution process is clearly identified and rigorously proved in Section 3 through discussions of partial RFID deployment scenarios. We conclude the introduction with a brief literature review.

1.1. Literature Review

Our models assume stochastic lead times. In standard single-stage stochastic inventory models, the lead time is considered either as a known deterministic constant or a random variable with known distribution. In these models, Kaplan 1970, Nahmias 1979, Ehrhardt 1984, the lead time is assumed to be time-independent with known distribution. The non-crossover property is also assumed in order to make the study tractable. In our analysis, we borrow concepts from multi-echelon systems. The seminal work on the serial multi-echelon inventory problem was conducted by Clark and Scarf 1960. In their research, the global system is decomposed into separate sub-systems. At each echelon, it is optimal to follow the base stock policy with respect to the echelon inventory.

RFID is the most promising technology providing complete and comprehensive supply chain visibility. It is a surprisingly simple computing and communication architecture since only two basic building blocks are needed - a tag and a reader (AIM Inc., 1999, Clampitt et al. 2006). We have already argued that RFID is an enabling technology for visibility that is assumed by our model. There are estimates about the value of RFID in supply chain management, including labor cost savings, reduced inventory holding costs, and stock-outs, Lee and Özer 2007, Hardgrave 2005. Most studies regarding RFID in inventory control concentrate on supply chain simulations, Lee, Cheng and Leung 2004, Fleisch and Tellkamp 2005, Kang and Stanley 2005.

Bottani and Rizzi 2008 describe profitability of deploying RFID in a three-tier supply chain. They show by a real world case in a fast-moving consumer goods market the benefits of pallet-level tagging, and much more lenient results of case- and item-level tagging. Ustundag and Tanyas 2009 investigate impacts of different factors, such as product value, lead time, and uncertain demand on the supply chain cost performance at echelon levels in conjunction with RFID tagging.

Among the few studies that analytically deal with RFID in inventory control, Song and Zipkin 1996 provide a modeling framework for the inventory control problem with supply information. While this study dates back to pre-era of modern RFID, it requires data available only through today's RFID deployments. The replenishment lead time is time-dependent and evolves over time. We borrow their modeling technique and enrich their study by focusing on the distribution side, thus dealing with two concurrent decisions. We also present results addressing the value of RFID in such distribution systems. Gaukler, Özer and Hausman 2006 quantify the benefits of RFID in the supply system of a retailer who faces uncertain demand and the option of emergency orders. They develop an order progress information model to study the optimal policies for both regular and emergency orders, under the assumption of a single outstanding order. Szmerekovsky and Zhang 2008 discuss impacts of item-level RFID tagging in a two-tier system under vendor managed inventory. First, the demand processes are characterized in both RFID and non-RFID systems. Then they study the control policies for each entity. Second, they study channel coordination efforts through sharing of the RFID costs by comparing centralized and decentralized systems. Atali, Lee and Özer 2006 analytically study inventory inaccuracy, which is a joint effect of transaction errors, shrinkage, and misplacements. RFID yields more accurate inventory records and easier audits. Bottani, Montanari, and Rizzi 2009 examine the impact of RFID on out-of-stocks of promotional items in the fast-moving consumer goods context. They show that by reducing the main causes of unavailability of sales through RFID can yield substantial savings. In addition, a reengineering process is exploited to compare the reduction of stock-outs. Results of an experiment suggest that RFID has the potential to reduce losses and improve profits in fast-moving consumer goods.

New processes hinging on RFID are of particular interest. Expediting on the supply side is one such process that can benefit from RFID. Kim et al. 2006 address expediting strategies based on RFID data. Pricing strategies based on added value on goods are discussed in Schneider 2007. RFID is used as a technology to allow dynamic pricing.

2. Comprehensive Inventory Model

Most of prior inventory models deal with supply information, i.e., the focus is on the procurement side of supply chains. Such models are appropriate for retailers and the endpoints of supply chains. We extend these models to capture both supply and distribution sides. In this section, a comprehensive inventory control model is first proposed. It contains both the supply and distribution sides of the entire chain. Next, we give the state reduction result for the pure distribution setting (upstream installations are neglected). This result is then extended by including upstream installations. Finally, optimal control policies are studied.

Given several installations as shown in Figure 1, let u_i denote the i'th upstream installation and let d_j denote the j'th downstream installation. There are two special installations: d_n represents the point-of-sale installation, and $u_n = d_0$ represents the main decision making installation. It will be called the main installation. We can identify the main installation as a supplier to Wal-Mart and $d_1, \cdots, d_{n-1}$ as installations owned by Wal-Mart. The main installation ships orders to Wal-Mart by directly shipping through all these installations. Upstream installations $u_0, \cdots, u_{n-1}$ represent the supply side of the main installation. The on-hand inventory IO_t at the main installation, net inventory I_{t,d_n} at the point-of-sale, and the net inventory $I_{t,k}$ for every $k \in \{u_1, \cdots, u_{n-1}\} \bigcup \{d_1, \cdots, d_{n-1}\}$ describe the system. There are two important differences between standard multi-echelon inventory systems and our model. We focus on both the supply and distribution sides of a decentralized system. Traditional multi-echelon models deal only with the supply side of a centralized system. In a centralized system, holding cost is accounted for at each installation. Meanwhile, in our decentralized system, it is accounted for only at a single installation. We consider such a firm with its physical location corresponding to installation d_0, Figure 1. Downstream Installations $d_1, \cdots, d_n$ represent another player, namely the retailer and upstream installations $u_0, \cdots, u_{n-1}$ attribute to different players as well. If RFID is deployed end to end in this serial supply chain, then at any point in time, the location of all outstanding orders placed with the supplier corresponding to installation u_0 are known. Likewise, the location of outstanding shipments is also known at any point in time. RFID is an enabling technology for cost effective tracking of goods and as such it provides real-time visibility, including recording quantities of outstanding orders and shipments. The arcs in Figure 1 represent possible order/shipment movements in a time period. There is no order/shipment moving into an installation if no incoming arcs attached to the installation appear. Each installation has a single outgoing arc, which could also be a self loop representing the scenario of the order staying put. The outgoing arc from installation d_n represents the exogenous end customer demand D_t.

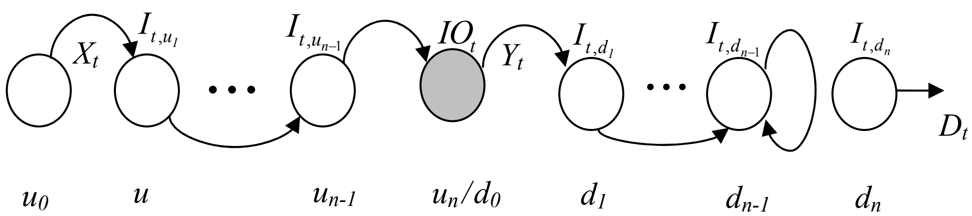

Figure 1: The Comprehensive Inventory Model

In order to model downstream shipment transitions, we introduce a random variable W. For each realization w of W there is a transition vector $M(w)$ that specifies where the current outstanding shipments at each stage move next. This vector has n elements and it encodes the transitions in downstream shipments. In Figure 2 we show two such possible transitions. Self loops correspond to the events that the outstanding shipments stay put at the installation. For example, under the top realization we have $I_{t+1,d_{n-1}} = I_{t,d_3} + I_{t,d_4}$ and under the bottom realization we obtain $I_{t+1,d_4} = I_{t,d_3} + I_{t,d_4}$. Similarly, we introduce a random variable R to encode the upstream stochastic order transitions. The corresponding transition function is denoted by $Q(r)$, where r is a realization of R.

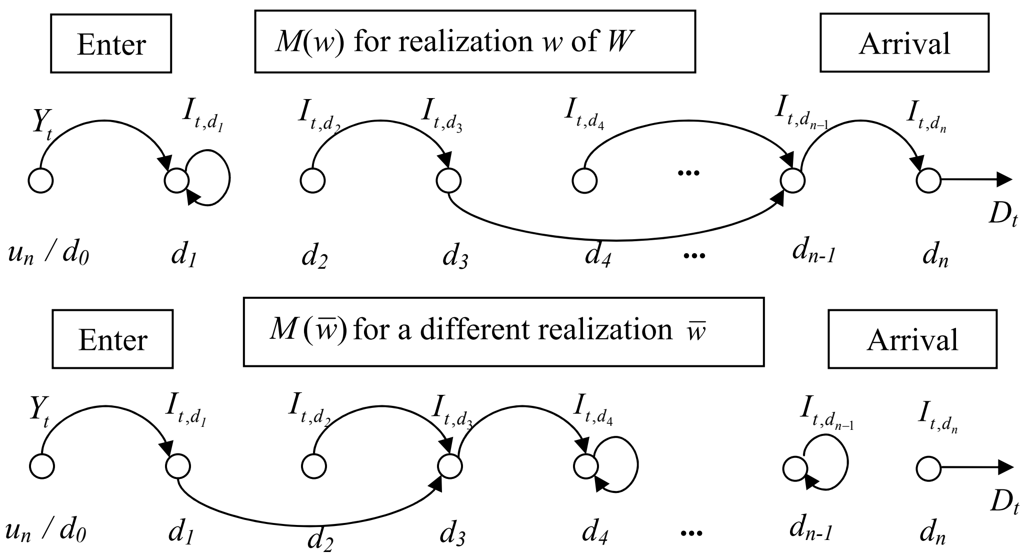

Figure 2: Two Possible Outstanding Shipment Transitions

2.1. Model

Throughout the document, we use the following notation.

C_t	Procurement cost function in time t
S_t	Shipping cost function in time t
h_t	Per unit holding cost in time t at the main installation u_n
p_t	Per unit shortage cost in time t occurred at the point-of-sale installation d_n

$Q(u_k, R)$ One step upstream transition at installation u_k, i.e., the coordinate of $Q(R)$ corresponding to installation u_k

$Q^p(u_k, R)$ p-step upstream transition at installation u_k, i.e., transition in p time periods corresponding to installation u_k

$M(d_k, W)$ One step downstream transition at installation d_k, i.e., the coordinate of $M(W)$ corresponding to installation d_k

$M^p(d_k, W)$ p-step downstream transition at installation d_k, i.e., transition in p time periods corresponding to installation d_k

L^u Upstream stochastic lead time $L^u = Min\ \{l > 1 : Q^l(u_0, R) = u_n\}$

L^d Downstream stochastic lead time $L^d = Min\ \{l > 1 : M^l(d_0, W) = d_n\}$

I_t State vector in time t: $I_t = (I_{t,u_1}, I_{t,u_2}, \cdots, I_{t,u_{n-1}}, IO_t, I_{t,d_1}, I_{t,d_2}, \cdots, I_{t,d_n})$

D_t Stochastic demand in time period t

D_t^{l-1} Cumulative demand over l periods starting from time t: $D_t^{l-1} = \sum_{k=t}^{t+l-1} D_k$

We make the following assumptions.

- We assume that installations u_0 and d_0 respectively denote the entering point for the replenishment and distribution process and installation d_n is the point-of-sale location. In addition, u_n and d_0 are physically the same installation.
- We assume that the unmet demand is backlogged at the point-of-sale installation d_n.
- Both upstream and downstream stochastic lead times are at least equal to or greater than 2. Formally, we require that $u_0 < Q(u_0, R) < u_n$ and $d_0 < M(d_0, W) < d_n$.
- We assume that the outstanding orders and shipments never cross in time. The current order and shipment sequence is preserved, or at least not reversed in both supply and distribution networks. Formally,

$$Q(u_k, r) \geq Q(u_{k-1}, r) \text{ for every } k = 1, 2, \cdots, n \text{ and any realization } r \text{ of } R,$$

$$M(d_k, w) \geq M(d_{k-1}, w) \text{ for every } k = 1, 2, \cdots, n \text{ and any realization } w \text{ of } W.$$

- Orders and shipments can not be broken apart. Once an order or shipment is placed, all corresponding units travel together.

Note that the lead times of greater than 1 are just a technical condition to rule out the trivial case. It requires that both inbound and outbound orders must go through at least one intermediate installation before arriving to the final destinations. In addition, it is easy to see that the non-crossover property implies that $Q^p(u_k, r) \geq Q^p(u_{k-1}, r)$ for every realization r of R, $k = 1, 2, \cdots, n,$ and $p = 1, 2, \cdots, L^u$, and $M^p(d_k, w) \geq M^p(d_{k-1}, w)$ for every realization w of W, $k = 1, 2, \cdots, n,$ and $p = 1, 2, \cdots, L^d$.

Within time period t, events occur as follows:

- the state vector I_t is observed at the beginning of time period t, next
- the shipping decision is made, then
- the replenishment decision is made,

- replenishment orders due in time period t arrive, and
- shipment orders due in time period t arrive,
- demand D_t in time period t occurs, and finally
- holding and penalty costs are assessed.

It is important to point that at the end of each time period the holding and penalty costs in the proposed model are evaluated differently from standard inventory models. In our proposed model, the holding cost is assessed at the main installation at the end of each time period. On the other hand, the penalty cost is evaluated at the point-of-sale installation where the customer demand might not have been satisfied. It is crucial to note that we only account for the costs incurred at the main installation (the supplier) even though there are other costs incurred by the remaining entities, for example the retailer. As for the replenishment orders and shipments, we charge the procurement and shipping costs when they are placed. We note that all these costs are actually incurred by the main installation (even though the penalty cost is accounted for at the point-of-sale installation). This reflects our initial assumption of a decentralized system focusing on the main installation. This also implies that there is no holding cost at other installations but $u_n = d_0$. Let $X_t \geq 0$ denote the procurement quantity and $Y_t \geq 0$ the shipping quantity in time period t, Figure 1. For ease of notation, we use $I_{t,u_0} = X_t$ and $I_{t,d_0} = Y_t$ in Section 2.

The underlying system dynamics are

$$IO_{t+1} = IO_t + \sum_{\substack{u_k : Q(u_k, R) = u_n \\ 1 \leq k \leq n-1}}^{u_{n-1}} I_{t,u_k} - I_{t,d_0},$$

and for $u_1 \leq u_i \leq u_{n\text{-}1}$ we have

$$I_{t+1,u_i} = \sum_{\substack{u_k : Q(u_k, R) = u_i \\ 0 \leq k \leq n-1}} I_{t,u_k}.$$

On the distribution side, for $d_1 \leq d_j \leq d_{n\text{-}1}$ we obtain

$$I_{t+1,d_j} = \sum_{\substack{d_k : M(d_k, W) = d_j \\ 0 \leq k \leq n-1}} I_{t,d_k}$$

and

$$I_{t+1,d_n} = I_{t,d_n} + \sum_{\substack{d_k : M(d_k, W) = d_n \\ 1 \leq k \leq n-1}} I_{t,d_k} - D_t.$$

Note that due to the non-crossover property, if $Q(u_k, R) = u_n$ for every k, $1 \leq k \leq n\text{-}1$, then $Q(u_{k+1}, R) = Q(u_{k+2}, R) = \cdots = Q(u_{n-1}, R) = u_n$. Similar property holds for M. The on-hand inventory IO_{t+1} in time t+1 equals the on-hand inventory in time t plus the outstanding orders which have just arrived at the main installation, minus the shipment Y_t made in time t. The on-hand inventory at installation u_i equals the arrived outstanding orders and potential X_t. Downstream installation inventory I_{t+1,d_j} in time t+1 is equal to the outstanding shipments which move to installation d_j plus shipment Y_t if transition $M(d_0, W) = d_j$ occurs. Finally, the net inventory at the point-of-sale installation in time t+1 equals the net inventory in time t plus the outstanding shipments, which arrive at the point-of-sale, minus the customer demand in time t.

Let $\widehat{V}_t$ be the expected total minimum system cost starting from time t up to T under an optimal policy. The optimality equation over finite time horizon $[0,T]$ reads

$$\widehat{V}_t(I_t) = \underset{\substack{I_{t,u_0}\geq 0 \\ I_{t,d_0}\geq 0 \\ I_{t,d_0}\leq IO_t}}{Min} \left\{ \begin{aligned} & C_t(I_{t,u_0}) + S_t(I_{t,d_0}) + h_t \cdot E^R(IO_t + \sum_{\substack{u_k:Q(u_k,R)=u_n \\ 1\leq k\leq n-1}}^{u_{n-1}} I_{t,u_k} - I_{t,d_0}) \\ & + p_t \cdot E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k:M(d_k,W)=d_n \\ 1\leq k\leq (n-1)}} I_{t,d_k})^+] + E^{D,W,R}[\widehat{V}_{t+1}(I_{t+1})] \end{aligned} \right\}. \quad (1)$$

The action space constraint $I_{t,d_0} \leq IO_t$ guarantees that we do not ship more than we have on hand.

This comprehensive model is complex. We instead first analyze a simplified version and tackle the comprehensive model after learning from the analysis in such a setting. In the following section, we study a special case of the comprehensive model where we neglect the replenishment stochastic lead time, i.e., the replenishment order is instantaneous. After better understanding such a distribution model, we decompose the comprehensive model into two problems, which can be handled separately. This decomposition is shown in Section 2.3.

2.2. Distribution Side

This section demonstrates how the supplier could make use of the real-time information to improve the distribution operations. Imagine that a supplier to Wal-Mart has access to the distribution information of its goods via Retailink (http://retaillink.wal-mart.com). In a traditional setting, only point-of-sale information is available. Soon after first RFID enabled pallets started arriving to the distribution centers, suppliers had complete visibility of their shipments through the same system.

To recapture, each shipment goes through a stream of installations labeled as d_0, d_1, $\cdots$, d_n. Each installation corresponds to a transportation node or a specific stockpiling point. Installation d_0 stands for the supplier itself, i.e., the main decision-making installation and, d_n represents the point-of-sale. Furthermore, each of the nodes is deployed with RFID. Without RFID, it would not be accessible to observe the installation inventories and the exogenous stochastic information W. To better understand the distribution process, instantaneous replenishment is assumed.

Two models are discussed in this section. The full model captures all installation inventories, while the reduced model only includes the inventory on-hand and the downstream echelon inventory with respect to the point-of-sale. The full model serves as the starting point and it models slap-and-ship. The reduced model is simpler, nevertheless, it is equivalent in a certain sense as shown here. Quantities V_t represent the expected total minimum cost over time periods $t, t+1, \cdots, T$ in the full model or the cost-to-go. Similarly, $\overline{V}_t$ represent the expected total minimum cost over time periods $t, t+1, \cdots, T$ in the reduced model. Since we do not consider the costs incurred beyond time point T, the terminal costs V_{T+1} and $\overline{V}_{T+1}$ are assumed to be zero.

An illustration of the full model is shown in Figure 3. In addition to previously defined terms, we depict that the order replenishment is instantaneous. The rest of the model configuration is identical to the comprehensive model.

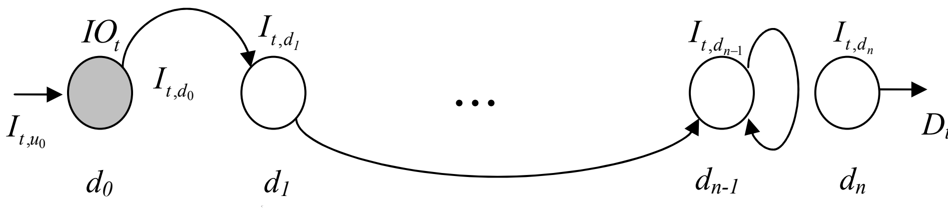

Figure 3: Full Model with Distribution Information

The optimality equation of the full model reads

$$V_t(\, I_{t,d}) = \underset{\substack{I_{t,u_0} \geq 0 \\ I_{t,d_0} \geq 0 \\ I_{t,d_0} \leq IO_t}}{\text{Min}} \left\{ \begin{aligned} & C_t(I_{t,u_0}) + S_t(I_{t,d_0}) + h_t(IO_t \; + I_{t,u_0} - I_{t,d_0}) \\ & + p_t \cdot E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k : M(d_k, W) = d_n \\ 1 \leq k \leq n-1}} I_{t,d_k})^+] + E^{D,W}[V_{t+1}(I_{t+1,d})] \end{aligned} \right\}, \tag{2}$$

where the state vector is $I_{t,d} = (IO_t, I_{t,d_1}, I_{t,d_2}, \cdots, I_{t,d_n})$. The underlying system dynamics equations are

$$\begin{aligned} IO_{t+1} &= IO_t + I_{t,u_0} - I_{t,d_0} \\ I_{t+1,d_j} &= \sum_{\substack{d_k : M(d_k, W) = d_j \\ 0 \leq k \leq n-1}} I_{t,d_k} \qquad d_1 \leq d_j \leq d_{n-1} \\ I_{t+1,d_n} &= I_{t,d_n} + \sum_{\substack{d_k : M(d_k, W) = d_n \\ 1 \leq k \leq n-1}} I_{t,d_k} - D_t. \end{aligned}$$

The first equation above is different from the corresponding equation in the comprehensive model since there is no lead time on the supply side. The inventory on-hand in time t+1 equals the inventory on-hand in time t plus instantaneous replenishment order I_{t,u_0}, minus the shipment I_{t,d_0} in time t. The remaining two equations are identical.

The full model contains many state variables. The multi-dimensional state vector results in difficulties when computing inventory control policies. We next introduce the reduced model with distribution information as shown in Figure 4. The reduced model has only two variables: the inventory on-hand IO_t and downstream echelon inventory $r_{t,1}$ with respect to the point-of-sale. Formally, we define $r_{t,1} = \sum_{k=1}^{n} I_{t,d_k}$.

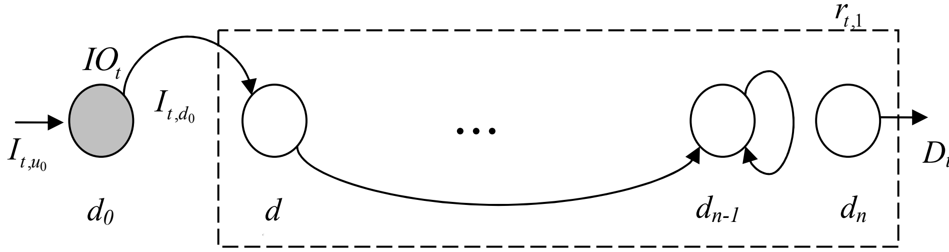

Figure 4: Reduced Model with Distribution Information

The optimality equation for the reduced model reads

$$\overline{V}_t(IO_t, r_{t,1}) = \underset{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}}{Min} \left\{ \begin{array}{l} C_t(I_{t,u_0}) + S_t(I_{t,d_0}) + h_t(IO_t + I_{t,u_0} - I_{t,d_0}) \\ + E^{L^d}(p_{t+l^d-1} \cdot E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t,1})^+]) + E^D[\overline{V}_{t+1}(IO_{t+1}, r_{t+1,1})] \end{array} \right\}, \quad (3)$$

where the underlying system dynamics equations are

$$IO_{t+1} = IO_t + I_{t,u_0} - I_{t,d_0}$$

$$\begin{aligned} r_{t+1,1} &= \sum_{k=1}^{n} I_{t+1,d_k} \\ &= I_{t+1,d_n} + \sum_{k=1}^{n-1} I_{t+1,d_k} \\ &= I_{t,d_n} + \sum_{\substack{d_k:M(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k} - D_t + \sum_{k=1}^{n-1} \sum_{\substack{d_g:M(d_g,W)=d_k\\ 0\le g\le n-1}} I_{t,d_g} \qquad (4) \\ &= I_{t,d_n} - D_t + \sum_{k=1}^{n} \sum_{\substack{d_g:M(d_g,W)=d_k\\ 0\le g\le n-1}} I_{t,d_g} \end{aligned}$$

(5)

$$\begin{aligned} &= I_{t,d_n} - D_t + I_{t,d_0} + \sum_{g=1}^{n-1} I_{t,d_g} \\ &= r_{t,1} + I_{t,d_0} - D_t. \end{aligned}$$

In (4) we use system dynamics equations for I_{t+1,d_i} for every $i = 1, \cdots, n$. The third term in (5) covers all the possibilities of shipment movements in time *t*. As a result, this term corresponds to the summation of all outstanding shipments, plus shipment I_{t,d_0}, which has just been made in time *t*.

Next, we state and prove the main result in this section. The following theorem, proved in Appendix 1, shows the relationship between the full and reduced models in the distribution process.

Theorem 1. We have

$$V_t(I_{t,d}) = \overline{V}_t(IO_t, r_{t,1}) + \alpha_t, \quad (6)$$

where $\alpha_t = E^{L^d} \left(\sum_{j=0}^{L^d-2} p_{t+j} \cdot E^{D,W}[(D_t^j - I_{t,d_n} - \sum_{p=1}^{j+1} \sum_{\substack{d_k:M^p(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+] \right)$ represents the expected total penalty cost over the shipping lead time.

By means of Theorem 1, we show that in a distribution process the original value function is divided into two parts: the reduced value function and a penalty cost term. This decomposition proves to be crucial in order to study the inventory control policies in Section 2.3. Accessing the echelon-level inventory statuses and the current inventory on-hand suffices to characterize the optimal polices since the penalty cost term only depends on the exogenous demand, shipment movements, and the current inventory. Furthermore, the penalty term is an explicit, closed form function.

2.3. Decomposition and Analysis of the Comprehensive Model

In the previous section, we discussed in detail the distribution process with a complete RFID deployment. By a system-wise deployment of RFID both supply and distribution sides can be tracked. The comprehensive inventory model with both supply and distribution information has already been presented in Section 2.1. In this section, the system is decomposed into two sub-problems in terms of the value function. Under some assumptions the inventory control policy of each sub-problem is determined separately. The optimal system control policy is the echelon base stock policy with two threshold numbers, as shown later in this section.

By applying the same techniques as those presented in Section 2.2 with respect to (1), we obtain the following result.

Theorem 2. We have

$$\hat{V}_t(I_t) = \alpha_t + \underset{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}}{Min} \left\{ \begin{array}{l} C_t(I_{t,u_0}) + S_t(I_{t,d_0}) + h_t \cdot E^R(IO_t + \sum\limits_{\substack{u_k:Q(u_k,R)=u_n\\ 1\le k\le n-1}}^{u_{n-1}} I_{t,u_k} - I_{t,d_0}) \\ +E^{L^d}(p_{t+L^d-1} \cdot E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t,1})^+]) + E^{D,W,R}[\tilde{V}_{t+1}(I_{t+1})] \end{array} \right\}, \qquad (7)$$

where $\alpha_t = E^{L^d}\ (\sum\limits_{j=0}^{L^d-2} p_{t+j} \cdot E^{D,W}[(D_t^j - I_{t,d_n} - \sum\limits_{p=1}^{j+1} \sum\limits_{\substack{d_k:M^p(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+]\ )$ and $r_{t,1}$ is defined in Section 2.2.

To derive an optimal policy, we focus on the minimization part in (7). We first define $N(u_i) = \min\{u_k : Q(u_k,R) = u_i, 0 \le k \le n-1\}$ for every $i = 1,2,\cdots,n$. We denote $IO_t^e = IO_t + r_{t,1}$, and $I_{t,u_i}^e = \sum\limits_{k=i}^{n-1} I_{t,u_k} + IO_t^e$ for every $i = 0,1,2,\cdots,n-1$. We represent the system state as $I_t^e = (I_{t,u_1}^e,\cdots,I_{t,u_{n-1}}^e, IO_t^e, r_{t,1})$. Next, we rewrite (7) in the echelon version as

$$\tilde{V}_t(I_t^e) = \underset{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t^e - r_{t,1}}}{Min} \left\{ \begin{array}{l} C_t(I_{t,u_0}) + S_t(I_{t,d_0}) + h_t \cdot E^R(I_{t,N(u_n)}^e - r_{t,1} - I_{t,d_0}) \\ +E^{L^d}(p_{t+L^d-1} \cdot E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t,1})^+]) + E^{D,W,R}[\tilde{V}_{t+1}(I_{t+1}^e)] \end{array} \right\}. \qquad (8)$$

In (8), the system dynamics are expressed as

$$r_{t+1,1} = r_{t,1} + I_{t,d_0} - D_t,$$
$$IO_{t+1}^e = I_{t,N(u_n)}^e - D_t,$$

and for every $i = 1,2,\cdots,n-1$ we have

$$I_{t+1,u_i}^e = I_{t,N(u_i)}^e - D_t.$$

Theorem 3. If the shipping cost functions S_t for $t = 1,\cdots,T$ are linear and $S_t \ge h_t$, then

$$\tilde{V}_t(I_t^e) = \bar{V}_t(r_{t,1}) + G_t(I_{t,u_1}^e,\cdots,I_{t,u_{n-1}}^e, IO_t^e)\ . \qquad (9)$$

In (9), the downstream sub-system follows

$$\bar{V}_t(\ r_{t,1}) \equiv \underset{r_{t,d_0}\ge r_{t,1}}{Min} \left\{ \begin{array}{l} S_t \cdot (r_{t,d_0} - r_{t,1}) - h_t \cdot r_{t,d_0} \\ +E^{L^d}(p_{t+L^d-1} \cdot E^D[(D_t^{L^d-1} - r_{t,d_0})^+]) + E^D[\bar{V}_{t+1}(\ r_{t+1,1})] \end{array} \right\}, \qquad (10)$$

where $r_{t+1,1} = r_{t,d_0} - D_t$ and action r_{t,d_0} is the downstream echelon inventory level after shipping in time *t*. Function G_t is a function of only the upstream echelon inventories. Moreover, the optimal decision rule in the downstream sub-system corresponding to shipping quantities follows the base-stock policy with the base stock levels r^*_{t,d_0} derived appropriately from (10).

Proof: The proof is provided in Appendix 2.

In Theorem 3, we show how to simplify the value function of the comprehensive model. We now discuss the underlying replenishment inventory control policy.

Corollary 1: If the procurement cost functions C_t for $t = 1, \cdots, T$ are all linear in addition to S_t being linear and $S_t \geq h_t$ for $t = 1, \cdots, T$, then the replenishment decision also follows the base stock policy.

Proof: Since the value function can be decomposed as in (9) and the sub-systems $\overline{V_t}$ and G_t are separable in terms of the state variables, we borrow concepts from the multi-echelon inventory control techniques, Scarf 1960. We first obtain the optimal downstream echelon stock levels r^*_{t,d_0} as discussed in Theorem 3. We next move upstream to focus on the replenishment process. It is easy to see by applying convexity that all G_t's are convex. Based on (19) in Appendix 2, we obtain the upstream base stock level r^*_{t,u_0} in time *t* as in Song and Zipkin 1996. If $I^e_{t,u_1} < r^*_{t,u_0}$, the optimal order quantity is $r^*_{t,u_0} - I^e_{t,u_1}$; otherwise, the optimal order quantity is zero. This completes the proof.

Under the conditions stated in Corollary 1, the optimal control policy for the comprehensive inventory model proposed in Section 2.1 follows the echelon base stock policy and has two threshold numbers r^*_{t,d_0} and r^*_{t,u_0} in any given time period *t*.

3. Value of RFID in Distribution

In order to identify the value of RFID in inventory control systems, we study partial RFID deployment scenarios, in which only some installations of the chain are covered by RFID. As a result, there is a possibility that the supplier does not know the location of shipments once they leave the RFID-enabled installations. We differentiate two types of shipments in transit: those with both location and age information and those with only age information. Note that the age is the number of periods since the order was shipped out from the supplier at installation d_0. By comparing the systems with different scope of RFID deployments, we are able to analytically identify the value of RFID in the system. For simplicity, we focus on the downstream distribution side, i.e., the replenishment lead time is zero.

We study the finite time horizon inventory problem with *T* time periods. In this section, two partial RFID deployment scenarios are proposed, as shown in Figure 5 and Figure 7. Scenario 1 represents the case where the first *s* downstream installations are RFID enabled, while in scenario 2 an additional downstream installation d_{s+1} is deployed with RFID. We assume that the remaining configuration in the two scenarios is identical with the aim to isolate the contribution of RFID. Furthermore, both time-labeled (age of outstanding shipments) and location-labeled inventory information is included in the two partial scenarios, while only location inventory information is captured in the complete RFID deployment model discussed in Section 2. RFID system deployed in the first s installations provides information of both types, however, only the time-labeled inventory information within the non-RFID zone is used since there the location cannot be captured.

3.1. Two Partial RFID Deployment Scenarios

In order to capture ages of outstanding shipments, the following new notation is required. In the first s installations the system dynamics are as in Section 2. Installations $d_{s+1}, d_{s+2}, \cdots, d_n$ do not have RFID and thus only age is captured. Once a shipment leaves installation d_s, the location and its progress is no longer known, only age is available. Shipments then arrive randomly based on age as in Kaplan 1970.

q_j	Probability that outstanding shipments, which are j periods old or more, arrive during the current time period
J	Random variable with the distribution based on q_j
$R_{t,k}$	Amount of the outstanding shipments in time t placed k periods ago; we set $R_{t,k} = 0$ if such shipments have arrived before time t.
$A_{t,j}$	Age of the "oldest" shipment at installation d_j in time t
$a_{t,j}$	Age of the "youngest" shipment at installation d_j in time t
$m(t)$	$\max\{A_{t,1}, A_{t,2}, \cdots, A_{t,s}\}$
$m'(t)$	$\max\{A_{t,1}, A_{t,2}, \cdots, A_{t,s}, A_{t,s+1}\}$; clearly, we have $m'(t) \geq m(t)$.
V_t^s	The value function with RFID deployed in the first s installations
I_t^s	State vector in time t with RFID deployed in the first s installations, $I_t^s = (IO_t, I_{t,d_n}, a_{t,1}, \cdots, a_{t,s}, A_{t,1}, \cdots, A_{t,s}, R_{t,1}, \cdots, R_{t,T-1})$

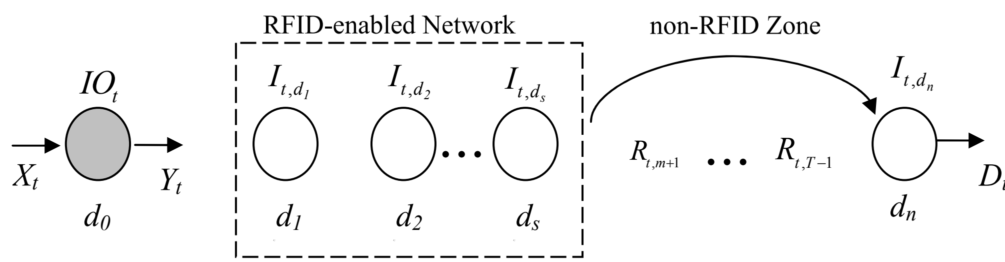

Figure 5: Partial RFID Implementation in Scenario 1

If no outstanding shipments appear at installation d_k, we set $a_{t,k} = A_{t,k} = 0$. We require $R_{t,0} = 0$. The state space ζ_t^s under s RFID-enabled installations in time t is defined as follows. Vector $I_t^s \in \zeta_t^s$ if and only if:

- $IO_t \geq 0$,

- for every $i = 1, 2, \cdots, s$ we have $0 \le a_{t,i} \le A_{t,i}$,
- if for any $i = 1, 2, \cdots, s$ it holds that $a_{t,i} = 0$, then $A_{t,i} = 0$,
- if for any $i = 1, 2, \cdots, s$ such that $a_{t,j} > 0$, then $R_{t,a_{t,i}} > 0$,
- if for any $i = 1, 2, \cdots, s$ such that $A_{t,i} > 0$, then $R_{t,A_{t,i}} > 0$,
- if for every pair (i, j) with $1 \le i < j \le s$, it holds (1) $A_{t,i} > 0$, $a_{t,j} > 0$ and (2) for every k with $i < k < j$ has $a_{t,k} = A_{t,k} = 0$, then we have $A_{t,i} \le a_{t,j} - 1$ and $R_{t,a_{t,k}} = R_{t,A_{t,k}} = 0$.

We observe that the inventory at installation d_i for every $1 \le i \le s$ is given by $\sum_{k=a_{t,i}}^{A_{t,i}} R_{t,k}$. Suppose $1 \le i < j \le s$. Given any two consecutive installations d_i and d_j with positive inventory, as shown in Figure 6, we have $0 < a_{t,i} \le A_{t,i} < a_{t,j} \le A_{t,j}$ and for every $l, A_{t,i} < l < a_{t,j}$ we have $R_{t,l} = 0$.

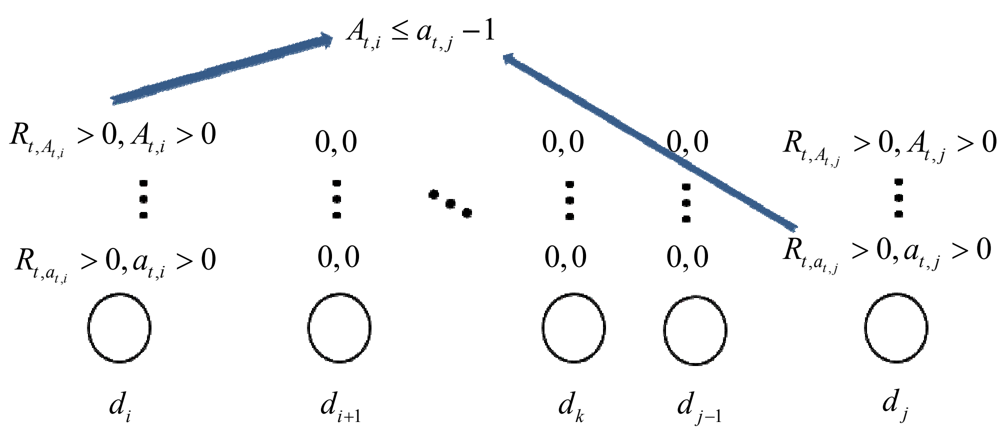

Figure 6: An illustration of the action space of scenario 1 in time t

The main goal in this section is to show that $V_t^{s+1} \le V_t^s$. This clearly shows that by installing RFID at installation d_{s+1} the system cost is reduced. We call the system with s, s+1 RFID enabled installations scenario 1, scenario 2, respectively. In addition, we call installations d_s, d_{s+1} the border in scenarios 1, 2, respectively. We make the following assumptions with respect to scenario 1.

- Transitions moving from non-RFID zone beyond installation d_s across the border back into the RFID-enabled network are not allowed.
- Random vector W describing transitions in the RFID-enabled portion is conditioned on random variable J describing the non-RFID area. In other words, we first have the observation of random variable J and then the realization of W must be compatible with J in terms of the shipment transitions. For example, if a realization of J requires all shipments of age p to arrive at d_n and for installation $k<s$ we have $a_{t,k} \le p \le A_{t,k}$, then clearly W cannot lead shipments at locations d_i for $k \le i \le s$ to a different installation but d_n.

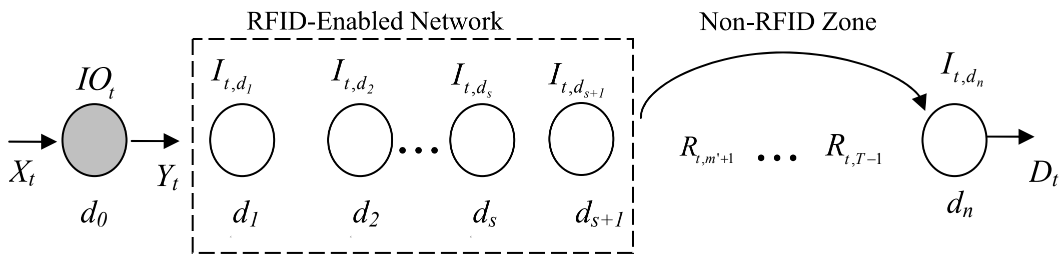

Figure 7: Partial RFID Implementation in Scenario 2

The shipment transitions originating from the RFID-enabled installations depend on the exogenous random vector W. The random vector W assures location-labeled installation inventory information in each time period. The difference from Section 2 is in the fact that the shipments governed by W have to be compatible with J. As seen in Figure 8, we first observe a realization j of J. It is obvious that all the outstanding shipments with age at least j must accordingly arrive at installation d_n. The realization of W provides additional information about the shipment transitions among installations d_0, d_1, d_2. The top portion of Figure 8 represents a possible and compatible shipment transition case, while the bottom portion represents a case in which the shipment transitions based on W and J are incompatible.

The system dynamics for scenario 1 are divided into two groups. Part I shows the transitions of installation inventories and outstanding shipments, while part II demonstrates the age transitions at installations. If j is a realization of J, they read

$$\begin{cases} IO^s_{t+1} = IO_t + X_t - Y_t \\ I^s_{t+1,d_n}(j) = \begin{cases} I_{t,d_n} + \sum\limits_{k=a_{t,i}}^{T-1} R_{t,k} - D_t & 1 \le a_{t,i} \le j \le A_{t,i} \le m(t) \text{ and } 1 \le i \le s \\ I_{t,d_n} + \sum\limits_{k=a_{t,i'}}^{T-1} R_{t,k} - D_t & 1 \le A_{t,i} < j < a_{t,i'} \le m(t) \text{ and for every } k \text{ with} \\ & 1 \le i < k < i' \le s \text{ we have } a_{t,k} = A_{t,k} = 0 \\ I_{t,d_n} + \sum\limits_{k=j}^{T-1} R_{t,k} - D_t & j \ge m(t)+1 \end{cases} \\ R^s_{t+1,1}(j) = Y_t \\ R^s_{t+1,k}(j) = \begin{cases} R_{t,k-1} & j > m(t) \text{ and } 2 \le k \le j \\ 0 & m(t) < j < k \text{ and } k \ge 2 \\ R_{t,k-1} & 2 \le k \le a_{t,g} \le j \le m(t) \\ 0 & \text{otherwise when } k \ge 2, \end{cases} \end{cases} \tag{I}$$

$$\begin{cases} a^s_{t+1,i}(j) = \begin{cases} \min\{a_{t,k}+1 : M(d_k,w)=d_i, d_0 \le d_k \le d_{g-1}(j), R_{t,a_{tk}} > 0\} & 1 \le i \le s \text{ and } 1 \le j \le m(t) \\ \min\{a_{t,k}+1 : M(d_k,w)=d_i, d_0 \le d_k \le d_s, R_{t,a_{tk}} > 0\} & 1 \le i \le s \text{ and } j > m(t) \end{cases} \\ A^s_{t+1,i}(j) = \begin{cases} \max\{A_{t,k}+1 : M(d_k,w)=d_i, d_0 \le d_k \le d_{g-1}(j), R_{t,A_{tk}} > 0\} & 1 \le i \le s \text{ and } 1 \le j \le m(t) \\ \max\{A_{t,k}+1 : M(d_k,w)=d_i, d_0 \le d_k \le d_s, R_{t,A_{tk}} > 0\} & 1 \le i \le s \text{ and } j > m(t). \end{cases} \end{cases} \quad \text{(II)}$$

Here, we denote $g(j) = \underset{i}{\operatorname{argmax}}\{a_{t,i} \mid j \ge a_{t,i}\}$. In (II), if the minimum or maximum is over an empty set, we define the corresponding quantity $a^s_{t+1,i}(j)$ or $A^s_{t+1,i}(j)$ to be 0.

The interface is defined as the position of the youngest outstanding shipment which arrives at installation d_n at the end of time t. In $I^s_{t+1,d_n}(j)$, $a^s_{t+1,i}(j)$, and $A^s_{t+1,i}(j)$ we distinguish two cases. The first case $1 \le j \le m(t)$ (for $I^s_{t+1,d_n}(j)$ it actually consists of two sub-cases) corresponds to the case where the interface is in the RFID-enabled area. Note that $m(t)$ is the oldest age in the RF-ID-enabled area and j is a realization of J, which implies that all shipments of age equal to or more than j arrive at d_n. It follows that the shipments arriving at d_n at the end of time period t include shipments in the RFID-enabled area and all outstanding shipments in the non-RFID zone because of the non-crossover property. In the other case of $j > m(t)$, the interface is beyond the RFID-enabled area. The arriving shipments at installation d_n in time period t are only from the non-RFID zone. We similarly distinguish two cases in the remaining equations. In $R^s_{t+1,k}(j)$ for $j > m(t)$, all outstanding shipments which are j periods old arrive in time t. However, in the case of $j \le m(t)$, outstanding shipments which are $a_{t,g}(j)$ old or more arrive because we have more detailed information from RFID and shipments are not broken. By definition, $g(j)$ describes the installation beyond which all outstanding shipments arrive during current time period t.

The optimality equation for partial RFID deployment scenario 1 reads

$$V^s_t(I^s_t) = \underset{\substack{X_t \ge 0 \\ X_t \ge 0 \\ Y_t \le IO_t}}{Min} \begin{Bmatrix} C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t) \\ + p_t \cdot \sum_{j=1}^{m(t)} q_j \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+] \\ + p_t \cdot \sum_{j=m(t)+1}^{T-1} q_j \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \\ + \sum_{j=1}^{T-1} q_j \cdot E^{D,W/J=j}[V^s_{t+1}(I^s_{t+1})] \end{Bmatrix}. \quad (11)$$

Notation $W/J = j$ denotes the random variable W conditional on a realization j of J.

In order for the optimality equation to be well defined, we need to show the following lemma, which is proved in Appendix 3.

Lemma 1. If $I^s_t \in \zeta^s_t$, then $I^s_{t+1} \in \zeta^s_{t+1}$.

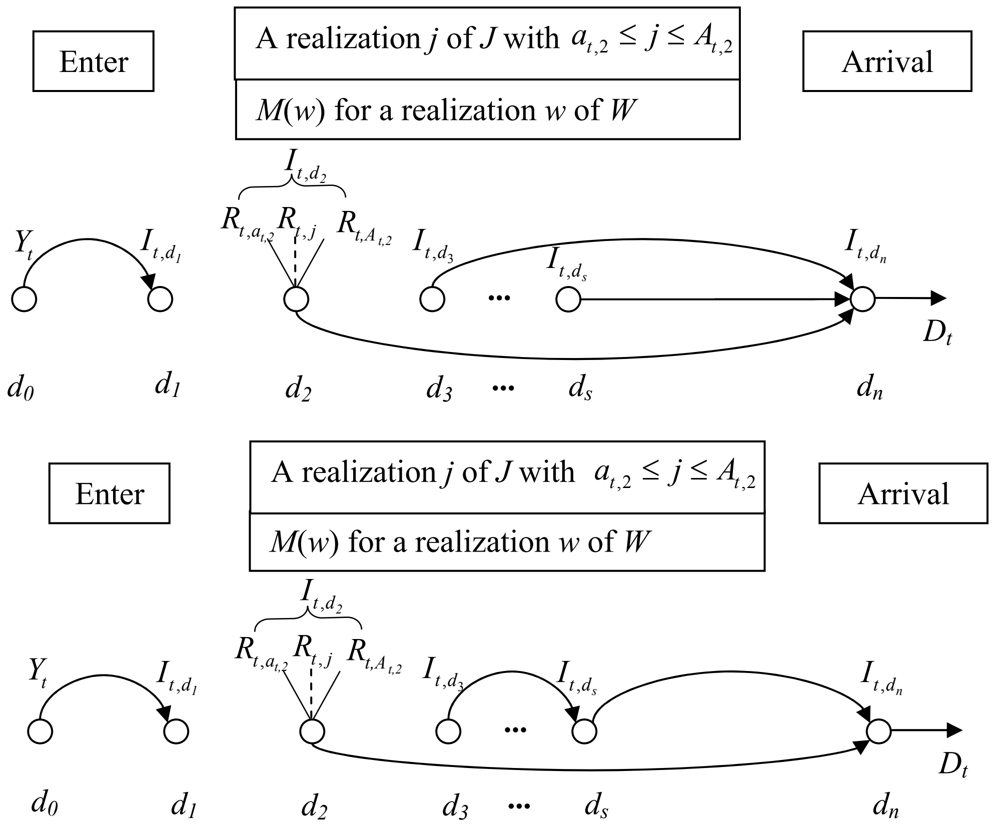

Figure 8: Shipment Transition Compatibility between W and J

In the partial RFID deployment scenario 2, the RFID-enabled network is extended to capture the information of one more downstream installation d_{s+1}. It is clear that state vector I_t^{s+1} in scenario 2 has two more coordinates $a_{t+1,s+1}^{s+1}$ and $A_{t+1,s+1}^{s+1}$. The state transitions in scenario 2 are also divided into two parts. Part I captures the same transitions of inventories as in scenario 1, except that we replace s with $s+1$ and $m(t)$ with $m'(t)$ in the state equations for scenario 2. Part II equations of system dynamics regarding $a_{t+1,i}^{s+1}$ and $A_{t+1,i}^{s+1}$ for $i = 1, 2, \cdots, s$ have exactly the same transitions as shown in scenario 1 due to the assumption of restricted backward movements from non-RFID area beyond installation d_{s+1} to RFID-enabled area. The additional transitions special to scenario 2 regarding installation d_{s+1} read

$$a_{t+1,s+1}^{s+1}(j) = \begin{cases} \min\{ a_{t,k} + 1 : M(d_k, w) = d_{s+1};\ d_0 \le d_k \le d_{g-1}(j),\ R_{t,a_{tk}} > 0\} & 1 \le j \le m'(t) \\ \min\{ a_{t,k} + 1 : M(d_k, w) = d_{s+1};\ d_0 \le d_k \le d_{s+1},\ R_{t,a_{tk}} > 0\} & j > m'(t) \end{cases}$$

$$A_{t+1,s+1}^{s+1}(j) = \begin{cases} \max\{ A_{t,k} + 1 : M(d_k, w) = d_{s+1};\ d_0 \le d_k \le d_{g-1}(j),\ R_{t,A_{tk}} > 0\} & 1 \le j \le m'(t) \\ \max\{ A_{t,k} + 1 : M(d_k, w) = d_{s+1};\ d_0 \le d_k \le d_{s+1},\ R_{t,A_{tk}} > 0\} & j > m'(t). \end{cases}$$

We accordingly replace s with $s+1$ and $m(t)$ with $m'(t)$ in the optimality equation for partial scenario 2.

Shipments moving to installation d_{s+1} at the end of time period t are only from the RFID-enabled area. They can come from installation d_{s+1} itself if the outstanding shipments at installation d_{s+1} in time period t stay put. Similarly, as before, we distinguish two cases: $j \le m^{'}(t)$ and $j > m^{'}(t)$. Note that starting with the same outstanding shipments in time period t in scenario 1 and 2, the two systems may possibly end up with different states in time period t+1. From here on, we denote the state difference between scenario 1 and 2 by using the superscripts s and s+1. For example, $a^{s}_{t+1,i}$ denotes the youngest age of outstanding shipments at installation d_i in time period t+1 corresponding to scenario 1. Likewise, $A^{s+1}_{t+1,i}$ denotes the oldest age of outstanding shipments at installation d_i in time period t+1 corresponding to scenario 2.

3.2. Value of RFID in the Serial Distribution Process

Intuitively, the broader the RFID deployment is in the distribution network, the greater should the benefits be. If the RFID deployment and maintenance costs are neglected, we are able to show that additional benefits are obtained by a broader RFID deployment because of availability of additional information. We compare two partial RFID deployment scenarios, based on identical model configurations and assumptions, e.g., the same lead time distributions and non-crossover of the outstanding shipments in time.

The following is our main result.

Theorem 4. For any $I^{s}_{t} \in \zeta^{s}_{t}$ and $(a_{t,s+1}, A_{t,s+1})$, we have

$$V^{s}_{t}(I^{s}_{t}) \ge V^{s+1}_{t}(IO_t, I_{t,d_n}, a_{t,1}, \cdots, a_{t,s}, a_{t,s+1}, A_{t,1}, \cdots, A_{t,s}, A_{t,s+1}, R_{t,1}, \cdots, R_{t,T-1}). \tag{12}$$

The proof of this result is very technical and is provided in Appendix 4. It clearly states that under optimal policies, the total system cost of s+1 RFID deployments is equal to or lower than the cost of having only s RFID deployments. In other words, RFID reduces the total inventory cost. A weakness of our result is the fact that the deployment cost in not taken into consideration. The significance of the cost reduction depends on the RFID deployments.

4. Concluding Remarks

After the initial lab experiments and case studies, the real breakthrough of RFID applications came with the RFID mandates imposed by Wal-Mart and the U.S. Department of Defense. Most of their suppliers are required to apply at least pallet level RFID transponders to the products being shipped to selected distribution centers. The well-known slap-and-ship strategy provides valuable information and benefits to the retailers. However, it is much harder to identify a return on investment for suppliers. We first present a comprehensive inventory model and show that the underlying multivariate cost-to-go decomposes into two lower dimensional functions. Under certain assumptions, the optimal control policy for both replenishments and shipments is obtained. Furthermore, we analytically show that larger RFID deployments yield reduced overall expected cost. This clearly establishes that there is potential benefit of using RFID if novel processes are used. There are several important contributions of this work. Prior research mainly studies applications of RFID internally within a company or inventory control problems in a multi-firm setting with RFID supply information. This research is the first one to analytically study the inventory control policies with explicit RFID information at the distribution side. This is a nontrivial task since we have to simultaneously deal with two actions (replenishment and shipping quantities). Another contribution is, first, modeling a system with only a partial RFID deployment, and, second, comparing the expected cost of the resulting models with respect to the extent of the RFID deployment. Perhaps the most important contribution concerns the next generation business applications built on top of RFID data. We show that even by employing slap-and-ship, it is possible to generate additional value. This research is closing the gap between the ever increasing supply chain data and challenging analytical tools to process such information. The main contribution of our work is to establish models and policies for slap-and-ship that benefit suppliers.

It is clear that in short-term slap-and-ship provides a quick and relatively low cost solution compared with a full-scale RFID deployment to achieve compliance with a retailer's mandate. On the other hand, experiences from slap-and-ship can shape suppliers' future internal deployments and even further upstream, where suppliers can reap all the benefits from RFID. In this work, we show such possibility by an analytical approach exploiting the vast richness of RFID data. We first propose a comprehensive inventory model and then evaluate a serial distribution process with RFID deployments, which mimics the slap-and-ship processes of suppliers facing RFID mandates. The inventory control problem in the serial distribution chain is modeled as a dynamic program. Furthermore, the original problem is reduced in size to a simpler model without losing optimality. Based on this state reduction result, the optimal inventory control policy is the echelon base stock policy. It establishes that by knowing the downstream flow of goods through the distribution channel the supplier can improve decision making with respect to inventory control. The proposed comprehensive inventory model contains both supply and distribution information in serial systems. It asserts that armed with entire supply chain status information from using RFID, suppliers could better manage their internal and external processes and make integrated decisions that would improve their bottom line.

The value of RFID in inventory control systems is also discussed in detail within the studied context. It is usually claimed that slap-and-ship is a cost-bearing solution. We argue that beyond the cost associated with slap-and-ship, suppliers could indeed find the benefits of RFID deployments downstream. We show analytically how RFID can improve the system performance in terms of the expected overall cost over a finite time horizon. The partial RFID deployment scenario with $s+1$ installations covered by RFID yields lower procurement and shipping costs than the costs based on s installations covered by RFID. The cost difference can be regarded as the true value of RFID in the system, setting aside the RFID system deployment and maintenance costs.

The most important revelation and message of our paper is the fact that even by employing slap-and-ship, which is a basic deployment, benefits in inventory control are possible. Such benefits could offset the underlying costs and thus establish the elusive ROI for an RFID deployment. Clearly a positive ROI is not possible by simply slapping the tag and then shipping. An information system with analytical capabilities demonstrated here must be put together. RFID leaves an enormous trail of data that can be creatively analyzed and used to improve decision making. The results in Section 3 clearly demonstrate that wider RFID deployments bring additional benefits. Firms must carefully leverage deployment costs vs. benefits when deciding the scope of an RFID deployment. We demonstrated that additional benefits are definitely possible with broader deployments.

A drawback of our study is that the actual deployment costs are not explicitly captured. We do not claim of showing a positive ROI, but without the analytical aspects presented herein, a positive ROI is definitely not possible under the slap-and-ship strategy.

Many suppliers to Wal-Mart are struggling with the underlying costs of slap-and-ship. Through its power, Wal-Mart is able to dictate the RFID terms, while risking insolvency of smaller suppliers. Instead of these relentless pressures, Wal-Mart could more tightly collaborate with suppliers by using its tremendous IT resources and knowledge. To this end, it should show the suppliers how to gain benefits from its RFID mandate, and not only incur cost. By using the presented findings, the big-box retailer could show the suppliers how to improve inventory management simply by using the feedback data from Wal-Mart; thus not going deeper into the processes of the suppliers and more complex deployments of this pervasive technology.

With the real-time information generated from RFID, many new research directions are possible. This research addresses a partial RFID deployment issue in order to study the value of RFID in inventory control. We assume RFID is deployed from the supplier downstream. An alternative setting not addressed here is to start deploying RFID at the point-of-sale and expanding it upward. It would be interesting to investigate the system differences between a forward and backward deployment. All partial RFID deployment scenarios in this work are based on perfect RFID read rates. This immediately raises the question of the impact of imperfect RFID read rates on system performance.

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Appendix 1

Proof of Theorem 1: We use induction. We define $\alpha_{T+1}=0$. Since all the terminal costs are zero, we clearly have $V_{T+1}=\overline{V}_{T+1}+\alpha_{T+1}=0$.

We assume that (6) holds for t+1. In other words,

$$V_{t+1}(I_{t+1,d})=\overline{V}_{t+1}(IO_{t+1},r_{t+1,1})+\alpha_{t+1},$$

where $\alpha_{t+1}=E^{L^d}(\sum_{j=0}^{L^d-2}p_{t+j+1}\cdot E^{D,W}[(D_{t+1}^{j}-I_{t+1,d_n}-\sum_{p=1}^{j+1}\sum_{\substack{d_k:M^p(d_k,W)=d_n\\1\le k\le n-1}}I_{t+1,d_k})^+])$. Based on the system dynamics of the full model, we have

$$\begin{aligned}\sum_{\substack{d_k:M^p(d_k,W)=d_n\\1\le k\le n-1}}I_{t+1,d_k}&=\sum_{\substack{d_k:M^p(d_k,W)=d_n\\1\le k\le n-1}}\sum_{\substack{d_g:M(d_g,W)=d_k\\0\le k\le n-1}}I_{t,d_g}\\&=\sum_{\substack{d_g:M^{p+1}(d_g,W)=d_n\\0\le k\le n-1}}I_{t,d_g}.\end{aligned}\tag{13}$$

Next, we manipulate α_{t+1}. We substitute $I_{t+1,d_n}=I_{t,d_n}+\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-D_t$ and use (13) in α_{t+1} to obtain

$$\begin{aligned}\alpha_{t+1}&=E^{L^d}\left\{\sum_{j=0}^{L^d-2}p_{t+j+1}\cdot E^{D,W}[(D_t^{j+1}-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-\sum_{p=1}^{j+1}\sum_{\substack{d_g:M^{p+1}(d_g,W)=d_n\\0\le k\le n-1}}I_{t,d_g})^+]\right\}\\
&=E^{L^d}\left\{\begin{array}{l}\sum_{j=0}^{L^d-3}p_{t+j+1}\cdot E^{D,W}[(D_t^{j+1}-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-\sum_{p=1}^{j+1}\sum_{\substack{d_g:M^{p+1}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g})^+]\\+p_{t+L^d-1}\cdot E^{D,W}[(D_t^{L^d-1}-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-\sum_{p=1}^{L^d-1}\sum_{\substack{d_g:M^{p+1}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g}-I_{t,d_0})^+]\end{array}\right\}\\
&=E^{L^d}\left\{\begin{array}{l}\sum_{i=1}^{L^d-2}p_{t+i}\cdot E^{D,W}[(D_t^{i}-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-\sum_{q=2}^{i+1}\sum_{\substack{d_g:M^{q}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g})^+]\\+p_{t+L^d-1}\cdot E^{D,W}[(D_t^{L^d-1}-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n\\1\le k\le n-1}}I_{t,d_k}-\sum_{q=2}^{L^d}\sum_{\substack{d_g:M^{q}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g}-I_{t,d_0})^+]\end{array}\right\}\\
&=E^{L^d}\left\{\begin{array}{l}\sum_{i=1}^{L^d-2}p_{t+i}\cdot E^{D,W}[(D_t^{i}-I_{t,d_n}-\sum_{q=1}^{i+1}\sum_{\substack{d_g:M^{q}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g})^+]\\+p_{t+L^d-1}\cdot E^{D,W}[(D_t^{L^d-1}-I_{t,d_n}-\sum_{q=1}^{L^d}\sum_{\substack{d_g:M^{q}(d_g,W)=d_n\\1\le k\le n-1}}I_{t,d_g}-I_{t,d_0})^+]\end{array}\right\}.\end{aligned}$$

In α_{t+1} above, the summation over j is decomposed into two groups: from 0 to L^d -3 and $j=L^d$ -2. Moreover, the non-crossover property is applied for $j=L^d$ -2 as follows. Term I_{t,d_0} can be taken out and term $\sum_{\substack{d_g:M^{L^d}(d_g,W)=d_n \\ 1\le g\le n-1}} I_{t,d_g}$ can be folded into the summation term $\sum_{q=1}^{L^d}\sum_{\substack{d_g:M^q(d_g,W)=d_n \\ 1\le g\le n-1}} I_{t,d_g}$ because of $M^{L^d}(d_k,W)\ge M^{L^d}(d_0,W)=d_n$ for every $k\ge 1$.

In order to establish the result, we analyze $\alpha_{t+1}-\alpha_t$. We have

$$
\begin{aligned}
&\alpha_{t+1}-\alpha_t \\
&= E^{L^d}\left\{\sum_{i=1}^{L^d-2} p_{t+i}\cdot E^{D,W}[(D_t^i-I_{t,d_n}-\sum_{q=1}^{i+1}\sum_{\substack{d_g:M^q(d_g,W)=d_n \\ 1\le k\le n-1}} I_{t,d_g})^+]\right\} \\
&\quad + E^{L^d}\left\{p_{t+L^d-1}\cdot E^{D,W}[(D_t^{L^d-1}-I_{t,d_n}-\sum_{q=1}^{L^d}\sum_{\substack{d_g:M^q(d_g,W)=d_n \\ 1\le k\le n-1}} I_{t,d_g}-I_{t,d_0})^+]\right\} \\
&\quad - E^{L^d}\left\{\sum_{j=0}^{L^d-2} p_{t+j}\cdot E^{D,W}[(D_t^j-I_{t,d_n}-\sum_{p=1}^{j+1}\sum_{\substack{d_k:M^p(d_k,W)=d_n \\ 1\le k\le n-1}} I_{t,d_k})^+]\right\} \\
&= E^{L^d}\left\{p_{t+L^d-1}\cdot E^{D,W}[(D_t^{L^d-1}-I_{t,d_n}-\sum_{q=1}^{L^d}\sum_{\substack{d_g:M^q(d_g,W)=d_n \\ 1\le k\le n-1}} I_{t,d_g}-I_{t,d_0})^+]\right\} \\
&\quad - p_t\cdot E^{D,W}[(D_t-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n \\ 1\le k\le n-1}} I_{t,d_k})^+].
\end{aligned}
\tag{14}
$$

In (14), the penalty cost terms from t+1 to $t+L^d$ -2 cancel out. We end up with two terms for time period t and $t+L^d-1$.

It is clear that $I_{t,d_n}+\sum_{q=1}^{l^d}\sum_{\substack{d_g:M^q(d_g,w)=d_n \\ 1\le g\le n-1}} I_{t,d_g}=I_{t,d_n}+\sum_{j=1}^{n-1} I_{t,d_j}=r_{t,1}$ for every realization l^d of L^d and w of W. We are now ready to show the induction step starting from (2). For ease of notation, let $\beta_t=C_t(I_{t,u_0})+S_t(I_{t,d_0})+h_t(IO_t+I_{t,u_0}-I_{t,d_0})$. We have

$$
\begin{aligned}
&V_t(I_{t,d}) \\
&= \underset{\substack{I_{t,u_0}\ge 0 \\ I_{t,d_0}\ge 0 \\ I_{t,d_0}\le IO_t}}{Min}\left\{\beta_t+p_t\cdot E^{D,W}[(D_t-I_{t,d_n}-\sum_{\substack{d_k:M(d_k,W)=d_n \\ 1\le k\le n-1}} I_{t,d_k})^+]+V_{t+1}(I_{t+1,d})\right\}
\end{aligned}
\tag{15}
$$

$$= \min_{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}} \left\{ \beta_t + p_t \cdot E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k:M(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+] + \overline{V}_{t+1}(IO_{t+1}, r_{t+1,1}) + \alpha_{t+1} \right\} \tag{16}$$

$$= \min_{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}} \left\{ \begin{array}{l} \beta_t + E^{L^d}(p_{t+L^d-1} \cdot E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t1})^+]) + \overline{V}_{t+1}(IO_{t+1}, r_{t+1,1}) + \alpha_{t+1} \\ -E^{L^d}(p_{t+L^d-1} \cdot E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t1})^+]) + p_t \cdot E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k:M(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+] \end{array} \right\}$$

$$= \min_{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}} \left\{ \begin{array}{l} \overline{V}_t(IO_t, r_{t1}) + \alpha_{t+1} \\ -E^{L^d}(p_{t+L^d-1} E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t1})^+]) + p_t E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k:M(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+] \end{array} \right\} \tag{17}$$

$$= \min_{\substack{I_{t,u_0}\ge 0\\ I_{t,d_0}\ge 0\\ I_{t,d_0}\le IO_t}} \left\{ \begin{array}{l} \overline{V}_t(IO_t, r_{t1}) + \alpha_t \ + \alpha_{t+1} - \alpha_t \\ -E^{L^d}(p_{t+L^d-1} E^D[(D_t^{L^d-1} - I_{t,d_0} - r_{t1})^+]) + p_t E^{D,W}[(D_t - I_{t,d_n} - \sum_{\substack{d_k:M(d_k,W)=d_n\\ 1\le k\le n-1}} I_{t,d_k})^+] \end{array} \right\} \tag{18}$$

$$= \overline{V}_t(IO_t, r_{t1}) + \alpha_t.$$

Equation (15) is the optimality equation of the distribution process. In (16) we use the induction hypothesis. The optimality equation (3) of the reduced model is applied in (17). In (18) we use the derived result for $\alpha_{t+1} - \alpha_t$ and the observation $I_{t,d_n} + \sum_{q=1}^{l^d} \sum_{\substack{d_g:M^q(d_g,w)=d_n\\ 1\le g\le n-1}} I_{t,d_g} = r_{t,1}$ for each realization l^d of L^d and w of W. After simplifying the remaining terms and pulling term α_t out of minimization because it is independent of the decision variables I_{t,u_0} and I_{t,d_0}, we conclude that $V_t(I_{t,d}) = \overline{V}_t(IO_t, r_{t1}) + \alpha_t$. This completes the proof.

Appendix 2

Proof of Theorem 3: We prove the statement by induction. It is straightforward to show the theorem for T+1. We assume that (9) holds for t+1, i.e.,

$$\widetilde{V}_{t+1}(I_{t+1}^e) = \overline{V}_{t+1}(r_{t+1,1}) + G_{t+1}(I_{t+1,u_1}^e, \cdots, I_{t+1,u_{n-1}}^e, IO_{t+1}^e).$$

By examining the downstream sub-system, standard arguments are applicable showing that $\overline{V}_t$'s are all convex. The base stock levels

$$r_{t,d_0}^* = \arg\min_{r_{t,d_0}\ge 0} \left\{ (S_t - h_t) r_{t,d_0} + E^{L^d}(p_{t+L^d-1} E^D[(D_t^{L^d-1} - r_{t,d_0})^+]) + E^D[\overline{V}_{t+1}(r_{t,d_0} - D_t)] \right\}$$

are used to determine the optimal shipping decisions to the distribution process. In other words, if $r_{t,1} < r_{t,d_0}^*$, the optimal shipping quantity in time t is $r_{t,d_0}^* - r_{t,1}$; otherwise, the optimal shipping quantity is zero. In order to show (9), we need to distinguish two cases.

Case I: Let first $r_{t,d_0}^* < IO_t^e$. We have

$$\widetilde{V}_t(I_t^e) = \underset{\substack{I_{t,u_0}\geq 0 \\ r_{t,d_0}\geq r_{t,1} \\ r_{t,d_0}\leq IO_t^e}}{Min} \left\{ \begin{array}{l} C_t(I_{t,u_0}) + S_t\cdot(r_{t,d_0} - r_{t,1}) + h_t\cdot E^R(I^e_{t,N(u_n)} - r_{t,d_0}) \\ +E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - r_{t,d_0})^+]) + E^{D,W,R}[\widetilde{V}_{t+1}(I^e_{t+1})] \end{array} \right\}$$

$$= \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\} + S_t\cdot(r^*_{t,d_0} - r_{t,1})$$
$$- h_t\cdot r^*_{t,d_0} + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - r^*_{t,d_0})^+]) + \overline{V}_{t+1}(r_{t+1,1}).$$

It implies that

$$\widetilde{V}_t(I_t^e) - \overline{V}_t(r_{t,1}) = \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\}.$$

Case II : Let now $r^*_{t,d_0} \geq IO^e_t$. It is clear that the optimal shipping decision satisfies $r^*_{t,d_0} = IO^e_t$ because of the convexity of function $S_t\cdot(r_{t,d_0} - r_{t,1}) - h_t\cdot r_{t,d_0} + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - r_{t,d_0})^+])$ and $\overline{V}_{t+1}$ in r_{t,d_0}. Finally, we derive that

$$\widetilde{V}_t(I_t^e) = \underset{\substack{I_{t,u_0}\geq 0 \\ r_{t,d_0}\geq r_{t,1} \\ r_{t,d_0}\leq IO_t^e}}{Min} \left\{ \begin{array}{l} C_t(I_{t,u_0}) + S_t\cdot(r_{t,d_0} - r_{t,1}) + h_t\cdot E^R(I^e_{t,N(u_n)} - r_{t,d_0}) \\ +E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - r_{t,d_0})^+]) + E^{D,W,R}[\widetilde{V}_{t+1}(I^e_{t+1})] \end{array} \right\}$$

$$= \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\} + S_t\cdot(IO^e_t - r_{t,1})$$
$$- h_t\cdot IO^e_t + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - IO^e_t)^+]) + \overline{V}_{t+1}(r_{t+1,1}).$$

It implies that

$$\widetilde{V}_t(I_t^e) - \overline{V}_t(r_{t,1}) = \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\} + S_t\cdot(IO^e_t - r_{t,1})$$
$$- h_t\cdot IO^e_t + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - IO^e_t)^+]) + E^D[\overline{V}_{t+1}(r_{t+1,1}) - S_t\cdot(r^*_{t,d_0} - r_{t,1})$$
$$+ h_t\cdot r^*_{t,d_0} - E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - r^*_{t,d_0})^+]) - E^D[\overline{V}_{t+1}(r_{t+1,1})$$
$$= \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + \theta_t(IO^e_t) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\},$$

where $\theta_t(IO^e_t) = (h_t - S_t)(r^*_{t,d_0} - IO^e_t) + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - IO^e_t)^+ - (D_t^{L^d-1} - r^*_{t,d_0})^+])$ is the penalty term corresponding to the shortage of on-hand inventory for an optimal shipment.

As a result, we have

$$\begin{aligned} \widetilde{V}_t(I_t^e) - \overline{V}_t(r_{t,1}) &= G_t(I^e_{t,u_1},\cdots,I^e_{t,u_{n-1}}, IO^e_t) \\ &= \underset{I_{t,u_0}\geq 0}{Min}\left\{ C_t(I_{t,u_0}) + h_t\cdot E^R(I^e_{t,N(u_n)}) + \theta_t(IO^e_t) + E^R[G_{t+1}(I^e_{t+1,u_1},\cdots,I^e_{t+1,u_{n-1}}, IO^e_{t+1})]\right\}, \end{aligned} \tag{19}$$

where the penalty term reads

$$\theta_t(IO^e_t) = \begin{cases} 0 & r^*_{t,d_0} < IO^e_t \\ (h_t - S_t)(r^*_{t,d_0} - IO^e_t) + E^{L^d}(p_{t+L^d-1}\cdot E^D[(D_t^{L^d-1} - IO^e_t)^+ - (D_t^{L^d-1} - r^*_{t,d_0})^+]) & r^*_{t,d_0} \geq IO^e_t. \end{cases}$$

This completes the proof of Theorem 3.

Appendix 3

Proof of Lemma 1: We need to show that if $I_t^s \in \zeta_t^s$, then $I_{t+1}^s \in \zeta_{t+1}^s$. First, we consider non-negativity requirements. It is clear from system dynamics that $IO_{t+1}^s = IO_t^s + X_t - Y_t \geq 0$ because $X_t \geq 0$, $Y_t \geq 0$ and $Y_t \leq IO_t^s$. Non-negativity of other components follows by definition and from system dynamics.

If $R_{t,a_{t,k}} > 0$ for a k, then $a_{t,k} > 0$ (otherwise $a_{t,k} = A_{t,k} = 0$, which implies $R_{t,a_{t,k}} = 0$). In turn we have $A_{t,k} > 0$ and thus $R_{t,A_{t,k}} > 0$. System dynamics imply that for every j $0 \leq a_{t+1,i}^s(j) \leq A_{t+1,i}^s(j)$.

If $a_{t+1,i}^s(j) = 0$, then the underlying set defining the quantity is $\varnothing$. The underlying set defining $A_{t+1,i}^s(j)$ is also empty, which in turn implies $A_{t+1,i}^s(j) = 0$.

If $a_{t+1,i}^s(j) > 0$, then we have $R_{t,a_{t,k}} > 0$ for some $1 \leq k \leq s$ and $a_{t+1,i}^s(j) = a_{t,k} + 1$. Since $a_{t+1,i}^s(j) > 0$, we obtain that j in time period t must be greater than $a_{t,k}$. Then either $j > m(t)$ or $a_{t,k} < j \leq m(t)$. System dynamics imply in both cases that $R_{t+1,a_{t+1,i}^s(j)}^s = R_{t,a_{t,k}} > 0$. The same argument applies to the case $A_{t+1,i}^s(j) > 0$ implying $R_{t+1,A_{t+1,i}^s(j)}^s > 0$.

It remains to show that if for $1 \leq i' < j' \leq s$ such that $A_{t+1,i'}^s > 0$, $a_{t+1,j'}^s > 0$ and for every $i' < k' < j'$ we have $a_{t+1,k'}^s = A_{t+1,k'}^s = 0$, then we have $A_{t+1,i'}^s \leq a_{t+1,j'}^s - 1$. A possible scenario of the order transitions from time period t to t+1 is shown in Figure 9. Outstanding orders in the installation range [*u,v*] in time period t move to installation i' in time period t+1. Similarly, outstanding orders in [*u',v'*] move to installation j' in time period t+1. The light-shaded installations correspond to installations with no inventory in time period t. Installations $\tilde{i}$ and i are the two farthest installations with positive inventory within [*u,v*], i.e., inventory in installations within $[\tilde{i}, i]$ is 0, while j and $\tilde{j}$ are two such installations within [*u',v'*]. As can be easily verified by the non-crossover assumption, the dark-shaded area represents installations with no outstanding orders in time period t. Formally, we obtain $A_{t+1,i'}^s = A_{t,i} + 1$ and $a_{t+1,j'}^s = a_{t,j} + 1$. From $A_{t,i} \leq a_{t,j} - 1$, we then obtain $A_{t+1,i'}^s \leq a_{t+1,j'}^s - 1$.

In conclusion, the presented dynamic program is well-defined. This completes the proof of Lemma 1.

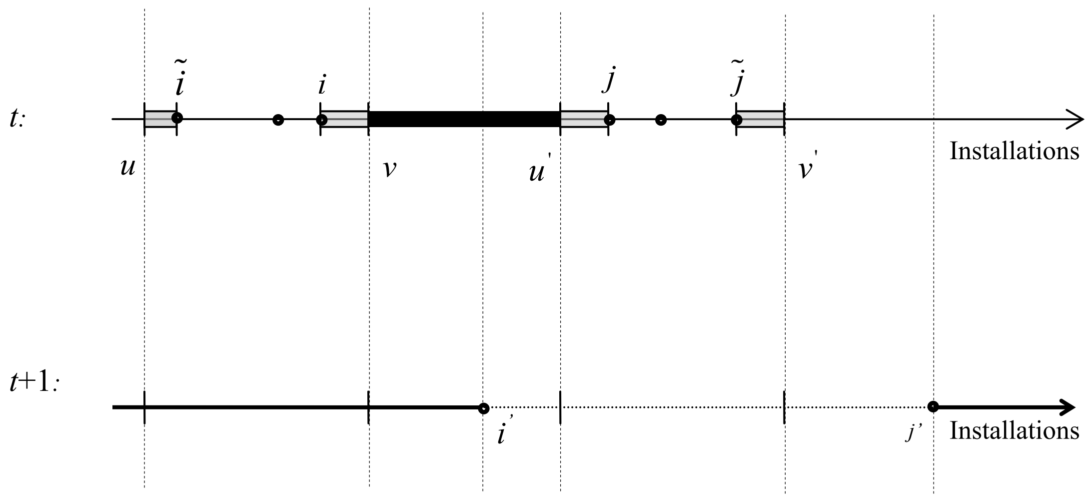

Figure 9: Graphical Representation of Order Transitions

Appendix 4

In order to show Theorem 4, we need the following lemma.

Lemma 2. For simplicity, we denote

$V_{t+1}^{s+1}(I_{t+1,d_n}^{s+1}, R_{t+1}^{s+1}) = E[V_{t+1}^{s+1}(IO_{t+1}^{s+1}, I_{t+1,d_n}^{s+1}, a_{t+1,1}^{s+1}, \cdots, a_{t+1,s+1}^{s+1}, A_{t+1,1}^{s+1}, \cdots, A_{t+1,s+1}^{s+1}, R_{t+1,1}^{s+1}, \cdots, R_{t+1,T-1}^{s+1})]$, where the state variables are derived from the recursion in scenario 2, and

$V_{t+1}^{s+1}(I_{t+1}^{s}, a_{t+1,s+1}^{s+1}, A_{t+1,s+1}^{s+1}) = E[V_{t+1}^{s+1}(IO_{t+1}^{s}, I_{t+1,d_n}^{s}, a_{t+1,1}^{s}, \cdots, a_{t+1,s}^{s}, a_{t+1,s+1}^{s+1}, A_{t+1,1}^{s}, \cdots, A_{t+1,s}^{s}, A_{t+1,s+1}^{s+1}, R_{t+1,1}^{s}, \cdots, R_{t+1,T-1}^{s})]$,

where state variable $I_{t+1}^{s} \in \zeta_{t+1}^{s}$ is derived from the recursion in scenario 1. If $I_{t+1,d_n}^{s} \le I_{t+1,d_n}^{s+1}$, then for every $k = \{1, 2, \cdots, T\text{-}1\}$ we have $R_{t+1,k}^{s} \ge R_{t+1,k}^{s+1}$, $I_{t+1,d_n}^{s} + \sum_{k=1}^{T-1} R_{t+1,k}^{s} = I_{t+1,d_n}^{s+1} + \sum_{k=1}^{T-1} R_{t+1,k}^{s+1}$ and

$$V_{t+1}^{s+1}(I_{t+1}^{s}, a_{t+1,s+1}^{s+1}, A_{t+1,s+1}^{s+1}) \ge V_{t+1}^{s+1}(I_{t+1,d_n}^{s+1}, R_{t+1}^{s+1}) . \tag{20}$$

Proof: Let us first show that for every $k = \{1, 2, \cdots, T\text{-}1\}$ we have $R_{t+1,k}^{s} \ge R_{t+1,k}^{s+1}$. We distinguish three cases.

1)Let $j > m'(t)$ and $k \ge 2$. Following the state transitions in partial scenarios 1 and 2, for $j > m'(t)$ and $k \le j$, we have $R_{t+1,k}^{s}(j) = R_{t,k-1}$ and $R_{t+1,k}^{s+1}(j) = R_{t,k-1}$, which implies $R_{t+1,k}^{s}(j) = R_{t+1,k}^{s+1}(j)$. If $m'(t) < j < k$, we have $R_{t+1,k}^{s}(j) = 0$ and $R_{t+1,k}^{s+1}(j) = 0$, which also implies $R_{t+1,k}^{s}(j) = R_{t+1,k}^{s+1}(j)$.

2)Let $m(t) < j \le m^{'}(t)$ and $k \ge 2$. For $m(t) < a_{t,g} \le k \le j \le m^{'}(t)$, we have $R^{s}_{t+1,k}(j) = R^{s}_{t,k-1}(j)$ and $R^{s+1}_{t+1,k}(j) = 0$, which implies $R^{s}_{t+1,k}(j) \ge R^{s+1}_{t+1,k}(j)$. Otherwise, we clearly have $R^{s}_{t+1,k}(j) = R^{s+1}_{t+1,k}(j)$.

3)Finally, let $1 \le j \le m(t)$ and $k \ge 2$. For $k \le a_{t,g} \le j \le m(t) \le m^{'}(t)$, we have $R^{s}_{t+1,k}(j) = R^{s}_{t,k-1}(j)$ and $R^{s+1}_{t+1,k}(j) = R^{s}_{t,k-1}(j)$, which implies $R^{s}_{t+1,k}(j) = R^{s+1}_{t+1,k}(j)$. Otherwise, we have $R^{s}_{t+1,k}(j) = 0$ and $R^{s+1}_{t+1,k}(j) = 0$, which again implies $R^{s}_{t+1,k}(j) = R^{s+1}_{t+1,k}(j)$.

We also observe that $R^{s}_{t+1,1}(j) = R^{s+1}_{t+1,1}(j) = Y_t$. Thus, we conclude that for every $k = \{1, 2, \cdots, T\text{-}1\}$ we have $R^{s}_{t+1,k} \ge R^{s+1}_{t+1,k}$.

Next, let us show $I^{s}_{t+1,d_n} \le I^{s+1}_{t+1,d_n}$. We similarly distinguish cases as follows.

1)Let first $1 \le a_{t,i} \le j \le A_{t,i} \le m(t)$ and $1 \le i \le s$. By using the state transitions in scenarios 1 and 2, we have $I^{s}_{t+1,d_n}(j) = I^{s+1}_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=a_{t,i}}^{T-1} R_{t,k} - D_t$.

2)Let $1 \le A_{t,i} < j < a_{t,i'} \le m(t)$. For every g such that $1 \le i < g < i' \le s$ and $a_{t,d_g} = A_{t,d_g} = 0$, we have $I^{s}_{t+1,d_n}(j) = I^{s+1}_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=a_{t,i'}}^{T-1} R_{t,k} - D_t$.

3)Let now $m(t) < j \le m^{'}(t)$. We have

$$I^{s}_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=j}^{T-1} R_{t,k} - D_t,$$

$$I^{s+1}_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=m(t)+1}^{T-1} R_{t,k} - D_t.$$

4)Finally, let $j > m^{'}(t)$. We have $I^{s}_{t+1,d_n}(j) = I^{s+1}_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=j}^{T-1} R_{t,k} - D_t$.

Therefore, we obtain $I^{s}_{t+1,d_n}(j) \le I^{s+1}_{t+1,d_n}(j)$ because $\sum_{k=j}^{T-1} R_{t,k} \le \sum_{k=m(t)+1}^{T-1} R_{t,k}$.

Now, we show $I^{s}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s}_{t+1,k} = I^{s+1}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s+1}_{t+1,k}$. We first investigate $I^{s}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s}_{t+1,k}$ and distinguish two cases.

1) Let $1 \le j \le m(t)$ and $g = \operatorname*{argmax}_i \{a_{t.i} \mid j \ge a_{t.i}, d_1 \le d_i \le d_s\}$. By state transitions in scenario 1 we have

$$\sum_{k=2}^{T-1} R^{s}_{t+1,k}(j) = \sum_{k=2}^{a_{t,g}} R^{s}_{t+1,k}(j) + \sum_{k=a_{t,g}+1}^{T-1} R^{s}_{t+1,k}(j) = \sum_{p=1}^{a_{t,g}-1} R_{t,p}.$$

Let $e = \min\{d : M(d,W) = d_n, d_1 \le d \le d_s\}$ and $f = \max\{d : M(d,W) = d_n, d_1 \le d \le d_s\}$. Let also B be the subset of $\{e, e+1, \cdots, f\}$ such that for every $d_b \in B$ we have $R_{t,a_{t,b}} > 0$ and for every $d_b \in \{e, e+1, \cdots, f\} \setminus B$ we have $R_{t,a_{t,b}} = R_{t,A_{t,b}} = 0$. We also denote

$d_{b_1} = \min\{d_b : d_b \in B\}$ and $d_{b_2} = \max\{d_b : d_b \in B\}$. We have $a_{t,b_1} = a_{t,g}$ and $m(t) = \max_{b\in B}\{A_{t,b}\} = A_{t,b_2}$. In addition, $R^s_{t+1,1}(j) = Y_t$. Adding $I^s_{t+1,d_n}(j)$ and $\sum_{k=1}^{T-1} R^s_{t+1,k}(j)$, we obtain

$$\begin{aligned}
I^s_{t+1,d_n}(j) + \sum_{k=1}^{T-1} R^s_{t+1,k}(j) &= (I_{t,d_n} + \sum_{p=a_{t,b_1}}^{A_{t,b_2}} R_{t,p} + \sum_{k=m(t)+1}^{T-1} R_{t,k} - D_t) + (Y_t + \sum_{k=2}^{T-1} R^s_{t+1,k}(j)) \\
&= I_{t,d_n} + \sum_{p=a_{t,b_1}}^{A_{t,b_2}} R_{t,p} + \sum_{k=m(t)+1}^{T-1} R_{t,k} - D_t + Y_t + \sum_{p=1}^{a_{t,g}-1} R_{t,p} \\
&= I_{t,d_n} + \sum_{p=1}^{m(t)} R_{t,p} + \sum_{k=m(t)+1}^{T-1} R_{t,k} - D_t + Y_t \\
&= I_{t,d_n} + \sum_{k=1}^{T-1} R_{t,k} - D_t + Y_t.
\end{aligned}$$

2) Let now $j > m(t)$. We have $I^s_{t+1,d_n}(j) = I_{t,d_n} + \sum_{k=j}^{T-1} R_{t,k} - D_t$, $R^s_{t+1,1}(j) = Y_t$, and $\sum_{k=2}^{T-1} R^s_{t+1,k}(j) = \sum_{k=2}^{j} R^s_{t+1,k}(j) = \sum_{p=1}^{j-1} R_{t,p}$. Adding $I^s_{t+1,d_n}(j)$ and $\sum_{k=1}^{T-1} R^s_{t+1,k}(j)$, we have

$$\begin{aligned}
I^s_{t+1,d_n}(j) + \sum_{k=1}^{T-1} R^s_{t+1,k}(j) &= I_{t,d_n} + \sum_{k=j}^{T-1} R_{t,k} - D_t + Y_t + \sum_{k=2}^{T-1} R_{t+1,k}(j) \\
&= I_{t,d_n} + \sum_{k=j}^{T-1} R_{t,k} - D_t + Y_t + \sum_{p=1}^{j-1} R_{t,p} \\
&= I_{t,d_n} + \sum_{k=1}^{T-1} R_{t,k} - D_t + Y_t.
\end{aligned}$$

We conclude that in both cases $I^s_{t+1,d_n} + \sum_{k=1}^{T-1} R^s_{t+1,k} = I_{t,d_n} + \sum_{k=1}^{T-1} R_{t,k} - D_t + Y_t$.

By using almost identical steps as those shown above it can be derived that $I^{s+1}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s+1}_{t+1,k} = I_{t,d_n} + \sum_{k=1}^{T-1} R_{t,k} - D_t + Y_t$. It follows that $I^s_{t+1,d_n} + \sum_{k=1}^{T-1} R^s_{t+1,k} = I^{s+1}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s+1}_{t+1,k}$ $= I_{t,d_n} + \sum_{k=1}^{T-1} R_{t,k} - D_t + Y_t$.

Next, we argue that if $I^s_{t+1,d_n} \le I^{s+1}_{t+1,d_n}$, $R^s_{t+1,k} \ge R^{s+1}_{t+1,k}$ and $I^s_{t+1,d_n} + \sum_{k=1}^{T-1} R^s_{t+1,k} = I^{s+1}_{t+1,d_n} + \sum_{k=1}^{T-1} R^{s+1}_{t+1,k}$, then $I^s_{t+2,d_n} \le I^{s+1}_{t+2,d_n}$, $R^s_{t+2,k} \ge R^{s+1}_{t+2,k}$ and $I^s_{t+2,d_n} + \sum_{k=1}^{T-1} R^s_{t+2,k} = I^{s+1}_{t+2,d_n} + \sum_{k=1}^{T-1} R^{s+1}_{t+2,k}$. We note that from time period t+1 to time period t+2 the system follows the recursions in scenario 2. It is clear that if $R^s_{t+1,k} \ge R^{s+1}_{t+1,k}$ for every $k = \{1, 2, \cdots, T\text{-}1\}$, then $R^s_{t+2,k} \ge R^{s+1}_{t+2,k}$. We next show $I^s_{t+2,d_n} \le I^{s+1}_{t+2,d_n}$ by distinguishing two cases.

1) Let $j \le m^{'}(t+1)$. We note that for every $k=\{1,2,\cdots,s\}$, it holds $a^{s}_{t+1,k}=a^{s+1}_{t+1,k}$ and $A^{s}_{t+1,k}=A^{s+1}_{t+1,k}$. By state transitions in scenario 2, we obtain

$$I^{s}_{t+2,d_n}(j)=I^{s}_{t+1,d_n}+\sum_{l=a^{s}_{t+1,k}(j):M(d_k,w)=d_n}^{T-1} R^{s}_{t+1,l}-D_{t+1},$$

$$I^{s+1}_{t+2,d_n}(j)=I^{s+1}_{t+1,d_n}+\sum_{l=a^{s+1}_{t+1,k}(j):M(d_k,w)=d_n}^{T-1} R^{s+1}_{t+1,l}-D_{t+1}.$$

By using $R^{s}_{t+1,k}\ge R^{s+1}_{t+1,k}$ for every $k=\{1,\ 2,\cdots,T\text{-}1\}$ and $I^{s}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+1,k}=I^{s+1}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+1,k}$, it follows that $I^{s}_{t+2,d_n}(j)\le I^{s+1}_{t+2,d_n}(j)$.

2) Let $j>m^{'}(t+1)$. It is clear that $I^{s}_{t+1,d_n}+\sum_{k=j}^{T-1}R^{s}_{t+1,k}\le I^{s+1}_{t+1,d_n}+\sum_{k=j}^{T-1}R^{s+1}_{t+1,k}$ by using $R^{s}_{t+1,k}\ge R^{s+1}_{t+1,k}$ and $I^{s}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+1,k}=I^{s+1}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+1,k}$.

Thus, we obtain $I^{s}_{t+2,d_n}(j)\le I^{s+1}_{t+2,d_n}(j)$ for every j.

It is clear that

$$I^{s}_{t+2,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+2,k}=I^{s}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+1,k}-D_{t+1}+Y_{t+1},$$

$$I^{s+1}_{t+2,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+2,k}=I^{s+1}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+1,k}-D_{t+1}+Y_{t+1}.$$

We obtain $I^{s}_{t+2,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+2,k}=I^{s+1}_{t+2,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+2,k}$ since $I^{s}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s}_{t+1,k}=I^{s+1}_{t+1,d_n}+\sum_{k=1}^{T-1}R^{s+1}_{t+1,k}$.

Now, we are ready to prove the last part of the lemma by induction. If $I^{s}_{t+2,d_n}\le I^{s+1}_{t+2,d_n}$, and $R^{s}_{t+2,k}\ge R^{s+1}_{t+2,k}$ for every $k=\{1,\ 2,\cdots,T\text{-}1\}$, we assume that the lemma holds for t+2, i.e., $V^{s+1}_{t+2}(I^{s+1}_{t+2},a^{s+1}_{t+2,s+1},A^{s+1}_{t+2,s+1})\ge V^{s+1}_{t+2}(I^{s+1}_{t+2,d_n},R^{s+1}_{t+2})$. The state variables in $V^{s+1}_{t+2}(I^{s+1}_{t+2},a^{s+1}_{t+2,s+1},A^{s+1}_{t+2,s+1})$ evolve from values of $(I^{s}_{t+1},a^{s+1}_{t+1,s+1},A^{s+1}_{t+1,s+1})$, while variables in $V^{s+1}_{t+2}(I^{s+1}_{t+2,d_n},R^{s+1}_{t+2})$ transition from values of $(I^{s+1}_{t+1,d_n},R^{s+1}_{t+1})$. For ease of notation, we define

$$\begin{aligned}LHS&=C_{t+1}(X_{t+1})+S_{t+1}(Y_{t+1})+h_{t+1}(IO_{t+1}+X_{t+1}-Y_{t+1})\\&+p_{t+1}\cdot\sum_{j=1}^{m^{'}(t+1)}q_j\cdot E^{D,W/J=j}[(D_{t+1}-I^{s}_{t+1,d_n}-\sum_{l=a^{s}_{t+1,k}(j):M(d_k,w)=d_n}^{T-1}R^{s}_{t+1,l})^{+}]\\&+p_{t+1}\cdot\sum_{j=m^{'}(t+1)+1}^{T-1}q_j\cdot E^{D}[(D_{t+1}-I^{s}_{t+1,d_n}-\sum_{k=j}^{T-1}R^{s}_{t+1,k})^{+}],\end{aligned}$$

$$RHS = C_{t+1}(X_{t+1}) + S_{t+1}(Y_{t+1}) + h_{t+1}(IO_{t+1} + X_{t+1} - Y_{t+1})$$
$$+ p_{t+1} \cdot \sum_{j=1}^{m^{'}(t+1)} q_j \cdot E^{D,W/J=j}[(D_{t+1} - I^{s+1}_{t+1,d_n} - \sum_{l=a^{s+1}_{t+1,k}(j):M(d_k,w)=d_n}^{T-1} R^{s+1}_{t+1,l})^+]$$
$$+ p_{t+1} \cdot \sum_{j=m^{'}(t+1)+1}^{T-1} q_j \cdot E^D[(D_{t+1} - I^{s+1}_{t+1,d_n} - \sum_{k=j}^{T-1} R^{s+1}_{t+1,k})^+].$$

Recall that for $j \le m^{'}(t+1)$, we have $I^s_{t+1,d_n} - \sum_{l=a^s_{t+1,k}(j):M(d_k,w)=d_n}^{T-1} R^s_{t+1,l} \le I^{s+1}_{t+1,d_n} - \sum_{l=a^{s+1}_{t+1,k}(j):M(d_k,w)=d_n}^{T-1} R^{s+1}_{t+1,l}$ and for $j > m^{'}(t+1)$ it holds $I^s_{t+1,d_n} + \sum_{k=j}^{T-1} R^s_{t+1,k} \le I^{s+1}_{t+1,d_n} + \sum_{k=j}^{T-1} R^{s+1}_{t+1,k}$. Thus, we have $LHS \ge RHS$. Finally, we obtain

$$\begin{aligned}
&V^{s+1}_{t+1}(I^s_{t+1}, a^{s+1}_{t+1,s+1}, A^{s+1}_{t+1,s+1}) \\
&= E[V^{s+1}_{t+1}(IO^s_{t+1}, I^s_{t+1,d_n}, a^s_{t+1,1}, \cdots, a^s_{t+1,s}, a^{s+1}_{t+1,s+1}, A^s_{t+1,1}, \cdots, A^s_{t+1,s}, A^{s+1}_{t+1,s+1}, R^s_{t+1,1}, \cdots, R^s_{t+1,T-1})] \\
&= \min_{\substack{X_{t+1} \ge 0 \\ Y_{t+1} \ge 0 \\ Y_{t+1} \le IO_{t+1}}} \{LHS + V^{s+1}_{t+2}(I^{s+1}_{t+2}, a^{s+1}_{t+2,s+1}, A^{s+1}_{t+2,s+1})\} \\
&\ge \min_{\substack{X_{t+1} \ge 0 \\ Y_{t+1} \ge 0 \\ Y_{t+1} \le IO_{t+1}}} \{LHS + V^{s+1}_{t+2}(I^{s+1}_{t+2,d_n}, R^{s+1}_{t+2})\} \qquad (21) \\
&\ge \min_{\substack{X_{t+1} \ge 0 \\ Y_{t+1} \ge 0 \\ Y_{t+1} \le IO_{t+1}}} \{RHS + V^{s+1}_{t+2}(I^{s+1}_{t+2,d_n}, R^{s+1}_{t+2})\} \\
&= V^{s+1}_{t+1}(I^{s+1}_{t+1,d_n}, R^{s+1}_{t+1}).
\end{aligned}$$

where (21) follows by the induction hypothesis. This completes the proof of Lemma 2. ■

Proof of Theorem 4. We take a closer look at the current one-period cost in time period *t* in scenarios 1 and 2. For ease of notation, we denote

$$\widehat{C}^{s+1}_t = C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t)$$
$$+ p_t \cdot \sum_{j=1}^{m^{'}(t)} q_j \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+]$$
$$+ p_t \cdot \sum_{j=m^{'}(t)+1}^{T-1} q_j \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+].$$

$$\widehat{C}^s_t = C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t)$$
$$+ p_t \cdot \sum_{j=1}^{m(t)} q_j \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+]$$
$$+ p_t \cdot \sum_{j=m(t)+1}^{T-1} q_j \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+],$$

Quantity $\widehat{C}_t^{s}$ denotes the one-period cost in scenario 1 and $\widehat{C}_t^{s+1}$ denotes the one-period cost in scenario 2. We first show that

$$\widehat{C}_t^{s} \geq \widehat{C}_t^{s+1}. \tag{22}$$

We start by showing

$$\sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \geq \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m(t)+1}^{T-1} R_{t,k})^+]. \tag{23}$$

To show (23), we distinguish three cases.

1) Let first $m'(t) = m(t)$. Then it follows

$$\sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] = \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m(t)+1}^{T-1} R_{t,k})^+] = 0.$$

2) Next let $m'(t) = m(t)+1$. We have

$$\begin{aligned}\sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] &= \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m(t)+1}^{T-1} R_{t,k})^+] \\ &= q_{m'(t)} \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m'(t)}^{T-1} R_{t,k})^+].\end{aligned}$$

3) Finally, let $m'(t) > m(t)+1$. For every $m(t)+1 \leq j \leq m'(t)$, we have $\sum_{k=j}^{m'(t)} R_{t,k} \leq \sum_{k=m(t)+1}^{m'(t)} R_{t,k}$

and $\sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \geq \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m(t)+1}^{T-1} R_{t,k})^+].$

Suppose that there exists $k \in \{m(t)+1, \cdots, m'(t)\}$ such that $R_{t,k} > 0$. It is clear that $m(t) = \max\{A_{t,1}, A_{t,2}, \cdots, A_{t,s}\}$ and $m'(t) = \max\{m(t), A_{t,s+1}\} = A_{t,s+1}$. By definition of ζ_t^s we conclude that $m(t)+1 \leq a_{t,s+1} \leq A_{t,s+1} = m'(t).$

Now, we perform the final step towards showing $\widehat{C}_t^s \geq \widehat{C}_t^{s+1}$. We have

$$\begin{aligned}
\widehat{C}_t^s &= C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t) \\
&\quad + \sum_{j=1}^{m(t)} q_j \cdot p_t \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+] \\
&\quad + \sum_{j=m(t)+1}^{T-1} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \\
&= C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t) \\
&\quad + \sum_{j=1}^{m(t)} q_j \cdot p_t \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+] \\
&\quad + \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] + \sum_{j=m'(t)+1}^{T-1} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+]
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\geq C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t) \\
&\quad + \sum_{j=1}^{m(t)} q_j \cdot p_t \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+] \\
&\quad + \sum_{j=m(t)+1}^{m'(t)} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=m(t)+1}^{T-1} R_{t,k})^+] + \sum_{j=m'(t)+1}^{T-1} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \\
&= C_t(X_t) + S_t(Y_t) + h_t(IO_t + X_t - Y_t) \\
&\quad + \sum_{j=1}^{m'(t)} q_j \cdot p_t \cdot E^{D,W/J=j}[(D_t - I_{t,d_n} - \sum_{l=a_{t,k}(j):M(d_k,w)=d_n}^{T-1} R_{t,l})^+] \\
&\quad + \sum_{j=m'(t)+1}^{T-1} q_j \cdot p_t \cdot E^D[(D_t - I_{t,d_n} - \sum_{k=j}^{T-1} R_{t,k})^+] \\
&= \widehat{C}_t^{s+1}.
\end{aligned} \tag{25}$$

In (24) we break the indices and apply (23) in (25).

We are now ready to carry out the complete proof by induction. Since we do not consider the cost incurred beyond the time horizon T, we have $V_{T+1}^s = V_{T+1}^{s+1} = 0$. By the induction hypothesis for t+1, if $I_{t+1}^s \in \zeta_{t+1}^s$, we assume that (12) holds, i.e.,

$$E[V_{t+1}^s(I_{t+1}^s)] \geq E[V_{t+1}^{s+1}(IO_{t+1}, I_{t+1,d_n}^s, a_{t+1,1}^s, \cdots, a_{t+1,s}^s, a_{t+1,s+1}^{s+1}, A_{t+1,1}^s, \cdots, A_{t+1,s}^s, A_{t+1,s+1}^{s+1}, R_{t+1,1}^s, \cdots, R_{t+1,T-1}^s)]. \tag{26}$$

Starting from (11), we obtain

$$V_t^s(I_t^s)$$

$$= \underset{\substack{X_t \geq 0 \\ X_t \geq 0 \\ Y_t \leq IO_t}}{Min} \left\{ \widehat{C}_t^s + E[V_{t+1}^s(I_{t+1}^s)] \right\}$$

$$\geq \underset{\substack{X_t \geq 0 \\ X_t \geq 0 \\ Y_t \leq IO_t}}{Min} \left\{ \widehat{C}_t^s + E[V_{t+1}^{s+1}(IO_{t+1}^s, I_{t+1,d_n}^s, a_{t+1,1}^s, \cdots, a_{t+1,s}^s, a_{t+1,s+1}^{s+1}, A_{t+1,1}^s, \cdots, A_{t+1,s}^s, A_{t+1,s+1}^{s+1}, R_{t+1,1}^s, \cdots, R_{t+1,T-1}^s)] \right\} \quad (27)$$

$$\geq \underset{\substack{X_t \geq 0 \\ X_t \geq 0 \\ Y_t \leq IO_t}}{Min} \left\{ \widehat{C}_t^{s+1} + V_{t+1}^{s+1}(I_{t+1}^s, a_{t+1,s+1}^{s+1}, A_{t+1,s+1}^{s+1}) \right\} \quad (28)$$

$$\geq \underset{\substack{X_t \geq 0 \\ X_t \geq 0 \\ Y_t \leq IO_t}}{Min} \left\{ \widehat{C}_t^{s+1} + V_{t+1}^{s+1}(I_{t+1,d_n}^{s+1}, R_{t+1}^{s+1}) \right\} \quad (29)$$

$$= V_t^{s+1}(IO_t, I_{t,d_n}^{s+1}, a_{t,1}^{s+1}, \cdots, a_{t,s+1}^{s+1}, A_{t,1}^{s+1}, \cdots, A_{t,s+1}^{s+1}, R_{t,1}^{s+1}, \cdots, R_{t,T-1}^{s+1}).$$

In (27), (28), (29), we have used (26), (22) and (20) respectively.

This completes the proof of Theorem 4.