

Pricing of Digital Goods vs. Physical Goods

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Abstract

E-commerce is growing rapidly and sales of digital goods represent a substantial portion of all online sales. Several goods such as music and books are now available in a physical and digital format. In a single-period case we compare pricing of digital vs. physical goods and derive an optimal pricing strategy for digital goods in both a general setting without a capacity constraint and a capacity constrained setting. We show that the optimal price for digital goods is usually lower in comparison with physical goods. We also investigate the optimal pricing problem for digital goods under externality in a multi-period case. We demonstrate that the optimal prices are decreasing over time. A relationship between optimal prices in the single- and multi-period cases is established.

1 Introduction

Firms have traditionally provided standard products in a physical format, but are now actively pursuing digital options. We are in the midst of an atoms-to-bits shift. The market shift from physical to digital goods is described by a causal pattern of the progress in operations management in Geoffrion (2002). The advance of technology is the initial momentum, business practice follows and academic research is required. We are now in the stage of flourishing market practices. An increasing number of customers began shopping online for digital goods. For several goods there exist two different formats, e.g., music, magazines and photography. As an example, music can be released and sold to consumers in the form of either physical albums or digital downloads. In our context, we always refer to physical and digital goods of the same product. Digital goods are differentiated from physical goods in several dimensions. They are expensive to produce but cheap to reproduce, since the unit cost of reproduction is negligible and virtually zero. Digital goods can be consumed simultaneously by more than one user.

In the context of digital music, it is costly for artists and record labels to produce new tracks. However, it is very cheap to distribute to mass customers after the production of the first copy. The sales of physical CDs were still in general the main channel of sales in year 2010. Music CD retailers have gone online to sell music CDs via internet. They have aligned pricing and operations strategies to optimize their markups. They can improve sales by optimizing service quality and product attributes, Rabinovich *et al.* (2008). However, the advent of music downloads has increased the competition among internet CD retailers and the total sales of physical CDs have decreased. The introduction of iTunes provided a spurt of online music sales. Contrary to the rapid decline of sales of physical CDs, digital downloads enjoyed a continuous increase during past years, Sisario (2011). Digital music revenue was 4.6 billion in 2010, up 6 percent from the year before. In 2009, it grew 12 percent from the year before, and in 2008 it was up 25 percent. According to a Nielsen and Billboard report, digital music purchases accounted for 50.3% of music sales in 2011,

Segall (2012). For the first time in history, digital music sales topped the physical sales of music. The British Phonographic Industry (BPI) reveals that digital music accounted for 55.5% of total music sales in the first quarter of 2012, Sweney (2012). The fast growing digital music market caught a lot of attention from both providers and consumers and led to different pricing models. Recently, Spotify started offering a subscription based service for digital music. When iTunes first launched, the pricing strategy mimicked the physical goods fixed pricing strategy except that it was by song. Now that the digital market has matured and there are multiple compelling offerings, more pricing flexibility is available (iTunes has a multi-tier pricing strategy).

Many magazine publishers have joined the digital world after the introduction of the iPad by providing digital content. A recent app Next Issue started offering digital magazine subscription on the all-you-can-eat basis. A March 2010 study by the Boston Consulting Group (2010) lists that consumers in the United States are willing to pay \$2 to \$4 for a single issue of an online magazine that costs \$5 in print. Customers perceive that the selling price of an issue of a digital magazine should be further lowered because of the cheaper supply cost and easy replication. It is now the time to consider how customers value digital goods and how to price digital goods accordingly.

The main goal of this work is comparing the optimal price between physical and digital distributions of the same goods from the seller's perspective in various settings. In a general single-period setting without any capacity constraints there exists an optimal price for digital goods that is lower than an optimal price for physical goods. Under mild conditions the same property also holds when considering a maximum order quantity for the physical goods. We further extend the result to the multi-period setting by showing that the optimal prices are ordered in each period. For digital goods that can be easily shared by many consumers, we consider externality as part of the pricing strategy. Externality is the effect influenced by other consumers who own the same goods. For example, in digital photography, the more copies of a photography are in circulation the less valuable each copy is. Another paramount

result is an optimal pricing strategy for digital goods under externality. We use pure linear externality and incorporate it into the cumulative distribution function of the customer's reserve price.

We focus on the difference between pricing strategies of physical and digital goods. Pricing of both physical and digital goods have already been studied, but most studies do not compare pricing of the two formats of the same goods. We fill this gap by investigating the pricing strategy for physical and digital goods at the same time. A major contribution of this paper is the derivation of the pricing strategy by using distinct features of digital goods. We are able to show that the optimal price for digital goods is generally lower. Another contribution of this work is modeling externality in the context of optimal pricing. We show the structure of optimal prices for digital goods under externality. In the multi-period setting of pricing digital goods under externality, a novelty of our dynamic program (DP) is that the customer's reserve price changes in time, i.e., it depends on the number of customers who previously bought the goods. A common theme in the pricing literature is the exploration of the relationship between the demand and pricing decision. We do not assume any aggregated demand and instead start from the perspective of an individual customer with a reserve price distribution. In our setting, each consumer is treated individually from a finite population set. We develop a general Bernoulli-based demand model, in which the aggregated demand follows the binomial distribution.

The paper is structured as follows. In Section 2, we compare the optimal pricing strategy for both digital and physical goods in different settings. In particular, we distinguish two cases: purchase-to-order and purchase-to-stock. We explore the pricing strategy for digital goods under externality in the multi-period setting in Section 3. We conclude the introduction by a brief literature review and a review of the newsvendor model.

1.1 Literature review

A building block of this work is the optimal pricing model in a single-period setting. The standard inventory problem in a single-period setting is well studied and referred to as the newsvendor model. Porteus (2002) provides an excellent review of the newsvendor model. In general, a decision maker in the newsvendor setting faces a stochastic demand and needs to determine an optimal order quantity while the price is exogenous and fixed. In retail and manufacturing, it is necessary and essential to take production and distribution decisions into account, Dana & Petruzzi (2001) and Cachon & Kok (2007). We combine pricing with production and distribution decisions of physical goods and further provide a comparison with the optimal pricing strategy of digital goods. With the advance of information technology, innovative pricing strategies, e.g., dynamic pricing, are employed by retailers and manufactures to learn the customer demand and reduce demand variability, e.g., Mattioli (2012). We apply the dynamic pricing technique to digital goods under externality in the multi-period setting and exhibit the structure of optimal prices.

An emerging topic in the pricing literature is the pricing of digital goods. The pricing strategy for digital goods needs to be re-evaluated due to their unique cost structure based on the large set-up cost and negligible marginal cost. For example, the traditional strategy of pricing by the marginal cost is no longer applicable because of the negligible marginal cost of digital goods. Digital goods are often priced by discrimination, e.g., versioning and bundling. Varian (2000) shows different pricing strategies for digital goods and demonstrates when a strategy is better than another. Sundararajan (2004) shows the power of the mixed strategy of fixed-fee and usage-based pricing in digital goods. For selling a large variety of digital goods, bundling is often a good strategy. Bakos & Brynjolfsson (1999) investigate the profits of a bundling strategy applied to a large variety of digital goods. We focus on the differences of pricing strategies between digital and physical goods. To the best of our knowledge this is the first paper comparing the pricing strategies of digital vs. physical goods.

Externality becomes an important factor in pricing when sharing of digital goods is possible. Schmitz (2002) shows the optimal licensing strategy for sharable goods. It is clear that licensing exhibits externalities. The classical newsvendor pricing model determines jointly the price and order quantity. However, existence of externality complicates the demand-price relationship. For example, in a two-sided market the value of the product on the one side is correlated to the number of users on the other side, Chou *et al.* (2012). This is called indirect network externality. Thus, we incorporate externality into the optimal pricing strategy in our work.

Externality is similar to the snob effect in pricing. Snobbish consumers care about not only the functional effect of the product but also the social effect. They prefer exclusiveness. Leibenstein (1950) first introduces the snob effect and studies its impact on demand and price in a single-period setting. We differ by addressing the multi-period pricing problem under externality. Rodriguez & Locay (2002) model heterogeneity in consumer valuation. In their context, consumers are homogeneous and their appreciation for the same product is stationary, i.e., the reserve price does not change over time. A consumer's utility depends upon his appreciation of the product, and the number of consumers that have bought the product at the moment of the purchase. Both papers assume that consumers' valuations remain the same over time periods. However, in our work we update the distribution of the reserve price in each time period. Amaldoss & Jain (2005) analyze the impact of the snobbish consumer behavior on purchasing decisions and establish an equilibrium price in a one-period setting. They assume that the consumers are heterogeneous, i.e., there are two segments of consumers consisting of snobs and conformists. We assume the consumers are homogeneous and study the externality in pricing in both single- and multi-period settings.

1.2 Review of the newsvendor problem

In the newsvendor setting, the retail price is fixed and exogenous, and the firm has to choose a quantity to produce or order. Let us assume that the physical good is sold at unit price

p and unit cost w_p in the newsvendor (i.e. purchase-to-stock) setting. The demand for the good is normally distributed with mean μ and standard deviation σ . For simplicity, let us assume that the salvage price is zero. We denote by ϕ and Φ the density and cumulative distribution functions of the normal distribution, respectively. The optimal expected profit is $\pi_P = (p - w_p)\mu - p\phi(z^*)\sigma$. It is well known that the optimal order quantity is given by $Q^* = \mu + \sigma z^*$, where $z^* = \Phi^{-1}\left(\frac{p-w_p}{p}\right)$.

2 Pricing of digital vs. physical goods

We discussed the difference between digital and physical goods of the same product in the introduction. We compare pricing of physical and digital goods in this section. The physical good is assumed to be the good currently prevailing in the market and the fixed cost associated with the physical good is a sunk cost. The digital counterpart is a new entrant. There is a fixed cost of c_d for preparing the digital good. For example, physical music albums have been the main stream for quite some time. When digital downloads were introduced to compete with CDs, there was the cost of preparing the digital format (converting to digital, preparing iTunes, etc). To formalize this, we assume that the unit replenishment cost of a digital good is w_d where $w_d < w_p$. Recall that w_p is the unit replenishment cost of the physical good. The replenishment of $Q > 0$ units generates a total cost of $TC_d(Q) = w_d \cdot Q + c_d$ for the digital good while the corresponding cost for the physical good is $TC_P(Q) = w_p \cdot Q$.

Let us first assume that there is a finite population of N buyers. We consider two pricing scenarios: physical and digital goods. The customers have random reserve prices R with a known distribution F , which is either continuous or discrete. We denote $\bar{F}(\cdot) = 1 - F(\cdot)$. In the discrete case, we assume $P(R = a_i) = f_i$, where $0 < a_1 < a_2 < \dots < a_n < \infty$, $f_i > 0$ and $\sum_{i=1}^n f_i = 1$. We assume a positive density function f on a bounded domain $[p, \bar{p}]$ and an increasing hazard rate for the reserve price distribution if F is a continuous distribution.

A purchase takes place if the offered price p is lower than or equal to the reserve price of the customer, i.e., with probability $P(R \geq p)$. In reality, the reserve price distributions of digital and physical goods are distinct. For instance, a customer who likes a particular song may prefer the digital format over the physical CD format because s/he could buy the individual song in the digital format rather than the entire album in the CD format. We however assume that the distributions of the reserve price for both the physical product and the digital counterpart are the same in order to compare the optimal prices.

We model pricing of digital and physical products in two cases: purchase-to-order and purchase-to-stock. In the case of purchase-to-order, the firm who owns the physical and digital goods is a monopolist, and it has no capacity constraints. In other words, the firm is completely flexible in satisfying customers given that their reserve prices are higher than the selling price. We refer to this case which does not take the inventory decision into account as the general pricing scenario. The objective of the firm is to maximize the expected profit functions π_d and π_p from selling to N buyers. We consider both single- and two-period problems. In the single-period setting, the profit functions are

$$\pi_d(p_1) = N(p_1 - w_d)P(R \geq p_1) - c_d$$

for the digital good and

$$\pi_p(p_1) = N(p_1 - w_p)P(R \geq p_1)$$

for the physical good. In the two-period environment, they are

$$\pi_d(p_1, p_2) = N[(p_1 - w_d)P(R \geq p_1) + (p_2 - w_d)P(p_2 \leq R < p_1)] - c_d$$

and

$$\pi_p(p_1, p_2) = N[(p_1 - w_p)P(R \geq p_1) + (p_2 - w_p)P(p_2 \leq R < p_1)],$$

respectively. We show that there always exist a pair of optimal prices in each period and

each setting such that the optimal price for selling the physical goods is greater than or equal to the optimal price for the digital good.

In the case of purchase-to-stock, the firm requires to decide both the order quantity Q and the retail price p to optimize its profit. The individual demand is $q = P(R > p) = \bar{F}(p)$. It implies that given retail price p and reserve price distribution F , the aggregated demand D follows the binomial distribution with parameters N and q , i.e., $D \sim \text{BIN}(N, q)$. We use G to represent the cumulative distribution function of D . In addition, we assume that all customers are myopic. They immediately buy the product as long as the reserve price is higher than the selling price. Note that if N is sufficiently large and p is small at the same time, the binomial distribution can be approximated by the normal distribution. In this case, the profit function π_d for the digital good remains the same. However, the profit function π_p for the physical good differs and it reads

$$\begin{aligned}
& \pi_p(p, Q) \\
&= N \cdot p \cdot P(\text{an item is available to the customer}) \cdot P(\text{the customer makes a purchase}) \\
&\quad - \text{supply cost} \\
&= Np \frac{p - w_p}{p} \bar{F}(p) - w_p Q \\
&= N(p - w_p) \bar{F}(p) - w_p Q.
\end{aligned}$$

2.1 Purchase-to-order mode

In this section we compare pricing of physical and digital goods in both single- and two-period settings.

Proposition 1. *In the single-period environment, there are optimal prices p_1^d and p_1^p associated with digital goods and physical goods, respectively such that $p_1^d < p_1^p$.*

Proof. We distinguish two cases: discrete and continuous reserve price distributions. We show the proof in the continuous case here and defer the proof in the discrete case to

Appendix. Let R be continuous and thus by assumption the reserve price distribution F has a strictly positive density function, $f(\cdot) > 0$ for all $[p, \bar{p}]$. Consider the two profit functions π_d and π_p . Let us denote by p_1^p and p_1^d the maximizers of the corresponding profit functions. Then, from the first order condition p_1^p solves

$$p_1 = w_p + \frac{\bar{F}(p_1)}{f(p_1)},$$

and p_1^d solves

$$p_1 = w_d + \frac{\bar{F}(p_1)}{f(p_1)}.$$

A sufficient condition for monotonicity of the optimal prices $p_1^p > p_1^d$ is that the reserve price function $F(p)$ is regular, i.e., $p - \frac{\bar{F}(p)}{f(p)}$ is strictly increasing in p . Our assumption of the monotone hazard rate function of F in the continuous case implies regularity of F . If $w(p) = p - \frac{1}{h(p)}$, then given $w_d < w_p$, it is straightforward to see that $p_1^d < p_1^p$. \square

In order to guarantee that the first order condition is sufficient for determining the optimal prices, unimodality of profit functions is needed, i.e., $(p - w_p)\bar{F}(p)$ and $(p - w_d)\bar{F}(p)$ are strictly unimodal. A sufficient condition for unimodality in the continuous distribution case is the monotone hazard function $h(p)$, which implies that the two profit functions are concave.

Substituting the first order conditions into the profit functions, we obtain that digital goods yield a higher profit in general if

$$\frac{\bar{F}(p_1^d)}{h(p_1^d)} - \frac{\bar{F}(p_1^p)}{h(p_1^p)} \geq \frac{c_d}{N}.$$

In particular, when the fixed cost of delivery of digital goods c_d is zero and $h(p)$ is increasing, it is more profitable to sell the digital goods.

We argued earlier that the optimal prices in the single-period setting are ordered. Now, we show the same property holds in the two-period setting.

Proposition 2. *In the two-period dynamic pricing environment, there exist optimal prices*

(p_1^d, p_2^d) and (p_1^p, p_2^p) associated with digital and physical goods, respectively, such that $p_1^p \geq p_1^d$ and $p_1^p \geq p_2^d$.

Proof. We distinguish two cases: discrete and continuous distributions. We prove the case of the continuous distribution here and defer the case of the discrete distribution to Appendix. Let R be continuous. Consider the profit functions $\pi_d(p_1, p_2)$ and $\pi_p(p_1, p_2)$ in the two-period setting. The first order conditions are

$$(I) : \begin{cases} p_2^d = p_1^d - \frac{1-F(p_1^d)}{f(p_1^d)} \\ w_d = p_2^d - \frac{F(p_1^d)-F(p_2^d)}{f(p_2^d)}, \end{cases}$$

and

$$(II) : \begin{cases} p_2^p = p_1^p - \frac{1-F(p_1^p)}{f(p_1^p)} \\ w_p = p_2^p - \frac{F(p_1^p)-F(p_2^p)}{f(p_2^p)}. \end{cases}$$

Optimal prices (p_1^d, p_2^d) satisfy the system of equations (I). We first show that $p_2^p \geq p_2^d$. Inequality $\pi_d(p_1^d, p_2^d) \geq \pi_d(p_1, p_2)$ for all p_1 and $p_2 \leq p_2^d$ implies that

$$\begin{aligned} & p_1^d[1 - F(p_1^d)] + p_2^d[F(p_1^d) - F(p_2^d)] - p_1[1 - F(p_1)] + p_2[F(p_1) - F(p_2)] \\ & \geq w_d[F(p_2) - F(p_2^d)] \\ & \geq w_p[F(p_2) - F(p_2^d)]. \end{aligned}$$

Hence, we have $\pi_p(p_1^d, p_2^d) \geq \pi_p(p_1, p_2)$ for all p_1 and $p_2 \leq p_2^d$. In particular, we have $\pi_p(p_1^d, p_2^d) \geq \pi_p(p_1^p, p_2)$ for all $p_2 \leq p_2^d$. By optimality of (p_1^p, p_2^p) , we have $\pi_p(p_1^p, p_2^p) \geq \pi_p(p_1^d, p_2^d) \geq \pi_p(p_1^p, p_2)$ for all $p_2 \leq p_2^d$. Thus, we conclude that $p_2^p \geq p_2^d$.

From $p_2^p \geq p_2^d$, it is straightforward to show $p_1^p \geq p_1^d$ using the first order conditions (I), (II) and the monotone hazard rate property. \square

We have shown the existence of ordered optimal prices (p_1^p, p_2^p) and (p_1^d, p_2^d) in the two-period setting. The existence of ordered optimal prices between physical and digital goods

is general. However, we do not claim that all optimal prices are ordered in such a manner. Consider the following example. Suppose the discrete distribution R is

$$R = \begin{cases} \frac{1}{2} & \text{with probability } \frac{1}{2}, \\ 1 & \text{with probability } \frac{1}{4}, \\ \frac{3}{2} & \text{with probability } \frac{1}{4}. \end{cases}$$

We further suppose that $w_p = \frac{1}{4}$ and $w_d = 0$. It can be easily verified that all optimal prices (p_1^d, p_2^d) are $(\frac{3}{2}, \frac{1}{2})$ or $(1, \frac{1}{2})$ for digital goods while all optimal prices (p_1^p, p_2^p) are $(\frac{3}{2}, 1)$, $(\frac{3}{2}, \frac{1}{2})$ or $(1, \frac{1}{2})$ for physical goods. The optimal prices $(1, \frac{1}{2})$ for physical goods are not larger than the optimal prices $(\frac{3}{2}, \frac{1}{2})$ for digital goods.

2.2 Purchase-to-stock mode

In this case, we consider pricing of physical and digital goods in the single-period setting. The seller's problem remains the same when selling digital goods, i.e., $\pi_d(p_1) = N(p_1 - w_d)P(R \geq p_1) - c_d$. However, the seller's problem when selling physical goods is maximizing $\pi_p(p, Q) = N(p - w_p)\bar{F}(p) - w_p Q$, where both p and Q are decision variables. Note that the optimal price p^* is a function of Q , i.e., $p^* = p^*(Q)$. Comparing with the purchase-to-order mode, Q is an additional variable for physical goods in this case. We show the order of the optimal prices in the following theorem. In the purchase-to-stock case, a weaker assumption of regularity of distribution function F is sufficient, comparing with the assumption of the monotone hazard rate of F in the purchase-to-order case. Distribution function F is regular if $p - \frac{\bar{F}(p)}{f(p)}$ is strictly increasing in p .

Theorem 1. *Assume distribution function F of the reserve price R is regular. Let the aggregated demand D be normal which holds if N goes to infinity. Let N be sufficiently large, and let the reserve price R be continuous. There is an optimal price p^d for digital goods which is lower than an optimal price $p^p(Q^*)$ for physical goods, i.e., $p^d < p^p(Q^*)$,*

where Q^* is the optimal purchase quantity.

Proof. See Appendix for the proof. □

We now investigate the same pricing problem in a small market, i.e., N is small. Let $N = 2$, and we assume that R is continuous. The optimal price of digital goods remains the same: under the regularity condition of distribution F the unique optimal price p^d solves $p - \frac{1-F(p)}{f(p)} = w_d$. However, we can no longer approximate the aggregated demand by the normal distribution. With $N = 2$, we have $D \sim \text{BIN}(2, q)$, where $q = \bar{F}(p)$. The cumulative distribution function of D is as follows.

$$\begin{aligned} G(0) &= F^2(p) \\ G(1) &= 2F(p) - F^2(p) \\ G(2) &= 1 \end{aligned}$$

For a fixed price p the optimal order quantity $Q^*(p)$ is the smallest integer in $\{0, 1, 2\}$ such that $P(D \leq Q^*(p)) = G(Q^*(p)) \geq \frac{p-w_p}{p}$.

Case 1: $Q^*(p^p) = 0$ for an optimal price p^p . By the discrete version of newsvendor, we have $G(0) \geq \frac{p-w_p}{p}$. It implies

$$p[1 - F^2(p)] \leq w_p.$$

In this case, the profit $\pi_p(p, Q^*(p)) = 0$. If there is an optimal price p^p such that $p^p[1 - F^2(p^p)] \leq w_p$ holds, then \bar{p} is an optimal price and the condition above also holds. Hence, Theorem 1 holds when $Q^*(p^p) = 0$.

Case 2: $Q^*(p^p) = 1$ for an optimal price p^p . The newsvendor argument for price p is $F^2(p) < \frac{p-w_p}{p} \leq 2F(p) - F^2(p)$. It implies

$$p\bar{F}^2(p) \leq w_p < p[1 - F^2(p)].$$

In this case, the profit function is $\pi_p(p, Q^*(p)) = \pi_p(p, 1) = 2pF(p)\bar{F}(p) - w_p$. We further

have

$$\frac{d}{dp}\pi_p(p, Q^*(p)) = 2\bar{F}(p)[F(p) + pf(p)] - 2pf(p)F(p).$$

In summary, in order to have an optimal price p^p corresponding to optimal order quantity $Q^*(p^p) = 1$, the following three conditions must hold:

- p^p solves $\bar{F}(p)[F(p) + pf(p)] = pf(p)F(p)$,
- p^p satisfies $F^2(p) < \frac{p-w_p}{p} \leq 2F(p) - F^2(p)$,
- $\pi_p(p^p, 1) \geq 0$.

Case 3: $Q^*(p^p) = 2$ for an optimal price p^p . The newsvendor argument for price p in this case is $2F(p) - F^2(p) < \frac{p-w_p}{p} \leq 1$. It implies $w_p < p\bar{F}^2(p)$. In this case, the profit function is $\pi_p(p, Q^*(p)) = \pi_p(p, 2) = 2pF(p)\bar{F}(p) + 2p\bar{F}^2(p) - 2w_p$. We further have

$$\frac{d}{dp}\pi_p(p, Q^*(p)) = 2[\bar{F}(p) - pf(p)].$$

In summary, in order to have an optimal price p^p corresponding to optimal order quantity $Q^*(p^p) = 2$, the following must hold:

- p^p solves $\bar{F}(p) = pf(p)$,
- p^p satisfies $w_p < p\bar{F}^2(p)$,
- $\pi_p(p^p, 2) \geq 0$.

The following example shows that Theorem 1 fails in the small market of $N = 2$. Let $F \sim U[0.1, 1.1]$. We obtain that $f(p) = 1$, $F(p) = p - 0.1$ and $\bar{F}(p) = 1.1 - p$ for every $p \in [0.1, 1.1]$. In addition, let $0.1 \geq w_p > w_d > 0$. We obtain that $p^p = 0.55$ is the optimal price and $Q^*(p^p) = 2$. In other words,

- The unique price $p^p = 0.55$ solves $\bar{F}(p) = pf(p)$,
- The optimal price $p^p = 0.55$ satisfies $w_p < p\bar{F}^2(p)$,

- Inequality $\pi_p(0.55, 2) = 0.625 - 2w_p > 0$ holds. In addition, we need to check the two boundary points. If price $p = 0.1$, we obtain $Q^*(p) = 2$. We further have $\pi_p(0.1, 2) = 0.2 - 2w_p > 0$ and $\pi_p(0.55, 2) > \pi_p(0.1, 2)$. If price $p = 1.1$, we have $Q^*(p) = 0$, and we obtain $\pi_p(1.1, 0) = 0$. Hence, $\pi_p(0.55, 2) > \pi_p(1.1, 0)$.

Thus, we have shown that the example above has a unique optimal price $p^p = 0.55$. On the other hand, the digital goods case has a unique optimal price $p^d = \frac{w_d + 1.1}{2} > 0.55$.

3 Pricing of digital goods under externality

For goods that can be shared or split, it is common for customers to experience externality to some degree. For instance, a procurement contract split between two suppliers may generate disutility to each supplier when they compete in an upstream market. As a second example consider licensing an innovation. A losing firm which does not get any allocation may experience negative externality due to the increased market power of the competitor. Standard physical goods are usually not subject to externality because of the excludability of the physical goods. For digital goods that could be easily replicated and shared among customers, it is important to consider the incurred externality.

In this section, we extend the optimal pricing strategies to digital goods with externality. We assume a pure negative externality due to not possessing the good. For digital goods with externality, a customer values a good depending on the allocation outcome. In other words, a customer places a strict value if s/he obtains the good, otherwise an externality value is accumulated. There are a finite number N of customers to sell the goods to. We denote by R , F the strict valuation and its cumulative distribution function, respectively. Further, we assume that R is continuous and the density function $f(p) > 0$ for all $p \in [0, \bar{p}]$. We use c_e to represent externality and assume that the externality function is linear in the strict valuation, i.e., $c_e(r) = c_e \cdot r$, where $-1 \leq c_e \leq 0$. Since in this section we focus only on digital goods, we denote by w , π the unit replenishment cost and the profit function,

respectively. We examine the single- and multi-period settings.

In the single-period setting, we model the pricing problem as follows. In the pricing of digital goods without externality, the probability of making a purchase for a consumer is $1 - F(p)$, where p is the posted price. Instead, in the case of digital goods with externality, this probability becomes $P(R > p) + P(R \leq p, (1 - c_e)R > p)$. Hence the expected profit function with fixed cost c_d can be expressed as

$$\begin{aligned}\pi(p) &= N(p - w)[P(R > p) + P(R \leq p, (1 - c_e)R > p)] - c_d \\ &= N(p - w)\left[1 - F\left(\frac{p}{1 - c_e}\right)\right] - c_d.\end{aligned}$$

Assuming monotonicity of the hazard rate function of R , the expected profit $\pi(p)$ is a unimodal function since the monotone hazard rate implies that the first order derivative of the profit with respect to the demand is monotone decreasing.

In the multi-period setting, we model the pricing of digital goods under externality by a dynamic program. We drop the fixed cost c_d because it does not depend on the number of price adjustments within the horizon as long as at least one is made. We assume a finite time horizon with T periods. We consider the customer's strict valuation R as the initial reserve price. We further assume that customers' reserve prices R_t with distribution functions F_t change over time. In other words, the reserve prices R_t depend on the number of customers who already bought the goods in previous periods. We assume that customers always make their purchase in the earliest period with the posted price being lower than their reserve price. The customer pool is initially a sample from a random population. The price decisions provide additional information about the sample distribution which implies updates to the prior distribution.

A purchase occurs in period t if $p_t < R_t < \frac{p_t - 1}{1 - c_e}$ or $R_t \leq p_t$ and $(1 - c_e)R_t > p_t$, where p_t is the posted price in period t . With probability $\bar{F}_t(p_t)$ a customer makes a purchase in period t . Let us denote by $S_t(x_t, p_t, F_t())$ the total sales in period t which depends on

the remaining number of customers x_t , the price p_t and the reserve price distribution F_t . Quantity $S_t(x_t, p_t, F_t())$ has a binomial distribution with parameter x_t and $\bar{F}_t(p_t)$. The probability of total sales of y units in period t is expressed as

$$P(S_t(x_t, p_t, F_t()) = y) = \binom{x_t}{y} (\bar{F}_t(p_t))^y (1 - \bar{F}_t(p_t))^{x_t - y} \text{ for } y = 0, \dots, x_t.$$

Note that the expected sales are given by $E[S_t(x_t, p_t, F_t())] = x_t \bar{F}_t(p_t)$. We formulate the problem as a dynamic program. In the DP model, the state includes the remaining number of customers x_t at time t and the running minimal posted price r_t by time t . Let $v_t(x_t, r_t)$ be the maximum expected profit-to-go starting in period t and with x_t remaining customers. The DP formulation reads

$$v_t(x_t, r_t) = \max_{p_t \geq 0} \{ (p_t - w)x_t[1 - F_t(p_t)] + E[v_{t+1}(x_t - S_t(x_t, p_t, F_t(p_t)), r_{t+1})] \},$$

where $v_t(0, r_t) = 0$ for any t and r_t , and the boundary condition is $v_{T+1}(\cdot, \cdot) = 0$. The state transitions of the running minimal price follow

$$r_{t+1} = \begin{cases} r_t & \text{if } p_t \geq r_t, \\ \frac{p_t}{1 - c_e} & \text{if } p_t < r_t. \end{cases}$$

There are a couple of challenges in this formulation. We need to know how the reserve price distribution is updated as a function of p_t and x_t . In other words, we need to understand how F_{t+1} is related to F_t . Starting from a reserve price distribution F , we need to obtain F_t as a function of the prices p_τ , $\tau \leq t$. Under the assumption that all customers whose reserve prices exceed the posted price immediately make a purchase (i.e., customers are not time-strategic), all of the information that is needed is $\min_{\tau \leq t} p_\tau$. In fact, we know that the reserve prices of all remaining customers in period t are lower than $\min_{\tau \leq t} \frac{p_\tau}{1 - c_e}$. In general, updating the distribution based on observed realizations is non-trivial. We assume that

the updated distribution corresponds to the conditioned version of the initial distribution truncated to the interval $(0, \min_{\tau \leq t} \frac{p\tau}{1-c_e})$. As a result, we have

$$F_t(p) = \begin{cases} \frac{F(\frac{p}{1-c_e})}{F(r_t)} & \text{if } p < r_t, \\ 1 & \text{if } p \geq r_t. \end{cases}$$

Consequently, the optimality equation becomes

$$v_t(x_t, r_t) = \max_{p_t \geq 0} \left\{ (p_t - w)x_t \left[1 - \frac{F(\frac{p_t}{1-c_e})}{F(r_t)} \right] + E[v_{t+1}(x_t - S_t(x_t, p_t, F_t(p_t)), r_{t+1})] \right\}. \quad (1)$$

As an example, if we start with a uniform distribution between 0 and 1 in period 1 and post a price of p_1 , F_2 is a uniform distribution between 0 and $\frac{p_1}{1-c_e}$. Moreover, if we post a price of p_2 in period 2 with $p_2 < p_1$, then F_3 is a uniform distribution between 0 and $\frac{p_2}{1-c_e}$.

3.1 Single-period pricing

We show that the optimal price for digital goods with externality is at least as large as the optimal price in the case without externality. Recall that the optimal price p^1 in the scenario without externality solves $p = w + \frac{1-F(p)}{f(p)}$. The optimal price p^2 under externality solves

$$\frac{p - w}{1 - c_e} = \frac{1 - F(\frac{p}{1-c_e})}{f(\frac{p}{1-c_e})}.$$

We then have

$$\begin{aligned}
\frac{w}{1-c_e} &= \frac{p^2}{1-c_e} - \frac{1-F(\frac{p^2}{1-c_e})}{f(\frac{p^2}{1-c_e})} \\
&= \frac{p^1}{1-c_e} - \frac{1}{1-c_e} \frac{1-F(p^1)}{f(p^1)} \\
&\geq \frac{p^1}{1-c_e} - \frac{1-F(p^1)}{f(p^1)} \\
&\geq \frac{p^1}{1-c_e} - \frac{1-F(\frac{p^1}{1-c_e})}{f(\frac{p^1}{1-c_e})},
\end{aligned}$$

which implies that $p^1 \leq p^2$ since $\frac{f(\cdot)}{1-F(\cdot)}$ is strictly increasing.

Next we further investigate the optimal prices for different distributions subject to the hazard rate order. The monotone hazard rate order is related to the following stochastic order. Continuous random variable X is said to be smaller than continuous random variable Y with respect to the hazard rate order, $X \leq_{hr} Y$, if the function $\frac{\bar{F}_Y(t)}{\bar{F}_X(t)}$ is increasing in t . It has been shown in Muller & Stoyan (2002).

We denote by p_1^F an optimal price in the one-period problem with respect to distribution F_1 and p_2^F with respect to F_2 . Define $\psi(\cdot) = 1 - \frac{1-F(\cdot)}{f(\cdot)}$. The next proposition shows that these two optimal prices are ordered, i.e., $p_1^F \leq p_2^F$.

Proposition 3. *For two hazard rate ordered distributions F_1 and F_2 , i.e., $\frac{\bar{F}_2(\cdot)}{\bar{F}_1(\cdot)}$ is increasing, there exist optimal prices p_1^F and p_2^F , respectively, such that $p_1^F \leq p_2^F$.*

Proof. By the assumption of the hazard rate order, we observe $\psi_1(\frac{p}{1-c_e}) \geq \psi_2(\frac{p}{1-c_e})$ for all p . In addition, they are also continuous because the density function f is continuous and $f(\frac{p}{1-c_e}) > 0$ for all $p \in [0, \bar{p}]$. The Brouwer's Fixed Point Theorem establishes that there must exist unique fixed points p_1^F and p_2^F for $\psi_1(\frac{p}{1-c_e})$ and $\psi_2(\frac{p}{1-c_e})$, respectively. By the optimality condition, we have

$$\frac{w}{1-c_e} = \psi_1\left(\frac{p_1^F}{1-c_e}\right) = \psi_2\left(\frac{p_2^F}{1-c_e}\right).$$

Since ψ_i is monotone for all $i = 1, 2$, we have

$$\psi_1\left(\frac{p_2^F}{1 - c_e}\right) \geq \psi_1\left(\frac{p_1^F}{1 - c_e}\right),$$

$$\psi_2\left(\frac{p_2^F}{1 - c_e}\right) \geq \psi_2\left(\frac{p_1^F}{1 - c_e}\right).$$

This implies that $p_1^F \leq p_2^F$. □

Proposition 3 shows that a higher optimal selling price can be extracted from a reserve price distribution with the heavier tail. In other words, if the reserve price is distributed more toward the higher end, then a higher selling price can be reached.

3.2 Dynamic pricing

We modeled the pricing of digital goods under externality in the multi-period setting as a dynamic program. Next we investigate the structure of the optimal prices and the relationship of the optimal prices between the single- and multi-period settings.

3.2.1 The structure of optimal prices

We study the optimal pricing strategy p_t^d for $t = 1, \dots, T$ of the DP model, where p_t^d is the optimal price of period t in the multi-period setting. We first claim that there exists a decreasing optimal price pattern over time.

Theorem 2. *There exist optimal prices p_t^d such that $r_t > p_t^d > w$. Furthermore, we have $p_t^d > p_{t+1}^d$ for every t .*

Proof. The proof is deferred to Appendix. □

It is not surprising that we have the decreasing optimal price pattern in the case under externality due to the dynamics of the model. One of the state variables, the running minimal posted prices, supports the decreasing optimal prices. Next, we compare the optimal prices between the single- and multi-period problems.

Proposition 4. We have $p_1^d \geq p^d \geq p_T^d$.

Proof. See Appendix for the proof. □

The inequality of optimal prices between single- and multi-period settings demonstrates a high-low optimal pricing strategy, which is the advantage of the dynamic pricing model. The seller can manipulate the prices and extract additional buyer's payoffs in multiple periods.

Let us consider a specific case to illustrate how the updating and pricing decisions interact. We assume $N = 2$, $T = 2$, $c_e = -0.1$ and $w = 0$. The prior reserve price distribution F is uniform in $(0,1)$. Let us assume we start with an aggressive pricing policy with $p_1 = 0.9$. At this price, the sale probabilities are as follows.

$$\begin{aligned} P(S_1(2, 0.9, F) = 0) &= 0.9^2 \\ P(S_1(2, 0.9, F) = 1) &= 2 \cdot 0.9 \cdot 0.1 \\ P(S_1(2, 0.9, F) = 2) &= 0.1^2 \end{aligned}$$

Moving on to period 2, we update the reserve price distribution with the information that any customer who did not purchase in the first period must have a reserve price less than 0.9 (otherwise they would have purchased in the first period). Our prior distribution was uniform in $(0,1)$, but clearly our remaining customers are uniformly distributed in $(0,0.9)$. Therefore, the updated reserve price distribution F_2 is uniform in $(0,0.9/1.1)$ after taking externality into account. We can now find the optimal selling price in period 2. Of course, if the remaining number of customers is zero, we have $v_2(0, 0.9/1.1) = 0$. In addition, given that $F_2(p_2) = p_2/0.9$ for $p_2 \leq 0.9/1.1$, we obtain

$$v_2(1, 0.9/1.1) = \max_{0.9/1.1 > p_2 > 0} \{p_2 \bar{F}_2(p_2)\} = \max_{0.9/1.1 > p_2 > 0} \left\{p_2 \left(1 - \frac{p_2}{0.9}\right)\right\}.$$

We conclude that given the initial price 0.9 and 1 remaining customer in period 2, the optimal

selling price in period 2 is $p_2 = 0.45$. Similarly,

$$v_2(2, 0.9/1.1) = \max_{0.9/1.1 > p_2 > 0} \{2p_2 \bar{F}_2(p_2)\} = \max_{0.9/1.1 > p_2 > 0} \left\{2p_2 \left(1 - \frac{p_2}{0.9}\right)\right\}$$

which yields the optimal selling price $p_2 = 0.45$ with 2 remaining customers in period 2.

Note that if we had not updated the reserve price distribution, we would have assumed that the customers in period 2 have reserve prices in $(0,1/1.1)$, leading to a different price of $p_2 = 0.5$. The profit corresponding to the pricing policy of $p_1 = 0.9, p_2 = 0.45$ is 0.73. The optimal price strategy of our model turns out to be $p_1^d = 0.76$ and $p_2^d = 0.38$. The corresponding optimal profit is $v_1(2, 1) = 0.76$, which is greater than 0.73.

4 Summary and concluding remarks

Easy replication of digital goods makes the pricing problem simple in the sense that there is no inventory capacity constraint. However, a consequence of the negligible marginal production cost is externality. In other words, a customer's valuation of digital goods depends on the final allocation of products.

We first examine the differences between pricing of the standard physical goods and digital goods. We show that the optimal price of digital goods is usually lower than the optimal price of physical goods. Without taking the capacity into account, the optimal price p^p with regard to physical goods is at least as large as the optimal price p^d of digital goods. Pricing of digital goods under capacity constraint exhibits the challenge of combining price and production/order decisions. We show for physical goods that optimal prices $p^p \geq p^p(Q)$, where $p^p(Q)$ is an optimal price for physical goods with capacity constraint Q . However, inequality $p^p(Q) \geq p^d$ only holds in a large market when the customer base N is sufficiently large.

We analyze pricing of digital goods with externality in various scenarios. In the single-period setting, we show there is an optimal price which is no less than an optimal price in

the setting without externality. The optimal prices with respect to the hazard rate order distributions are also ordered. A dynamic model for a multi-period pricing is also introduced. We show the decreasing pattern of optimal prices and investigate the relationship between optimal prices of the single- and multi-period problems.

References

- Amaldoss, W., & Jain, S. 2005. Conspicuous Consumption and Sophisticated Thinking. *Management Science*, **51**(10), 1449–1466.
- Bakos, Y., & Brynjolfsson, E. 1999. Bundling Information Goods: Pricing, Profits and Efficiency. *Management Science*, **45**(12), 1613–1630.
- Cachon, G., & Kok, A. 2007. Implementation of the Newsvendor Model with Clearance Pricing: How to (and How Not to) Estimate a Salvage Value. *Manufacturing and Service Operations Management*, **9**(3), 276–290.
- Chou, M., Sim, C., Teo, C., & Zheng, H. 2012. Newsvendor Pricing Problem in a Two-Sided Market. *Production and Operations Management*, **21**(1), 204–208.
- Dana, J., & Petruzzi, N. 2001. Note: The Newsvendor Model with Endogenous Demand. *Management Science*, **47**(11), 1488–1497.
- Geoffrion, A. 2002. Progress in Operations Management. *Production and Operations Management*, **11**(1), 92–100.
- Leibenstein, H. 1950. Bandwagon, Snob and Veblen Effects in the Theory of Consumer Demand. *Quarterly Journal of Economics*, **64**(2), 183–207.
- Mattioli, D. 2012. On Orbitz, Mac Users Steered to Pricier Hotels. *The Wall Street Journal*, June 26.
- Muller, A., & Stoyan, D. 2002. *Comparison Methods for Stochastic Models and Risks*. England: West Sussex: John Wiley and Sons, Ltd.
- Porteus, E. 2002. *Foundations of Stochastic Inventory Theory*. Stanford University Press: Englewood Cliffs, NJ.

- Rabinovich, E., Maltz, A., & Sinha, R. 2008. Assessing Markups, Service Quality, and Product Attributes in Music CDs' Internet Retailing. *Production and Operations Management*, **17**(3), 320–337.
- Rodriguez, A., & Locay, L. 2002. Two Models of Intertemporal Price Discrimination. *Journal of Economics*, **76**(3), 261–278.
- Schmitz, P. 2002. On Monopolistic Licensing Strategies under Asymmetric Information. *Journal of Economic Theory*, **106**(1), 177–189.
- Segall, L. 2012 (January). *Digital Music Sales Top Physical Sales*. <http://money.cnn.com>.
- Sisario, B. 2011 (January). *Digital Music Sales are Starting to Slow, Report Says*. NY-Times.com.
- Sundararajan, A. 2004. Nonlinear Pricing of Information Goods. *Management Science*, **50**(12), 1660–1673.
- Sweeney, M. 2012 (May). *Digital Music Spending Greater than Sales of CDs and Records for First Time*. <http://www.guardian.co.uk/>.
- Varian, H. 2000. Buying, Sharing and Renting Information Goods. *The Journal of Industrial Economics*, **48**(4), 473–488.

Appendix

Proof of Proposition 1. Let now R be discrete. Suppose p_1^d is an optimal price in the digital goods setting, i.e.,

$$p_1^d = \arg \max_{a_i} \{(a_i - w_d)(1 - f_1 - \dots - f_{i-1})\}.$$

We next show that there is an optimal price $p_1^p \geq p_1^d$ in the physical goods setting. Since $\pi_d(p_1^d) \geq \pi_d(p_1)$ for all $p_1 \leq p_1^d$ by optimality of p_1^d , i.e., $(p_1^d - w_d)P(R \geq p_1^d) \geq (p_1 - w_d)P(R \geq$

p_1), we have

$$\begin{aligned} p_1^d P(R \geq p_1^d) - p_1 P(R \geq p_1) &\geq -w_d P(p_1 \leq R \leq p_1^d) \\ &\geq -w_p P(p_1 \leq R \leq p_1^d). \end{aligned}$$

It implies that $\pi_p(p_1^d) \geq \pi_p(p_1)$ for all $p_1 \leq p_1^d$. Hence, we must have an optimal price p_1^p in the physical goods scenario that is greater than or equal to p_1^d , i.e., $p_1^p \geq p_1^d$. \square

Proof of Proposition 2. In the discrete case, we argue separately that $p_1^p \geq p_1^d$ and $p_2^p \geq p_2^d$, where (p_1^p, p_2^p) and (p_1^d, p_2^d) are optimal prices for the physical and digital goods, respectively.

Inequality $\pi_d(p_1^d, p_2^d) \geq \pi_d(p_1, p_2)$ for all (p_1, p_2) with $p_2 \leq p_2^d$ implies that

$$\begin{aligned} &p_1^d P(R \geq p_1^d) + p_2^d P(p_2^d \leq R < p_1^d) - p_1 P(R \geq p_1) - p_2 P(p_2 \leq R < p_1) \\ &\geq w_d [P(R \geq p_2^d) - P(R \geq p_2)] \\ &\geq w_p [P(R \geq p_2^d) - P(R \geq p_2)]. \end{aligned}$$

The first inequality follows by optimality of (p_1^d, p_2^d) for digital goods and the second inequality holds by $w_p > w_d$. Thus, we have $\pi_p(p_1^d, p_2^d) \geq \pi_p(p_1, p_2)$ for all (p_1, p_2) with $p_2 \leq p_2^d$. In particular, $\pi_p(p_1^d, p_2^d) \geq \pi_p(p_1^p, p_2)$ for all $p_2 \leq p_2^d$. Thus, we obtain that $\pi_p(p_1^p, p_2^p) \geq \pi_p(p_1^p, p_2)$ for all $p_2 \leq p_2^d$, and therefore there exist optimal prices p_2^p and p_2^d such that $p_2^p \geq p_2^d$.

We next show that the optimal prices are also ordered in the first period, i.e., $p_1^p \geq p_1^d$. Suppose not, i.e., we have $p_1^p < p_1^d$. We compare $\pi_p(p_1^d, p_2^d)$ and $\pi_p(p_1^p, p_2^d)$. Inequality $\pi_d(p_1^d, p_2^d) \geq \pi_d(p_1^p, p_2^d)$ implies that

$$p_1^d P(R \geq p_1^d) + p_2^d P(p_2^d \leq R < p_1^d) - p_1^p P(R \geq p_1^p) - p_2^d P(p_2^d \leq R < p_1^p) \geq 0.$$

It further implies that $\pi_p(p_1^d, p_2^d) \geq \pi_p(p_1^p, p_2^d)$ for physical goods. Then, we obtain $p_1^d P(R \geq p_1^d) - p_1^p P(R \geq p_1^p) \geq -p_2^d P(p_2^d \leq R < p_1^d)$, which implies that $p_1^d P(R \geq p_1^d) - p_1^p P(R \geq p_1^p) \geq$

$-p_2^p P(p_2^p \leq R < p_1^d)$ since $p_2^p \geq p_2^d$. We conclude that $\pi_p(p_1^d, p_2^p) \geq \pi_p(p_1^p, p_2^p)$. Thus, p_1^d is also an optimal price for physical goods. We just found an optimal price p_1^d for physical goods, which is the same as the optimal price for digital goods. This contradicts the hypothesis and thus there exist optimal prices p_1^p and p_1^d such that $p_1^p \geq p_1^d$. \square

Proof of Theorem 1. Given a fixed order quantity Q , we examine a randomly observed individual customer with Bernoulli demand $q = \bar{F}(p)$. The inverse demand curve $p = F^{-1}(1 - q)$ is non-increasing. We can directly derive the marginal revenue from the revenue function $\pi(q) = p \cdot q = qF^{-1}(1 - q)$ as follows. We denote $MR(p)$ or $MR(q)$ to be the marginal revenue in the general pricing scenario where we do not consider the order quantity Q . If for a given quantity Q the aggregated demand $D \leq Q$,¹ then the marginal revenue is expressed as

$$\begin{aligned} MR(q) &= \frac{\partial \pi(q)}{\partial q} \\ &= F^{-1}(1 - q) + q \cdot \frac{dF^{-1}(1 - q)}{dq} \\ &= F^{-1}(1 - q) - \frac{q}{f[F^{-1}(1 - q)]}. \end{aligned}$$

Notice above that since $F[F^{-1}(1 - q)] = 1 - q$, we obtain that

$$\frac{dF[F^{-1}(1 - q)]}{dq} = f[F^{-1}(1 - q)] \cdot \frac{dF^{-1}(1 - q)}{dq},$$

which implies that

$$\begin{aligned} \frac{dF^{-1}(1 - q)}{dq} &= \frac{1}{f[F^{-1}(1 - q)]} \cdot \frac{dF[F^{-1}(1 - q)]}{dq} \\ &= \frac{1}{f[F^{-1}(1 - q)]} \frac{d(1 - q)}{dq} \\ &= -\frac{1}{f[F^{-1}(1 - q)]}. \end{aligned}$$

¹Note that this is the case in the general pricing scenario without taking the inventory decision into account.

Equivalently, in this case the marginal revenue is expressed in terms of p as

$$MR(p) = p - \frac{1 - F(p)}{f(p)}.$$

We can find the optimal Q for any p by the standard newsvendor argument. In other words, $Q^*(p) = G^{-1}(\frac{p-w_p}{p})$. In particular, at optimal $Q^*(p)$ for any p the expected profit from a randomly observed customer is

$$\begin{aligned} \pi_p(p, Q^*(p)) &= pP(\text{an item is available to the customer})P(\text{the customer makes a purchase}) \\ &\quad - \text{supply cost} \\ &= p \frac{p - w_p}{p} \bar{F}(p) - w \frac{Q^*(p)}{N} \\ &= \bar{F}(p)(p - w_p) - w \frac{Q^*(p)}{N}. \end{aligned}$$

Notice that we approximate the discrete binomial demand by continuous normal demand as stated in the assumption of the theorem. It follows that

$$P(\text{an item is available to the customer}) = P(D \leq Q^*) = \frac{p - w_p}{p}.$$

The expected profit without considering the order quantity Q is

$$\pi_p(p) = \bar{F}(p)(p - w_p).$$

We next compare the above two profit functions. Particularly, we consider the term $w_p \bar{F}(p)$ as the individual supply cost and $p \bar{F}(p) - \frac{w_p Q^*(p)}{N}$ and $p \bar{F}(p)$ as individual revenue terms, respectively. Then, we differentiate both the cost and revenue terms with respect to q in order to obtain the marginal cost and revenue. The marginal cost is the constant w_p in both cases, while the marginal revenue terms vary. To be specific, the marginal revenue $MR(p)$ in the general pricing scenario remains the same. In the joint pricing and inventory

scenario, the expected marginal revenue¹ is

$$E[MR(p, Q^*(p))] = p - \frac{1 - F(p)}{f(p)} - \frac{w_p}{N} \frac{\partial}{\partial q} G^{-1}\left(1 - \frac{w_p}{p}\right),$$

where G^{-1} is the inverse demand function. We reapply the technique in deriving $\frac{dF^{-1}(1-q)}{dq}$ and obtain that

$$\begin{aligned} \frac{\partial}{\partial q} G^{-1}\left(1 - \frac{w_p}{p}\right) &= \frac{-1}{f(p)} \frac{\partial}{\partial p} G^{-1}\left(1 - \frac{w_p}{p}\right) \\ &= \frac{-1}{f(p)} \frac{w_p}{p^2} \frac{1}{g[G^{-1}(1 - \frac{w_p}{p})]} \\ &= \frac{-1}{f(p)} \frac{w_p}{p^2} \frac{\sqrt{NF(p)\bar{F}(p)}}{\phi[\Phi^{-1}(1 - \frac{w_p}{p})]}, \end{aligned}$$

where g is the probability density function of the aggregated demand D . Functions ϕ , Φ and Φ^{-1} are the probability density, cumulative distribution and inverse cumulative distribution functions of the normal demand, respectively. It can be seen that the expected marginal revenue curve $E[MR(p, Q)]$ is above the curve of $MR(p)$, Figure 1. Since the last term in $E[MR(p, Q^*(p))]$ is strictly positive, we conclude that the expected marginal revenue in the joint pricing and inventory case is strictly greater than the marginal revenue in the other case. It can be observed from Figure 1 that the optimal price depending on Q is less than or equal to the optimal price without taking the decision Q into account, i.e., $p^p(Q^*) \leq p^p$.

We now consider the marginal profit in the joint pricing scenario as

$$\frac{\partial}{\partial q} \pi(p, Q^*(p)) = p - \frac{1 - F(p)}{f(p)} + \frac{w_p^2}{N} \frac{1}{p^2 f(p)} \frac{\sqrt{NF(p)\bar{F}(p)}}{\phi[\Phi^{-1}(1 - \frac{w_p}{p})]} - w_p.$$

We decompose the marginal profit and regard

$$MC(p, Q^*(p)) = w_p - \frac{w_p^2}{N} \frac{1}{p^2 f(p)} \frac{\sqrt{NF(p)\bar{F}(p)}}{\phi[\Phi^{-1}(1 - \frac{w_p}{p})]}$$

¹Expected value is with regard to the aggregated demand.

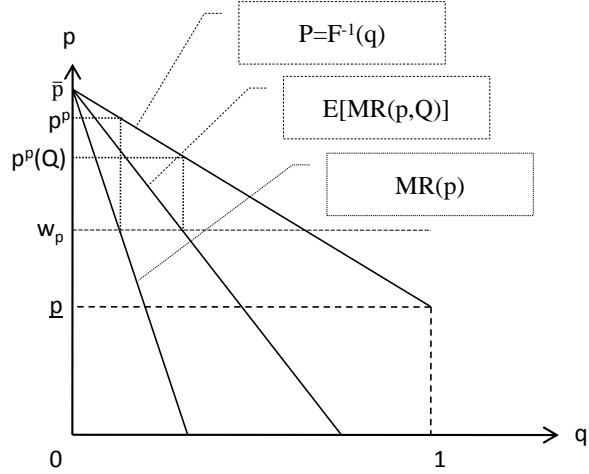


Figure 1: Construction of Optimal Prices

as the marginal cost while holding the marginal revenue term the same as in the general pricing scenario. Since by assumption $w_p > w_d$ and N is large, under the bounded domain of distribution F as assumed in Section 2, we obtain $MC(p, Q^*(p)) > w_d$. Thus by the regularity condition we further have $p^p(Q^*) > p^d$.

We have shown that there exists an optimal price for physical goods that is at least as large as an optimal price for the digital goods. \square

Proof of Theorem 2. We prove this theorem by induction. When $t = T$, it is clear that there exists an optimal price $p_{T+1}^d = 0$ since the terminal value is defined by $v_{T+1}(\cdot, \cdot) = 0$. Hence, we apparently have $p_T^d > p_{T+1}^d = 0$. We also have $p_T^d < r_T$ because

- if $p_T \geq r_T$, the profit-to-go at time T is zero, and
- if $w < p_T < r_T$, we have positive profit. In other words,

$$(p_T - w)x_T \left[\frac{F(r_T) - F\left(\frac{p_T}{1-c_e}\right)}{F(r_T)} \right] > 0 \text{ for all } x_T > 0.$$

Suppose now that the theorem holds for $t + 1$. For all x_{t+1} and r_{t+1} there exists an

optimal price p_{t+1}^d such that $p_{t+1}^d < r_{t+1}$ and

$$v_{t+2}(x_{t+1}, r_{t+1}) \leq (p_{t+1}^d - w)x_{t+1} \left[\frac{F(r_{t+1}) - F(\frac{p_{t+1}^d}{1-c_e})}{F(r_{t+1})} \right] + \mathbb{E}[v_{t+2}(x_{t+1} - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})].$$

We want to show that there exists an optimal price $p_t^d > w$ such that $p_t^d < r_t$ and

$$v_{t+1}(x_t, r_t) \leq (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+1}(x_t - S_t, \frac{p_t^d}{1-c_e})].$$

By the induction hypothesis, it can be seen that

$$\begin{aligned} \mathbb{E}[v_{t+2}(x_t - S_t, r_{t+1})] &\leq \mathbb{E} \left\{ (p_{t+1}^d - w)(x_t - S_t) \left[\frac{F(r_{t+1}) - F(\frac{p_{t+1}^d}{1-c_e})}{F(r_{t+1})} \right] \right\} \\ &\quad + \mathbb{E}[v_{t+2}(x_t - S_t - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})]. \end{aligned}$$

We further have

$$v_{t+1}(x_t, r_t) = (p_{t+1}^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_{t+1}^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+2}(x_t - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})].$$

In addition, suppose that there is an optimal price p_t^d in period t with $p_t^d < r_t$. Then we have

$$\begin{aligned} &(p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+1}(x_t - S_t, \frac{p_t^d}{1-c_e})] \\ = &(p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E} \left\{ (p_{t+1}^d - w)(x_t - S_t) \left[\frac{F(\frac{p_t^d}{1-c_e}) - F(\frac{p_{t+1}^d}{1-c_e})}{F(\frac{p_t^d}{1-c_e})} \right] \right\} \\ &+ \mathbb{E}[v_{t+2}(x_t - S_t - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})]. \end{aligned}$$

Let us denote

$$LPV_t = (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+1}(x_t - S_t, \frac{p_t^d}{1-c_e})].$$

Hence, we further have

$$\begin{aligned} LPV_t &= (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E} \left\{ (p_{t+1}^d - w)(x_t - S_t) \left[\frac{F(\frac{p_t^d}{1-c_e}) - F(\frac{p_{t+1}^d}{1-c_e})}{F(\frac{p_t^d}{1-c_e})} \right] \right\} \\ &\quad + \mathbb{E}[v_{t+2}(x_t - S_t - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})] \\ &\geq (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+2}(x_t - S_t, \frac{p_t^d}{1-c_e})]. \end{aligned}$$

We next show that $LPV_t \geq v_{t+1}(x_t, r_t)$. The seller is only allowed to make one price adjustment either at the beginning of t or $t + 1$. After choosing the decision, the seller has to skip the other period and proceed directly to the beginning of $t + 2$, see Figure 2.

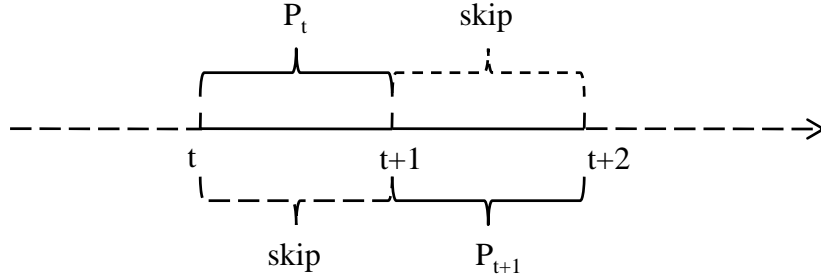


Figure 2: Illustration of restricted price adjustment

We use subscript 2 in addition to the subscripts of t and $t + 1$ to capture the two pricing scenarios and obtain that for all x_t and r_t we have

$$v_{2,t}(x_t, r_t) = (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+2}(x_t - S_t, \frac{p_t^d}{1-c_e})]$$

and

$$v_{2,t+1}(x_t, r_t) = (p_{t+1}^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_{t+1}^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+2}(x_t - S_{t+1}, \frac{p_{t+1}^d}{1-c_e})].$$

It can be seen that in the case of a single price adjustment within the considered two periods, $v_{2,t}(x_t, r_t) = v_{2,t+1}(x_t, r_t)$. Thus, we obtain

$$LPV_t \geq v_{2,t}(x_t, r_t) = v_{2,t+1}(x_t, r_t) = v_{t+1}(x_t, r_t).$$

Eventually, we conclude that $p_t^d < r_t$ and

$$v_t(x_t, r_t) = (p_t^d - w)x_t \left[\frac{F(r_t) - F(\frac{p_t^d}{1-c_e})}{F(r_t)} \right] + \mathbb{E}[v_{t+1}(x_t - S_t, \frac{p_t^d}{1-c_e})] \geq v_{t+1}(x_t, r_t).$$

In summary, we have shown that there exists a decreasing optimal price pattern over time in the *DP* model. □

Proof of Proposition 4. Recall that based on (1), we have

$$v_t(x_t, r_t) = \max_{0 \leq p_t \leq r_t} \left\{ (p_t - w)x_t \left[\frac{F(r_t) - F(\frac{p_t}{1-c_e})}{F(r_t)} \right] + E[v_{t+1}(x_t - S_t, r_{t+1})] \right\}.$$

We first show that $p^d \geq p_T^d$. For all $x_T > 0$ we have

$$\begin{aligned} v_T(x_T, r_T) &= \max_{0 \leq p_T \leq r_T} \left\{ (p_T - w)x_T \left[\frac{F(r_T) - F(\frac{p_T}{1-c_e})}{F(r_T)} \right]^+ \right\} \\ &= \max_{0 \leq p_T \leq r_T} \left\{ (p_T - w)x_T \left[\frac{F(r_T) - F(\frac{p_T}{1-c_e})}{F(r_T)} \right] \right\}. \end{aligned}$$

The first order condition implies that p_T^d satisfies

$$p_T = w + (1 - c_e) \frac{F(r_T) - F(\frac{p_T}{1-c_e})}{f(\frac{p_T}{1-c_e})}.$$

In the single-period setting, p^d satisfies

$$p = w + (1 - c_e) \frac{1 - F\left(\frac{p}{1 - c_e}\right)}{f\left(\frac{p}{1 - c_e}\right)}.$$

Hence, $p^d \geq p_T^d$ because

$$\begin{aligned} w &= p^d - (1 - c_e) \frac{1 - F\left(\frac{p^d}{1 - c_e}\right)}{f\left(\frac{p^d}{1 - c_e}\right)} \\ &= p_T^d - (1 - c_e) \frac{F(r_T) - F\left(\frac{p_T^d}{1 - c_e}\right)}{f\left(\frac{p_T^d}{1 - c_e}\right)} \\ &\geq p_T^d - (1 - c_e) \frac{1 - F\left(\frac{p_T^d}{1 - c_e}\right)}{f\left(\frac{p_T^d}{1 - c_e}\right)}. \end{aligned}$$

Note that the last inequality follows from the assumption of monotonicity of the hazard rate function.

Now we show $p_1^d \geq p^d$. The optimal expected profit at time $t = 1$ reads

$$v_1(N, \bar{p}) = \max_{0 \leq p_1 \leq \bar{p}} \left\{ N(p_1 - w) \left[\frac{F(\bar{p}) - F\left(\frac{p_1}{1 - c_e}\right)}{F(\bar{p})} \right] + \mathbb{E}[v_2(N - S_1, r_2)] \right\}.$$

Let us denote the profit function

$$\pi(p_1, p_2, \dots, p_T) \equiv (p_1 - w) \bar{F}\left(\frac{p_1}{1 - c_e}\right) + \sum_{i=2}^T (p_i - w) \left[F\left(\frac{p_{i-1}}{1 - c_e}\right) - F\left(\frac{p_i}{1 - c_e}\right) \right].$$

The first order condition of optimality with respect to p_1 reads

$$\frac{\partial}{\partial p_1} \pi(p_1, p_2, \dots, p_T) = \bar{F}\left(\frac{p_1}{1 - c_e}\right) - \frac{p_1 - p_2}{1 - c_e} f\left(\frac{p_1}{1 - c_e}\right) = 0.$$

It implies that

$$\frac{p_2^d}{1 - c_e} = \frac{p_1^d}{1 - c_e} - \frac{1 - F\left(\frac{p_1^d}{1 - c_e}\right)}{f\left(\frac{p_1^d}{1 - c_e}\right)}.$$

Recall that $\frac{w}{1-c_e} = \frac{p^d}{1-c_e} - \frac{1-F(\frac{p^d}{1-c_e})}{f(\frac{p^d}{1-c_e})}$. Since $p_2^d \geq w$, we further obtain that

$$\frac{p_1^d}{1-c_e} - \frac{1-F(\frac{p_1^d}{1-c_e})}{f(\frac{p_1^d}{1-c_e})} \geq \frac{p^d}{1-c_e} - \frac{1-F(\frac{p^d}{1-c_e})}{f(\frac{p^d}{1-c_e})}.$$

By regularity of the reserve price distribution F , we conclude that $p_1^d \geq p^d$. In summary, we showed that the seller adopts a high-low optimal price strategy where the single optimal price in the newsvendor setting falls in between.

□