Optimal Expediting Policies for a Serial Inventory System with Stochastic Lead Time and Visibility of Orders In Transit

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Abstract

Recent supply chains face higher volatility due to increased length and complexity. Cost pressure from severe competition drives industries to outsource its production to Asia, which in turn makes the supply chains long and complex. As a result, it is increasingly critical to manage supply chains actively to reduce uncertainty in delivery lead time and increase service rates. Often operating multiple delivery modes such as standard freight shipping and air is an effective way of addressing both delivery lead time uncertainty and service rates. We propose a model on how to operate multiple delivery modes in an optimal way and discuss a necessary order tracking system such as radio frequency identification as a prerequisite for expediting in a stochastic delivery lead time environment. We consider a serial supply chain and an expediting option from intermediate installations to the very downstream of the chain. The goods move stochastically among the installations and the system faces a stochastic demand. We identify systems that yield simple and tractable optimal policies, in which both regular ordering and expediting follow a variant of the base stock policy. We show that optimal expediting results in a significant reduction in the total logistics cost and the reduction increases as variability in delivery lead time increases. Furthermore, we show that expediting allows the system to be operated in a leaner way due to the reduced regular order amount and provide various managerial insights linking expediting, base stock levels, and expediting costs based on analytical and numerical analyses.

1 Introduction

One challenge of multi-stage global supply chains is to reduce the total logistics cost and increase the service rate when facing variability of delivery lead times which is the total time from placing an order until the final delivery and comprises of the transportation between stages and temporary storage along the supply chain. Such variability is caused by multiple sources such as an unexpected plane or cargo vessel arrival change, congestion due to work-load imbalances, extra work due to misshandling and incorrect stocking, and waisted time due to equipment failures. Since variability yields high unexpected cost in either inventory or backlogging, companies often expedite outstanding orders either from the supplier or from intermediate stages by using more expensive transportation modes such as air to reduce the total logistics cost. Large volume, high value electronics such as flat panels displays, high variety supplies such as special color paints, and seasonal products such as clothes are examples of industries where freight shipping is intense and air freight is used for expediting the delivery to reduce variability.

In order to expedite outstanding orders, proper visibility of orders in transit needs to be in place due to the stochastic nature of delivery lead times. There are a number of order tracking solutions available to improve visibility: from traditional approaches such as manual processing with barcodes to more recent technologies such as automatic processing with RFID[1]. Each solution has a different value proposition in terms of information accuracy, processing time, initial investment, maintenance, and labor cost. Optimally deployed expediting of outstanding orders results in an improvement of the total logistics cost. Supply chain managers face two problems. First, they need to decide to what extend to have visibility in the supply chain, and second, they need to decide when, how much, and from where to expedite based on given visibility. This paper addresses these two problems by studying optimal policies of expediting outstanding orders, its characteristics, and dynamics in multi-staged supply chains. The value of visibility can be estimated from the identified optimal policy, and, in turn, set the limit for an appropriate level of investment on visibility.

In particular, we consider a periodic review, single item inventory problem of a manufacturer. There is a single supplier where the manufacturer periodically places regular orders. The stochastic

[1] Radio Frequency IDentification

demand is fulfilled by the manufacturer and excessive demand is backlogged. The entire chain consists of multiple installations, and orders progress from one installation to another until delivered to the manufacturer. The movements of outstanding orders among installations are stochastic, hence the overall lead time is stochastic. Specifically, multiple movement patterns of outstanding orders are captured in the model, and one of the patterns is chosen stochastically in each time period. We assume that there exists an exogenous random variable with a known distribution that governs the movement pattern that occurs at the current time period. The manufacturer has the option to expedite orders from installations to the manufacturer at an extra per unit cost according to the current demand situation. Because the lead time is stochastic, expediting an order requires knowing its exact location. This location information can be provided, for example, by an order tracking technology such as RFID. We reiterate that we consider the manufacturer's inventory problem, where all decisions, e.g., regular ordering and expediting decisions, are made by the manufacturer and the corresponding logistics costs are also incurred by the manufacturer, including investment and operations costs of the order tracking system. As a real-world example of such a setting, consider a heavy equipment manufacturer with a plant near Shanghai which procures high value electronic parts from a supplier near Seoul. The procured parts are delivered by a 3rd party logistics provider through specified locations while the logistics cost is fully paid by the manufacturer. The regular route can be from the supplier, the port of Incheon (a major port of Korea), to Shanghai, while the expedited route can be either from a local airport near the supplier to Shanghai, or from the supplier, Incheon airport, to Shanghai. Since the locations are fixed, the manufacturer could install tracking mechanisms in all logistics locations to track the inventory level. In order to asses the value of visibility, it is important to develop models capable of exploiting data resulting from the tracking technology, and to find out optimal policies of expediting and regular ordering in such models. In the absence of optimal policies, it is hard to assess the value of visibility in the supply chain. As a result, we focus on deriving the optimal expediting and regular ordering policies under certain conditions. Since the setting of our model is quite general and the modeling scope is broad, finding the optimal policies in general is difficult. They generally depend on state variables, hence they are nonintuitive and complex. However, analytical results can be obtained for a certain subset of serial systems. We characterize conditions for a system to allow simple optimal policies, and call such systems *sequential* since orders do not cross in time under the optimal control. The sequential delivery property plays the key role in analyzing the optimal policies. We also provide sufficient and necessary conditions for identification of sequential systems. Within the realm of sequential systems, the optimal regular ordering and expediting policies are derived. The optimal regular ordering policy is the base stock policy with respect to the inventory position, and the optimal expediting policy is a variant of the base stock policy with respect to the echelon stock up to a certain installation. In addition, we find that as the expediting cost of a certain installation increases, the underlying expediting base stock level associated with the installation is non increasing, which is intuitive. Interestingly enough, we also derive that as the expediting cost for an installation increases, the expediting base stock levels for installations beyond the installation in question are nondecreasing.

The contributions of our work are several. First, to the best of our knowledge, the presented work is the first one to derive an optimal expediting policy of a multi-stage stochastic lead time inventory model, which is a clear distinction over the existing literature. Second, the proof technique is novel and nontraditional even though we rely on induction. After characterizing the sequential systems, we formulate the optimality equation suited for these systems using the sequential delivery property, and this leads to simple optimal policies. Optimality of these policies is proved in an induction loop by studying the difference in the cost-to-go for different states. Third, we find interesting directional dependencies of expediting base stock levels on expediting costs. Fourth, based on a numerical study, we provide various interesting insights on the dynamics of the optimal policy as parameters vary. Finally, an important managerial insight that the value of visibility can be elevated, if utilized actively with new processes such as expediting, is inferable from this work. Firms should strive for creative business processes in order to extract more value from visibility.

In the next section, we formally state the underlying model. We delineate the class of systems in which orders do not cross in time in Section 3, and discuss the scope of such sequential systems in the same section. We derive the corresponding optimal policies for the sequential systems in Section 4. In Section 5, we discuss directional dependencies of expediting base stock levels on expediting costs, and in Section 6, we discuss dynamics of the optimal policies by means of a numerical analysis. Section 7 summarizes the overall findings with a brief discussion on the selection criteria of a proper tracking technology suitable to a supply chain. We conclude the introduction with a literature review.

Literature Review

The most related models in the literature are divided in two groups: the stochastic lead time models and the multi supply mode models. Among the early works on the stochastic lead time models, Kaplan (1970), Nahmias (1979), and Ehrhardt (1984) consider stochastic lead time that is determined by a realization of a random variable. In their models they consider the age of orders to determine the stochastic movement, and if the age of an order exceeds the realized value of the random variable, then this order is considered to arrive at the destination. Song and Zipkin (1996) and Muharremoglu and Tsitsiklis (2008) are more recent publications on stochastic lead time models. In their models, the supply system is Markov modulated describing the supply condition. They also define an exogenous random variable, which determines the lead time of an order, but their modeling of the stochastic lead time is more comprehensive than the earlier works since the random variables determine the progress status of outstanding orders. Our model resembles the stochastic lead time description of Song and Zipkin (1996) and Muharremoglu and Tsitsiklis (2008), however, they do not consider expediting.

The multi supply mode models such as emergency ordering or expediting models with deterministic movement transitions include Barankin (1961), Neuts (1964), Daniel (1963), Fukuda (1964), and Veinott (1966) as the early works. They consider inventory systems with two supply modes of instantaneous and one period lead time. Models with emergency orders among others include Chiang and Gutierrez (1998) and Huggins and Olsen (2010), but their modeling of emergency orders is different from ours (emergency and expediting have different scopes). More related recent works are Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003). Lawson and Porteus (2000) extend the multi-echelon model by Clark and Scarf (1960) by allowing expediting between consecutive installations. Muharremoglu and Tsitsiklis (2003) generalize Lawson and Porteus (2000) by allowing supermodular expediting cost instead of a linear one.

Both Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003) allow expediting between arbitrary two installations. However, our model does not allow this since in our case orders can be expedited only to the manufacturer. As an example, consider the same heavy equipment manufacturer near Shanghai. The manufacturer may not be able to expedite from the local airport near the supplier to Incheon airport due to the lack of domestic flights between the two international airports. The manufacturer may only expedite orders to its own facility based on the inventory information (from order tracking system such as RFID) at each installation. It is important to note that it is nontrivial to prevent expediting between intermediate installations using the models of Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003). Therefore, our model addresses a different situation from their models. Furthermore, the stochastic lead time modeling considered herein is a fundamental leap from the deterministic cases in their models.

Kim et al. (2009) study a similar setting by allowing expediting from intermediate installations to the manufacturer. The most important distinction from the present work is that they consider a deterministic delivery lead time model, whereas the present work has stochastic delivery lead times, which poses non-trivial analytical challenges. Deterministic lead time is a special case of the stochastic lead time model, but the scope of sequential systems, in which analytical results on the optimal policy exist, is broader in the present work and it is a non-intuitive extension of the sequential systems of Kim et al. (2009) due to the introduction of movement patterns, as described later in detail in Section 3. Also, different movement patterns in the present work can address very wide variations in the stochastic behavior of order delivery, so the modeling power is significantly expanded in the present work. Furthermore, the deterministic model does not provide a model to measure the value of visibility, but the present work provides a practical model to determine whether to use a tracking system and to estimate how much to invest in a given setting.

Pei and Klabjan (2010) study the serial supply chain under the slap-and-ship RFID strategy. The entire chain is viewed from the perspective of the end supplier and thus the holding cost is not taken into account in the intermediate locations. The most clear distinction of Pei and Klabjan (2010) from the present work is that they only consider the regular flow of items with no expediting. In this paper, the supply chain is viewed from the point-of-sales location and it is considering expediting. The latter property requires completely different models, policies, and proof techniques.

Gaukler et al. (2008) consider emergency ordering under RFID in a supply chain with multiple stages, where the lead time is stochastic. RFID is used in a similar context as ours, i.e., to gain real-time location information. However, their model is simpler than ours since they allow at most one outstanding order at any point in time. Furthermore, rather than dealing with optimal policies, they confine their study to the set of base stock policies and study the best base stock levels. Therefore, the optimality of the base stock policy in their model is not guaranteed, and thus the nature of their work is distinct from ours. For a further literature review on RFID related inventory models, we refer the reader to Lee and Özer (2007).

2 Model statement

We consider a single supplier with a single-item manufacturing facility facing random demand which has a compact support with known distribution, and $\bar{K}-1$ serial intermediate installations between them. The supplier is denoted as installation $\bar{K}$ and the manufacturing facility is installation 0. The intermediate installations are numbered from 1 (next to the manufacturing facility) to $\bar{K}-1$ (next to the supplier). The manufacturer periodically reviews the inventory on hand and places a regular order at the supplier by paying per unit procurement cost c. Unsatisfied demand is backlogged and excessive inventory at the manufacturing facility is penalized. The planning horizon consists of T time periods. For simplicity, we assume that the system is stationary.

A movement pattern w describes the destination installation of outstanding orders for each installation in the next time period. We define multiple movement patterns. For example, consider a supply chain with $\bar{K}=5$, which has three illustrative movement patterns: slow, normal, and fast. In the normal pattern, orders at installation i move to installation $i-1$ for $i=1,\cdots,5$. In the slow pattern, orders at installations 1, 3, and 5 fail to progress, thus orders at these installations stay at the current location one more time period while orders at the remaining installations move to the next downstream installation. In the fast mode, orders in installations 2 and 3 move to installations

0 and 1 respectively while orders in the other installations move to the next downstream installation. Let us denote by $\mathbf{W}$ the set of all movement patterns, i.e., $\mathbf{W} = \{w_1, w_2, w_3, \cdots\}$. There is an exogenous random variable W with known distribution that selects a movement pattern in $\mathbf{W}$. At each time period, W realizes, and according to the realized movement pattern w, the outstanding orders at installation $i, 1 \leq i \leq \bar{K}$, move to installation $j = M(i, w)$, $0 \leq j \leq i$, where $M(\cdot)$ is a function that takes the origin installation i and the realized movement pattern w as arguments. Note that orders are not allowed to go backward to the upstream installations in this definition. We define $M(0, w) = 0$, and before W is realized we denote the corresponding random variable by $M(i, W)$. The lead time of a regular order is stochastic and determined by multiple realized movement patterns until delivery.

Let v_i be the amount of inventory at installation i for $0 \leq i \leq \bar{K}$ and $(v_0, v_1, v_2, \cdots, v_{\bar{K}})$ be the state vector. Based on the current state of the system from the order tracking system, the manufacturer expedites outstanding orders if need be by paying per unit delivery cost d_i for expediting orders from installation i.

The sequence of events in a time period is as follows. At the beginning of the time period, the state information is given. Next, the manufacturer places a regular order with the supplier (installation $\bar{K}$) and that order arrives to the supplier. Next, the manufacturer makes decisions on expediting for each installation, and the expedited orders arrive at the manufacturing facility instantaneously. After that, demand D realizes for the current time period. Inventory holding or backlogging cost is accounted for at the manufacturing facility after demand realization. Finally, W realizes and regular delivery occurs just before the end of the time period. Then the next time period begins.

We need the following assumption stating that regular orders should not cross in time. This assumption is standard in the stochastic lead time literature.

Assumption 1 (Orders not crossing in time)**.** *$M(i, w) \geq M(i-1, w)$ for all i and $w \in \mathbf{W}$.*

Let us define a related movement function $N(j, w) = \max\{i : M(i, w) \leq j, 0 \leq i \leq \bar{K}\}$ for all j and $w \in \mathbf{W}$, and let $N(j, W)$ be the corresponding random variable before W is realized. Under Assumption 1, a one-to-one mapping between M and N exists as the following example illustrates.

Example Consider an 8 installation system including the supplier and the manufacturer ($\bar{K} = 7$). At time t, assume that realized w of W drives the following movement.

i	0	1	2	3	4	5	6	7
$M(i, w)$	0	0	0	1	1	1	4	5

An equivalent information of the above movement can be expressed by $N(j, w)$ as follows (See Figure 1).

j	0	1	2	3	4	5	6	7
$N(j, w)$	2	5	5	5	6	7	7	7

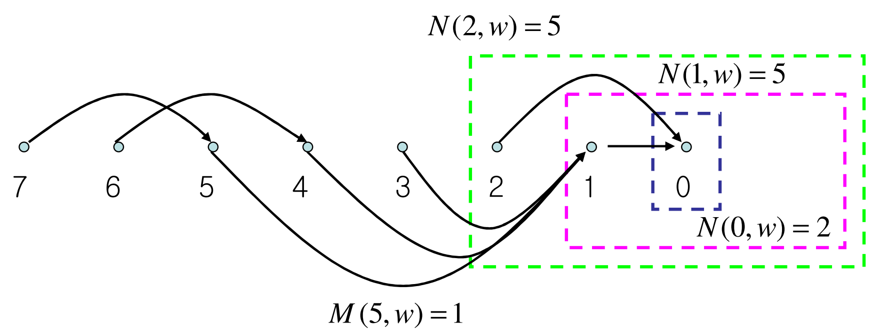

Figure 1: A regular movement driven by a realized w of W

Given installation j, we find $N(j, w)$ by observing the farthest installation whose movement leads to installation j or any downstream installation of j. □

Let us denote by $M^n(i, W)$ the n-period random movement function that represents the location (an installation) after n regular movements of the outstanding orders at installation i, where W is an n-dimensional random vector. We denote by w realizations of W which are also multi-dimensional vectors. The dimension of W can always be inferred from the underlying usage. Formally, $M^1(i, W) = M(i, W)$ where W is a random variable, and $M^n(i, W) = M(M^{n-1}(i, W'), W'')$ where $W = (W', W'')$ is a vector of length n, W' is a vector of length $n-1$, and W'' is a random variable. For convenience, we define $L(x) = E[r(x-D)]$, where $r(\cdot)$ is a convex holding/backlogging cost function, and let $Q^i(W)$ denote $N(M(\bar{K}, W) - i, W)$ for $i \leq M(\bar{K}, W)$. Random variable $Q^i(W)$ represents the maximum indexed installation that delivers its orders to the i'th downstream installation of the installation to which supplier $\bar{K}$ delivers its orders by a random movement. Random variable $Q^i(W)$ is necessary to simplify the general state transition equation since it describes regular movement from the last installation, i.e., the supplier. Note that $Q^i(W)$ is only defined when $i \leq M(\bar{K}, W)$. For example, $Q^i(W)$ has the following values based on Figure 1.

i	0	1	2	3	4	5	6	7
$Q^i(W)$	7	6	5	5	5	2	n/a	n/a

Let the echelon stock x^i be the sum of the inventory from installation 0 to installation i: $x^i = \sum_{j=0}^{i} v_j$, and let $\bar{0}^i = (0, 0, \cdots, 0)$ be a vector containing i zeros.

If there is no expediting, the state after a regular movement is a random vector $(x^{N(0,W)} - D, x^{N(1,W)} - x^{N(0,W)}, \cdots, x^{Q^1(W)} - x^{Q^2(W)}, x^{Q^0(W)} - x^{Q^1(W)} + u, \bar{0}^{\bar{K}-M(\bar{K},W)})$, where u is the regular ordering amount. Let e_i denote the expediting amount from installation i. Including expediting, the next state NS is

$$
\begin{aligned}
NS =& (x^{N(0,W)} + \sum_{i=N(0,W)+1}^{\bar{K}} e_i - D, x^{N(1,W)} - x^{N(0,W)} - \sum_{i=N(0,W)+1}^{N(1,W)} e_i, \cdots, \\
& x^{N(i,W)} - x^{N(i-1,W)} - \sum_{i=N(i-1,W)+1}^{N(i,W)} e_i, \cdots, \\
& x^{Q^1(W)} - x^{Q^2(W)} - \sum_{i=Q^2(W)+1}^{Q^1(W)} e_i, x^{Q^0(W)} - x^{Q^1(W)} + u - \sum_{i=Q^1(W)+1}^{Q^0(W)} e_i, \bar{0}^{\bar{K}-M(\bar{K},W)}).
\end{aligned}
$$

Figure 2 illustrates the inventory at installation i after a regular movement $x^{N(i,W)} - x^{N(i-1,W)} - \sum_{i=N(i-1,W)+1}^{N(i,W)} e_i$. The complete optimality equation of the dynamic program reads

$$
J_t(v_0, v_1, \cdots, v_{\bar{K}}) = \min_{\substack{u, e_1, \cdots, e_{\bar{K}} \\ 0 \leq e_{\bar{K}} \leq u + v_{\bar{K}} \\ 0 \leq e_i \leq v_i \\ i=1,\cdots,\bar{K}-1}} \{\sum_{i=1}^{\bar{K}} d_i e_i + L(x^0 + \sum_{i=1}^{\bar{K}} e_i) + cu + E[J_{t+1}(NS)]\}, \tag{1}
$$

where J_t is the cost-to-go at the beginning of time period t. For simplicity, let the terminal cost $J_{T+1}(v_0, v_1, \cdots, v_{\bar{K}})$ be a non-decreasing convex function in $x^{\bar{K}}$. Solving this optimality equation directly is difficult because of its high dimensionality and complex structure of constraints. In order to analyze (1), we need to introduce further assumptions.

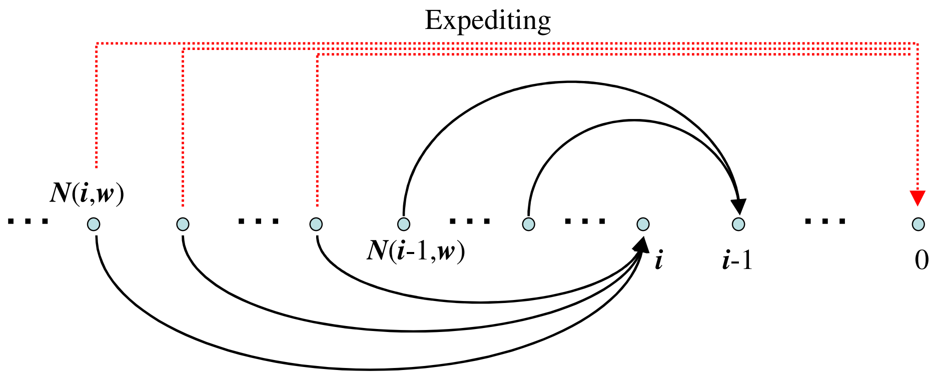

Figure 2: The next state transition

In the next section, we characterize a class of systems for which (1) has an alternative form that leads to tractable policies.

3 Sequential Systems

The following assumption requires that orders almost surely reach installation 0.

Assumption 2 (Eventual delivery of regular orders). $Prob[\cup_{n=1}^{\infty}\{w : M^n(i,w) = 0\}] = 1$ *for every installation* i*.*

In terms of the finite state Markov Chain theory, Assumption 2 requires that installation 0 is the only *recurrent installation*, and all the other installations are *transient installations*. In order to analyze the system, we need the following assumption.

Assumption 3 (Nondecreasing time value of delayed expediting). $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ *for all* i*, where* $d_0 = 0$*.*

Consider a unit at installation i. If we expedite it at the current time period, it costs d_i. If we defer expediting by a time period, the expected cost of expediting is $E[d_{M(i,W)}]$. Therefore, $d_i - E[d_{M(i,W)}]$ is *the time value of delayed expediting* of a unit at installation i by a time period. Assumption 3 implies that this time value of expediting does not decrease as installation number i increases. Next, we define a class of systems, in which all three assumptions hold.

Sequential systems A system is *sequential* if Assumptions 1, 2, and 3 hold.

The following lemma shows a characteristic of sequential systems, and its proof is in the appendix.

Lemma 1. *In sequential systems, $d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}]$, for any i and j, $i \geq j$, and $n \geq 1$.*

The following theorem shows a crucial property of sequential systems.

Theorem 1. *Under the optimal control of regular ordering and expediting, sequential systems preserve the sequence of orders in time, i.e., the no cross-over property holds.*

Assumption 1 guarantees that regular orders with no expediting do not cross in time. When expediting is introduced, in general, orders might easily cross even under Assumption 1. Theorem 1 states that this is not the case for sequential systems.

Proof of Theorem 1. We prove by induction on time. First, we show that the theorem holds at time $T+1$. Since J_{T+1} is a non-decreasing convex function in $x^{\bar{K}}$ by definition, zero regular ordering and expediting is optimal, and therefore the theorem holds.

Next, assume that the theorem holds for all time periods $t' \geq t$. Also let us assume that the realization of exogenous random variable W is observable with a proper tracking method after the expediting decision and demand realization at time t. Note that the expediting decision is made at the beginning of a time period, while the random movement realization happens at the end of time period t. Consider a decision maker at the manufacturing facility facing the following two options at time period t.

- Option 1: Expediting a set of units R_1 at time period t in a way that order crossing happens, i.e., there remains an outstanding unit after expediting all units ordered at an earlier time than the most recently ordered unit in R_1, and then following the optimal expediting policy from time period $t+1$ according to the induction hypothesis so that order crossing does not happen.

- Option 2: Expediting a set of units R_2 at time period t with $|R_2| = |R_1|$ so that no order crossing happens, i.e., there are no outstanding units that are ordered earlier than the most recently ordered unit in R_2, and then *replicating* Option 1 from time period $t+1$ onwards, i.e., placing the same amount of regular ordering as in Option 1, and expediting outstanding orders so that the manufacturing facility has the same level of inventory as in Option 1 at any time period from $t+1$.

Note that Option 2 is possible since the decision maker can simulate Option 1 with the knowledge from the random movement realization at each time period after the expediting decision at time t. Next, we show that the expected cost of Option 2 is less than or equal to that of Option 1 to prove the induction step. In other words, if the decision maker expedites so that order crossing happens at time period t, then there always exists a policy, Option 2, that leads to a non-higher cost, and therefore we conclude that the optimal expediting policy is to expedite in a way that no order crossing happens at time period t.

Note that units in R_1 are outstanding in Option 2, and similarly those in R_2 are outstanding in Option 1 at time period t. Consider a pair (u_1, u_2) such that $u_1 \in R_1$ and $u_2 \in R_2$, where u_1 is placed no earlier than u_2 in time. Since $|R_1| = |R_2| = R$, we can find R such disjoint pairs of units, that comprise R_1 and R_2. Let $\mathbf{P}$ be the set of such disjoint pairs. Formally, for any unit $u_1 \in R_1$, there is one and only one $u_2 \in R_2$, such that $(u_1, u_2) \in \mathbf{P}$. Consider the following algorithm for a pair $(u_1, u_2) \in \mathbf{P}$, where the respective installations of u_1 and u_2 are i and j in time period t, $i \geq j$. This algorithm enables Option 2 to replicate Option 1.

1. Set a new index k to be j before the random movement is realized at the current time period t. Also set a new index $\tau = 0$.

2. If the realized value of $M(k, W)$ is 0 at the end of time period $t + \tau$, then expedite u_1 at time period $t + \tau + 1$ and terminate the algorithm.

3. Otherwise, if u_2 is expedited under Option 1 at time period $t + \tau + 1$, then expedite u_1 at time period $t + \tau + 1$ and terminate the algorithm.

4. Otherwise, update $k \leftarrow M(k, W)$ and $\tau \leftarrow \tau + 1$. Go to step 2.

Let C_1 and C_2 be the cost under Option 1 and Option 2, respectively, to receive both u_1 and u_2 in the manufacturing facility eventually. We have $C_1 - C_2 = d_i + E[d_{M^{\tau+1}(j,W)}] - d_j - E[d_{M^{\tau+1}(i,W)}] \geq 0$, due to Lemma 1. Since this is true for all elements in $\mathbf{P}$, the total cost of Option 1 is no less than that of Option 2. This completes the induction step.

Recall that we assume that the random movement W is observable after the expediting decision at time t. For any realization of W, we could still operate Option 2 with the algorithm provided and it proves the existence of a non-inferior decision than Option 1. Therefore, regardless of actual observability of W we know that Option 1 is sub-optimal and the proof is completed. $\square$

For $1 \leq i \leq \bar{K}$, let $J_t^i(\cdot)$ be the optimal cost-to-go that can be achieved by a restricted control space, in which expediting from installations $i+1, i+2, \cdots, \bar{K}$ in time period t is not allowed. Note that the control space for J_t^i is restricted only in time period t, but unrestricted after time period t. Note also that $J_t^{\bar{K}}(\cdot) = J_t(\cdot)$. We utilize $J_t^i(\cdot)$ with respect to a fictitious state $(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}})$, where installation 0 has inventory x^{i-1}, and installations $1, 2, \cdots, i-1$ are empty. The optimality equation for $J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}})$, $1 \leq i \leq \bar{K}-1$, is given by

$$\begin{aligned} J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}}) = & \min_{x^{i-1} \leq y_i \leq x^i, z \geq x^{\bar{K}}} \{d_i y_i + L(y_i) - d_i x^{i-1} - c x^{\bar{K}} + cz \\ & + E[J_{t+1}(y_i - D, \bar{0}^{M(i,W)-1}, x^{N(M(i,W),W)} - y_i, x^{N(M(i,W)+1,W)} - x^{N(M(i,W),W)}, \\ & \cdots, x^{N(M(\bar{K},W)-1,W)} - x^{N(M(\bar{K},W)-2,W)}, z - x^{N(M(\bar{K},W)-1,W)}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\}, \end{aligned} \tag{2}$$

where y_i and z are decision variables: $y_i - x^{i-1}$ is the expediting amount from installation i and $z - x^{\bar{K}}$ is the regular ordering amount. For $i = \bar{K}$, the constraints in (2) become $x^{i-1} \leq y_i \leq z, z \geq x^{\bar{K}}$ in order to allow expediting regular orders that have just been placed. Note that the equation should be read appropriately, if $M(i,w) = 0$ for a realized value w of W.

By Theorem 1, in sequential systems expediting orders from installation i is never optimal before expediting all the outstanding orders at the downstream installation of installation i. With this fact, an alternative optimality equation equivalent to (1) is given by

$$
\begin{aligned}
J_t(v_0, v_1, v_2, \cdots, v_{\bar{K}}) = \min\{ & J_t^1(x^0, v_1, v_2, \cdots, v_{\bar{K}}), \\
& d_1 v_1 + J_t^2(x^1, 0, v_2, \cdots, v_{\bar{K}}), \\
& d_1 v_1 + d_2 v_2 + J_t^3(x^2, 0, 0, v_3, \cdots, v_{\bar{K}}), \\
& \cdots, \\
& \sum_{i=1}^{\bar{K}-1} d_i v_i + J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}), \\
& \sum_{i=1}^{\bar{K}} d_i v_i + J_t(x^{\bar{K}}, \bar{0}^{\bar{K}}), \}.
\end{aligned} \tag{3}
$$

The first term $J_t^1(\cdot)$ corresponds to expediting partially or fully from only installation 1, the second term $d_1 v_1 + J_t^2(\cdot)$ captures expediting everything from installation 1, expediting partially or fully from installation 2, and no expediting beyond, and so forth. The eventual optimal decisions for regular ordering and expediting are determined by the minimum term in (3) since the system is sequential. If the j-th term achieves the minimum in (3), the optimal decision for expediting is to expedite all outstanding orders in installations 1, 2, $\cdots, j-1$ and to expedite $y_j - x^{j-1}$ from installation j and nothing beyond installation j, where $y_j - x^{j-1}$ is derived from the j-th term. The optimal regular ordering decision is to place a regular order in the amount $z - x^{\bar{K}}$ that is determined in the j-th term.

Characterization of Sequential Systems

In this subsection, we discuss how to identify sequential systems. We derive first a necessary condition and then a sufficient condition for a system to be sequential. The following lemma, whose proof is given in the appendix, is used later.

Lemma 2. *Under Assumption 2, the following holds:*

(a) $\lim_{n\to\infty} Prob[M^n(i, W) = 0] = 1$ *for all* i,

(b) $\lim_{n\to\infty} Prob[M^n(i, W) = k] = 0$, $k \neq 0$ *for all* i.

The expediting costs should be nondecreasing in order for a system to be sequential as the next proposition states.

Proposition 1. *Sequential systems satisfy $d_i \geq d_{i-1}$, for all i.*

Proof. Using $j = i - 1$ in Lemma 1 results in $d_i - d_{i-1} \geq E[d_{M^n(i,W)} - d_{M^n(i-1,W)}]$. On the other hand, $E[d_{M^n(i,W)}] = \sum_k d_k Prob[M^n(i,W) = k] = \sum_{k \neq 0} d_k Prob[M^n(i,W) = k] + d_0 Prob[M^n(i,W) = 0]$. By taking $\lim_{n \to \infty}$ and applying Lemma 2 we get $lim_{n \to \infty} E[d_{M^n(i,W)}] = d_0 = 0$. Therefore, $d_i - d_{i-1} \geq 0$ for all i. □

Next we identify a sufficient condition.

Proposition 2. *Suppose Assumptions 1 and 2 hold and the followings are true for all i and $w \in \mathbf{W}$:*

- $d_i - d_{i-1} \geq d_{i-1} - d_{i-2}$, *and*
- $E[M(i,W) - M(i-1,W)] \leq 1$.

Then, the system is sequential.

Proof. Because of Assumption 1, $M(i,w) - M(i-1,w)$ is a nonnegative integer. Recall that orders do not go backward, i.e., $M(i,W) \leq i$. The first condition in the proposition implies $d_{M(i,w)} - d_{M(i-1,w)} \leq (M(i,w) - M(i-1,w))(d_i - d_{i-1})$. Therefore, by taking expectations on both sides, we have $E[d_{M(i,W)} - d_{M(i-1,W)}] \leq E[(M(i,W) - M(i-1,W))(d_i - d_{i-1})] \leq d_i - d_{i-1}$, which is Assumption 3. □

We call the first property in Proposition 2 convexity since it implies that the expediting cost differences are convex. Proposition 2 gives only sufficient conditions. We provide an example of a system that is sequential but nevertheless is not convex. In other words, sequential systems also include systems with non-convex expediting costs.

Example Consider a 5 installation system including the manufacturer and the supplier with four movement patterns: w_1, w_2, w_3, and w_4. More specifically,

- w_1: normal mode with probability p_1 such that $M(i, w_1) = i - 1$ for $i = 1, 2, 3, 4$,

- w_2: with probability p_2 such that $M(i, w_2) = i - 1$ for $i = 1, 3, 4$, and $M(2, w_2) = 0$,

- w_3: with probability p_3 such that $M(i, w_3) = i - 1$ for $i = 1, 2, 4$, and $M(3, w_3) = 1$, and

- w_4: with probability p_4 such that $M(i, w_4) = i - 1$ for $i = 1, 2, 3$, and $M(4, w_4) = 2$,

as shown in Figure 3. The associated probability distribution is $p_1 = \frac{1}{10}, p_2 = \frac{1}{10}, p_3 = \frac{3}{10}$, and

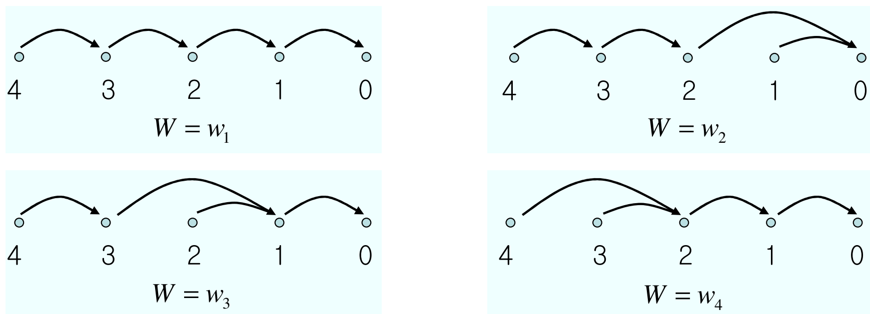

Figure 3: The movement patterns

$p_4 = \frac{1}{2}$. The system is clearly non-convex if the expediting costs are $d_1 = 10, d_2 = 19, d_3 = 27$, and $d_4 = 34$. To check that the system is sequential, let us compute $d_i - E[d_{M(i,W)}]$ for $i = 1, 2, 3$, and 4. We have $d_1 - E[d_{M(1,W)}] = d_1 - 0 = 10$, $d_2 - E[d_{M(2,W)}] = d_2 - p_1 d_1 - p_2 d_0 - p_3 d_1 - p_4 d_1 = 10$, $d_3 - E[d_{M(3,W)}] = d_3 - p_1 d_2 - p_2 d_2 - p_3 d_1 - p_4 d_2 = 10.7$, and $d_4 - E[d_{M(4,W)}] = d_4 - p_1 d_3 - p_2 d_3 - p_3 d_3 - p_4 d_2 = 11$. Since $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ for all i, the system is sequential. $\square$

4 Optimal Policies for Sequential Systems

In this section, we focus on identifying optimal policies for sequential systems.

4.1 Preliminaries

We frequently use the following lemma from Lawson and Porteus (2000), which originates in Karush (1959).

Lemma 3. *Let f be convex and have a finite minimizer on $\mathbb{R}$. Let $y^* = \arg\min f(x)$. Then,*

$\min_{x_1 \leq x \leq x_2} f(x) = a + g(x_1) + h(x_2)$, *where* $a = f(y^*)$, *and penalty functions* $g(x_1)$ *and* $h(x_2)$ *are*

$$g(x_1) = \begin{cases} 0 & x_1 \leq y^* \\ f(x_1) - a & x_1 > y^* \end{cases} \quad \text{and} \quad h(x_2) = \begin{cases} f(x_2) - a & x_2 \leq y^* \\ 0 & x_2 > y^* \end{cases}.$$

For a nondecreasing convex f, we define $a = 0$, $g(x) = f(x)$, and $h(x) = 0$. On the other hand, for a nonincreasing convex f, we define $a = 0$, $g(x) = 0$, and $h(x) = f(x)$.

In Lemma 3, g is nondecreasing convex, while h is nonincreasing convex. The following lemma, whose proof is given in the appendix, is an extension of Lemma 3.

Lemma 4. *Let f_1 be convex and $b \in \mathbb{R}$. We have $\min_{b \leq x \leq y} \{f_1(x) + f_2(y)\} = a_1 + g_1(b) + \min_{b \leq y}\{h_1(y) + f_2(y)\}$, where a_1, h_1, and g_1 are defined as in Lemma 3 with respect to f_1.*

The following functions are required later in the derivation of the optimal policy. For $1 \leq i \leq \bar{K}$ and $t \leq T$, let us recursively define

$$f_{i,t}(y) = d_i y + L(y) + E[S^1_{M(i,W),t+1}(y - D)], \tag{4}$$

$$S^0_{i,t} = a_{i,t} + E[S^0_{M(i,W),t+1}],$$

$$S^1_{i,t}(x) = g_{i,t}(x) - d_i x, \tag{5}$$

$$S^2_{i,t}(x) = h_{i,t}(x) - L(x) + E[S^2_{M(i,W),t+1}(x - D)],$$

where $S^0_{0,t} = S^1_{0,t}(\cdot) = S^2_{0,t}(\cdot) = 0$ for all t, and $S^0_{i,T+1} = S^1_{i,T+1}(x) = S^2_{i,T+1}(x) = 0$ for all i. Here, $a_{i,t}$, $g_{i,t}$, and $h_{i,t}$ are defined according to Lemma 3 with respect to $f_{i,t}$. Starting from the last time period T, functions $f_{i,t}$ and $S^j_{i,t}$ can be obtained recursively. In particular, from (4) we can compute $f_{i,T}$, then from (5) we obtain $S^1_{i,T}$ for all i. Next we compute $f_{i,T-1}$ from (4), and in turn, $S^1_{i,T-1}$ from (5) for all i. We repeat this procedure to define all $f_{i,t}$ and $S^1_{i,t}$. For $S^0_{i,t}$ and $S^2_{i,t}$ we use a similar procedure. It is easy to check for all i and t that $f_{i,t}(\cdot)$ is convex for sequential systems, and $S^0_{i,t} + S^1_{i,t}(x) + S^2_{i,t}(x) = 0$.

Functions $f_{i,t}$, $S^0_{i,t}$, $S^1_{i,t}$, and $S^2_{i,t}$ can be interpreted as follows. Fuction $f_{i,t}$ is the cost function from expediting from installation i in current and future time periods. Function $S^0_{i,t}$ represents the optimal cost attainable from echelon stock up to installation i at time period t and future time periods following random movements. Function $S^1_{i,t}$ can be interpreted as the increment in inventory cost resulting from expediting, and $S^2_{i,t}$ can be interpreted as the cost gains from expediting in current time period t and future time periods.

Let us denote by $y^*_{i,t}$ a minimizer of $f_{i,t}(x)$: $y^*_{i,t} \in \arg\min f_{i,t}(x)$. The following theorem is an important property of $f_{i,t}$ for sequential systems. The proof can be found in the appendix.

Theorem 2. *For sequential systems there exists a $y^*_{i,t}$ for every $y^*_{i+1,t}$ such that $y^*_{i,t} \geq y^*_{i+1,t}$ for all i and t.*

The lemma shown below is used later in the derivation of the optimal policy. The proof is in the appendix.

Lemma 5. *For sequential systems, function $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ is convex for all i and t, and for all $w \in \mathbf{W}$.*

4.2 Optimal Policies

The optimal policy for sequential systems is highly structured and given in the following theorem.

Theorem 3. *For sequential systems, the following policy is optimal.*

a. *Optimal expediting is determined by a set of base stock levels. Each base stock level corresponds to $y^*_{i,t}$ for expediting from installation i at time t. The expediting policy compares x^{i-1} and $y^*_{i,t}$ as follows.*

   - *If $x^{i-1} < y^*_{i,t}$, then we expedite $\min\{x^i - x^{i-1}, y^*_{i,t} - x^{i-1}\}$ from installation i.*
   - *Otherwise, if $x^{i-1} \geq y^*_{i,t}$, we do not expedite anything from installation i.*

b. *The optimal regular ordering policy is the base stock policy with respect to inventory position $x^{\bar{K}}$. Therefore, we place a regular order in the amount of $\max(0, z^*_t - x^{\bar{K}})$ at time period t. The optimal base stock level for regular ordering is given by the following statements.*

   - *The base stock level for regular ordering is $z^*_t = \arg\min\{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z - D) + S^2_{M(\bar{K},W),t+1}(z - D)]\}$ for all i and t.*

- *Function* $H_t(x)$ *is convex and follows the recursive equation* $H_t(x) = \min_{z \geq x}\{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z-D) + S^2_{M(\bar{K},W),t+1}(z-D)]\} - S^2_{\bar{K},t}(x) - cx$, *and* $H_{T+1}(x) = J_{T+1}(x, \bar{0}^{\bar{K}})$.

To better understand the policy, we consider the following illustrative example consisting of five installations. There are three movement patterns: w_1, w_2, and w_3 with probabilities p_1, p_2, and $1 - p_1 - p_2$, respectively. More specifically,

- w_1: $M(i, w_1) = i - 1$ for $i = 1, 2, 3, 4$,
- w_2: $M(i, w_2) = i - 1$ for $i = 1, 3$, and $M(2, w_2) = 0$ and $M(4, w_2) = 2$, and
- w_3: $M(i, w_3) = i - 1$ for $i = 2, 3$, and $M(1, w_3) = 1$ and $M(4, w_3) = 4$

as shown in Figure 4. Suppose that the regular ordering base stock level is $z^* = 210$ and the

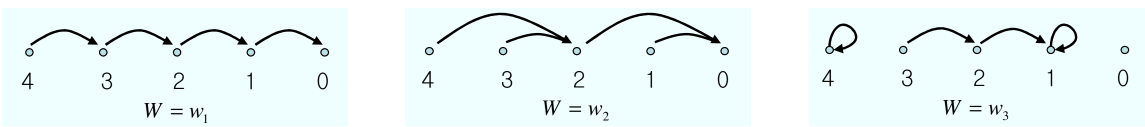

Figure 4: The movement patterns

expediting base stock levels are $y_4^* = 20$, $y_3^* = 50$, $y_2^*= 85$, and $y_1^* = 110$. In the following table, we summarize the mechanics of the optimal policy for a certain time period.

Installation i	4 (Suppl.)	3	2	1	0 (Manuf.)
v_i (x^i) before decisions	60 (185)	45 (125)	50 (80)	40 (30)	-10 (-10)
Regular Ordering $\max(z^* - x^4, 0)$	25				
Expediting $\max(\min(y_i^* - x^{i-1}, v_i), 0)$	0	0	50	40	
Realized demand D					65
v^i (x^i) after decisions and demand	85 (145)	45 (60)	0 (15)	0 (15)	15 (15)
v_i if the movement pattern $W = w_1$	0	85	45	0	15
v_i if the movement pattern $W = w_2$	0	0	130	0	15
v_i if the movement pattern $W = w_3$	85	0	45	0	15

In order to prove Theorem 3, we need the following proposition, which is proved concurrently with Theorem 3 within an induction loop in the subsequent proof. For convenience, we refer the items in the following proposition as (c) and (d).

Proposition 3. *For sequential systems, the following is true*

c. *For every $i = 1, \cdots, \bar{K}$, $0 \le e \le v_i$, and $t \ge \tilde{t}$, we have*

$$\begin{aligned} &J_t(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}}) - J_t(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}}) \\ &\quad = S^0_{i,t} + S^1_{i,t}(x^{i-1}) + S^2_{i,t}(x^{i-1} + e). \end{aligned}$$

d. *Function $S^2_{M(\bar{K},w),t}(x) + H_t(x)$ is convex for all t and $w \in \mathbf{W}$.*

Because x^i is nondecreasing in i and $y^*_{i,t}$ is nonincreasing in i, by Theorem 2, there exists at most one $i^* \in \{1, 2, \cdots, \bar{K}\}$ such that $x^{i^*-1} \le y^*_{i^*,t}$ and $x^{i^*} \ge y^*_{i^*+1,t}$. Theorem 3 states that we expedite everything up to installation $i^* - 1$, $\min\{x^{i^*} - x^{i^*-1}, y^*_{i^*,t} - x^{i^*-1}\}$ from installation i^*, and nothing beyond installation i^*. If such an i^* does not exist, then we do not expedite at all, or we expedite everything up to installation $\bar{K}$.

Proof of Theorem 3 and Proposition 3. We use induction. In the base case $t = T + 1$, the optimal expediting and regular ordering policies are null. We can safely set the base stock levels for expediting and regular ordering at negative infinity. Also, part (c) and (d) trivially hold when $t = T + 1$. In the proof, we also show that $H_t(x) = J_t(x, \bar{0}^{\bar{K}})$ for all t.

Let us assume that on and after time $t + 1 \le T + 1$, all parts in the theorem and proposition hold, and $H_{t+1}(x) = J_{t+1}(x, \bar{0}^{\bar{K}})$, and it is convex. Now we need to show the results for time period t. As the first step, we study $J_t(x, \bar{0}^{\bar{K}})$ in order to show $H_t(x) = J_t(x, \bar{0}^{\bar{K}})$ and that it is convex. In state $(x, \bar{0}^{\bar{K}})$, at the beginning of time period t we place a regular order of $z - x$ units and expedite

$y_{\bar{K}} - x$ units out of the just placed regular order. Therefore, we have

$$\begin{aligned}
J_t(x, \bar{0}^{\bar{K}}) &= \min_{x \le y_{\bar{K}} \le z} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) + cz + E[J_{t+1}(y_{\bar{K}} - D, \bar{0}^{M(\bar{K},W)-1}, z - y_{\bar{K}}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\} \\
&\quad - d_{\bar{K}} x - cx \\
&= \min_{x \le y_{\bar{K}} \le z} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^0_{M(\bar{K},W),t+1} \\
&\quad + S^1_{M(\bar{K},W),t+1}(y_{\bar{K}} - D) + S^2_{M(\bar{K},W),t+1}(z - D)]\} - d_{\bar{K}} x - cx \\
&= \min_{x \le y_{\bar{K}} \le z} \{f_{\bar{K},t}(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + E[S^0_{M(\bar{K},W),t+1}] - d_{\bar{K}} x - cx.
\end{aligned}$$

Note that the induction hypothesis on part (c) at $t + 1$ is used in the above derivation. By using Lemma 4, we have

$$\begin{aligned}
J_t(x, \bar{0}^{\bar{K}}) = \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
+ a_{\bar{K},t} + E[S^0_{M(\bar{K},W),t+1}] + g_{\bar{K},t}(x) - d_{\bar{K}} x - cx.
\end{aligned} \tag{6}$$

Note that $S^0_{\bar{K},t} + S^1_{\bar{K},t}(x) + S^2_{\bar{K},t}(x) = 0$ and $H_{t+1}(x) = J_{t+1}(x, \bar{0}^{\bar{K}})$ from the induction hypothesis. We have

$$J_t(x, \bar{0}^{\bar{K}}) = \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z - D) + S^2_{M(\bar{K},W),t+1}(z - D)]\} - S^2_{\bar{K},t}(x) - cx.$$

Since this coincides with the definition of $H_t(x)$, we conclude that $H_t(x) = J_t(x, \bar{0}^{\bar{K}})$. Furthermore, from the induction hypothesis on part (d) the right-hand side of (6) is convex, hence $H_t(x)$ is convex.

Let us now prove part (d). By adding $S^2_{M(\bar{K},w),t}(x)$ to both sides of (6), we get

$$\begin{aligned}
S^2_{M(\bar{K},w),t}(x) + J_t(x, \bar{0}^{\bar{K}}) &= \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + a_{\bar{K},t} + E[S^0_{M(\bar{K},W),t+1}] + S^2_{M(\bar{K},w),t}(x) + g_{\bar{K},t}(x) - d_{\bar{K}} x - cx,
\end{aligned}$$

which is convex because $S^2_{M(\bar{K},w),t}(x) + g_{\bar{K},t}(x)$ is convex by Lemma 5 and $E[J_{t+1}(z-D,\bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z-D)]$ is convex by the induction hypothesis. This completes the proof of part (d).

To prove parts (a) and (b), we apply part (c) to

$$J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots)$$

in (2) for $i < \bar{K}$ with a realized value w of W on and after time period $t+1$ repeatedly to obtain

$$\begin{aligned}
&J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots) \\
&\quad = S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D) \\
&\qquad + J_{t+1}(x^{N(M(i,w),w)} - D, \bar{0}^{M(i,w)}, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots) \\
&\quad = S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D) \\
&\qquad + \sum_{j=M(i,w)+1}^{M(\bar{K},w)-1} \{S^0_{j,t+1} + S^1_{j,t+1}(x^{N(j-1,w)} - D) + S^2_{j,t+1}(x^{N(j,w)} - D)\} \\
&\qquad + J_{t+1}(x^{N(M(\bar{K},w)-1,w)} - D, \bar{0}^{M(\bar{K},w)-1}, z - x^{N(M(\bar{K},w)-1,w)}, \bar{0}^{\bar{K}-M(\bar{K},w)}) \\
&\quad = S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D) \\
&\qquad + \sum_{j=M(i,w)+1}^{M(\bar{K},w)-1} \{S^0_{j,t+1} + S^1_{j,t+1}(x^{N(j-1,w)} - D) + S^2_{j,t+1}(x^{N(j,w)} - D)\} \\
&\qquad + S^0_{M(\bar{K},w),t+1} + S^1_{M(\bar{K},w),t+1}(x^{N(M(\bar{K},w)-1,w)} - D) + S^2_{M(\bar{K},w),t+1}(z - D) \\
&\qquad + J_{t+1}(z - D, \bar{0}^{\bar{K}}).
\end{aligned}$$

Let us gather in Q all of the terms in the above equation that only contain constants and state variables not involving any decision variables. Then we can rewrite

$$\begin{aligned}
&J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots) \\
&\quad = S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(\bar{K},w),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}}) + Q.
\end{aligned}$$

Substituting this into (2) and w for W, we obtain

$$\begin{aligned}
&J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}}) \\
&= \min_{x^{i-1} \le y_i \le x^i, z \ge x^{\bar{K}}} \{d_i y_i + L(y_i) - d_i x^{i-1} - c x^{\bar{K}} + cz \\
&\quad + E[S^1_{M(i,W),t+1}(y_i - D) + S^2_{M(\bar{K},W),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}}) + Q]\} \\
&= \min_{x^{i-1} \le y_i \le x^i, z \ge x^{\bar{K}}} \{f_{i,t}(y_i) + cz + E[S^2_{M(\bar{K},W),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}})]\} \\
&\quad + E[Q] - d_i x^{i-1} - c x^{\bar{K}},
\end{aligned} \tag{7}$$

for $i < \bar{K}$. When $i = \bar{K}$, we have

$$\begin{aligned}
J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}) &= \min_{x^{\bar{K}-1} \le y_{\bar{K}} \le z, z \ge x^{\bar{K}}} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) - d_{\bar{K}} x^{\bar{K}-1} - c x^{\bar{K}} + cz \\
&\quad + E[J_{t+1}(y_{\bar{K}} - D, \bar{0}^{M(\bar{K},W)-1}, z - y_{\bar{K}}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\} \\
&= \min_{x^{\bar{K}-1} \le y_{\bar{K}} \le z, z \ge x^{\bar{K}}} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) - d_{\bar{K}} x^{\bar{K}-1} - c x^{\bar{K}} + cz \\
&\quad + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^0_{M(\bar{K},W),t+1} + S^1_{M(\bar{K},W),t+1}(y_{\bar{K}} - D) \\
&\quad + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&= \min_{x^{\bar{K}-1} \le y_{\bar{K}} \le z, z \ge x^{\bar{K}}} \{f_{\bar{K},t}(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) \\
&\quad + S^2_{M(\bar{K},W),t+1}(z - D)]\} + E[S^0_{M(\bar{K},W),t+1}] - d_{\bar{K}} x^{\bar{K}-1} - c x^{\bar{K}}.
\end{aligned} \tag{8}$$

By applying Lemma 4, we have

$$\begin{aligned}
J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}) &= \min_{z \ge x^{\bar{K}}} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + a_{\bar{K},t} + E[S^0_{M(\bar{K},W),t+1}] + g_{\bar{K},t}(x^{\bar{K}-1}) - d_{\bar{K}} x^{\bar{K}-1} - c x^{\bar{K}}.
\end{aligned} \tag{9}$$

We now consider part (a) of the statement. From (7) for $i < \bar{K}$ the optimal expediting quantity is determined by

$$\min_{x^{i-1} \le y_i \le x^i} \{f_{i,t}(y_i)\},$$

and for $i = \bar{K}$ from (8) by

$$\min_{x^{\bar{K}-1} \le y_{\bar{K}} \le \max\{x^{\bar{K}}, z_t^*\}} \{f_{\bar{K},t}(y_{\bar{K}})\}.$$

Because $f_{i,t}(y_i)$ is convex for all i, this states that the base stock policy is optimal for expediting for every i. This completes the proof of part (a).

Next, we show part (b) of the theorem using (3). Note that the optimal regular ordering is determined by $J_t^i(\cdot)$, $1 \le i \le \bar{K}$ or $J_t(x^{\bar{K}}, \bar{0}^{\bar{K}})$ which corresponds to the minimum term in (3). Now we show that all of these lead to the same optimal decision. From (6) and (9), the optimal regular ordering quantity for $J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}})$ and $J_t(x^{\bar{K}}, \bar{0}^{\bar{K}})$ is determined by

$$\min_{z \ge x^{\bar{K}}} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\}. \tag{10}$$

On the other hand, if the minimum term is attained at $i < \bar{K}$, the optimal regular ordering quantity is determined from (7) by

$$\min_{z \ge x^{\bar{K}}} \{cz + E[S^2_{M(\bar{K},W),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}})]\}. \tag{11}$$

Note that $h_{\bar{K},t}(z)$ is nonincreasing convex, and $h_{\bar{K},t}(z) = 0$ for $z \ge y^*_{\bar{K},t}(\in \arg\min\{f_{\bar{K},t}(y)\})$. Therefore, if $z_t^* \ge y^*_{\bar{K},t}$, then (10) and (11) lead to the same minimizer z_t^*. If $z_t^* < y^*_{\bar{K},t}$, from Theorem 2, we have $z_t^* < y^*_{i,t}$ for all i, which results in expediting everything in the supply chain including the fresh regular order at the current time period by part (a). In this case, (10) determines the regular ordering quantity since we are expediting from the supplier. As a result, (10) always determines the optimal regular ordering. Because $H_{t+1}(z-D) = J_{t+1}(z-D, \bar{0}^{\bar{K}})$ and $H_{t+1}(z-D) + S^2_{M(\bar{K},w),t+1}(z - D)$ is convex for any realization w of W by part (d), the unconstrained minimizer z_t^* of (10) is well defined, and (10) states that the optimal regular ordering policy, which is the base stock policy with respect to $x^{\bar{K}}$. Hence part (b) is proved.

Finally, let us prove part (c). At time period t, we know that parts (a), (b), and (d) hold. Also, from the induction hypothesis, we assume (c) holds on and after time period $t+1$. We show in the appendix that (c) holds at time period t using all these results. Once part (c) is proved at time t with all the induction hypothesis, the induction step of the entire proof completes. $\square$

5 Results on the Expediting Base Stock Levels of Sequential Systems

In this section, we provide an insightful result on the variation of the magnitude of the expediting base stock levels as the expediting cost varies. As expediting cost varies, we expect the expediting base stock levels to also vary. For example, as d_i increases, $y^*_{i,t}$ should be nonincreasing to compensate for the higher cost of expediting. However, this increment in d_i might increase the need for expediting from elsewhere. Indeed, we show that the expediting base stock levels are nondecreasing for installations beyond installation i as d_i increases. On the other hand, the variation in d_i does not effect the base stock levels of the downstream installations.

The results in this section are applicable only when derivatives and integrals in expectations are interchangeable. If the holding and backlogging cost function has bounded derivatives, all functions under consideration have this interchangeability property, since all functions considered are Lipschitz. We assume in this section that this is the case. By Lemma 3.2 in Glasserman and Tayur (1995), derivatives and integrals in expectations are interchangeable. The main result of this section follows.

Theorem 4. *For a sequential system we have*

$$\frac{\partial y^*_{i,t}}{\partial d_i} \leq 0 \quad \text{and} \quad \frac{\partial y^*_{i,t}}{\partial d_j} \geq 0$$

for $j < i$.

The following diagram illustrates this theorem.

$$
\text{as} \quad d_i \uparrow \qquad \left\{ \begin{array}{ll} & \vdots \\ y^*_{i-2} & \text{no change} \\ y^*_{i-1} & \text{no change} \\ y^*_{i} & \downarrow \\ y^*_{i+1} & \uparrow \\ y^*_{i+2} & \uparrow \\ & \vdots \end{array} \right.
$$

Sequential systems have monotonic base stock levels as in Figure 5. As d_i increases, y^*_i decreases because higher d_i directly discourages expediting from installation i. However, the reduced y^*_i results in less safety stock in the manufacturing facility, which again calls for more expediting from beyond installation i, and hence increased y^*_j for $j > i$. The fact that y^*_t for $t \leq i-1$ do not change follows from their definition since in order to derive them, d_i is not needed. We prove this in several

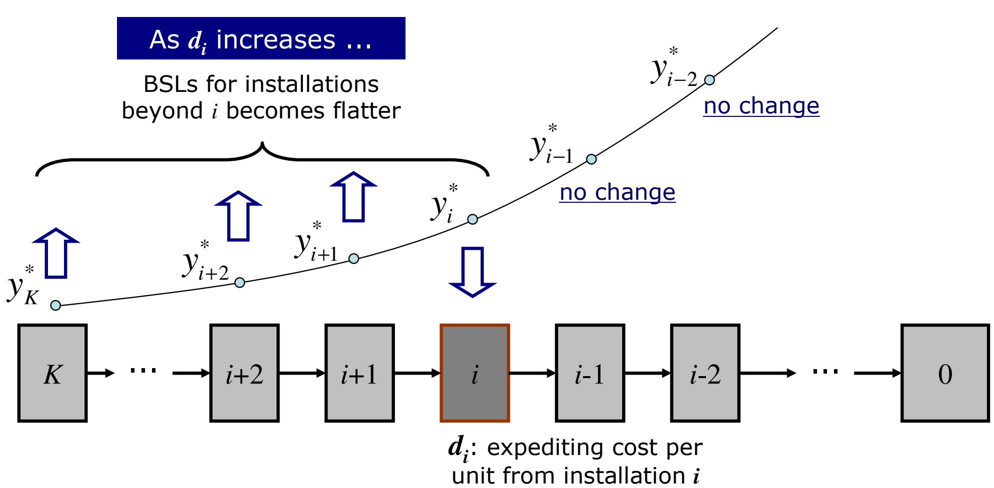

Figure 5: Directional sensitivity of base stock levels

steps using the following two lemmas.

Lemma 6. *In sequential systems, for $i \geq j \geq 1$,*

$$-1 \leq \frac{\partial}{\partial d_j}\frac{\partial S^1_{i,t}(y)}{\partial y} \leq 0.$$

Proof. We use induction. Note that $S^1_{i,T}(y) = g_{i,T}(y) - d_i y$ and $f_{i,T}(y) = d_i y + L(y)$. In the base case we have $-1 \leq \frac{\partial}{\partial d_j}\frac{\partial S^1_{i,T}(y)}{\partial y} \leq 0$ since when $y \leq y^*_T$ and $j = i$, $\frac{\partial}{\partial d_j}\frac{\partial S^1_{i,T}(y)}{\partial y} = -1$, and otherwise $\frac{\partial}{\partial d_j}\frac{\partial S^1_{i,T}(y)}{\partial y} = 0$.

Assume that $-1 \leq \frac{\partial}{\partial d_j}\frac{\partial S^1_{i,t+1}(y)}{\partial y} \leq 0$ for a given i and all j such that $i \geq j \geq 1$. We have

$$\frac{\partial f_{i,t}(y)}{\partial y} = d_i + \frac{\partial L(y)}{\partial y} + E[\frac{\partial}{\partial y}S^1_{M(i,W),t+1}(y-D)].$$

When $y \leq y^*_{i,t}$, we have $S^1_{i,t}(y) = -d_i y$ since $g_{i,t}(y) = 0$. Therefore $\frac{\partial}{\partial d_j}\frac{\partial S^1_{i,t}(y)}{\partial y} = 0$ for $j < i$, and $\frac{\partial}{\partial d_i}\frac{\partial S^1_{i,t}(y)}{\partial y} = -1$ for $j = i$. On the other hand, when $y > y^*_{i,t}$, we have $S^1_{i,t}(y) = f_{i,t}(y) - d_i y - a_{i,t}$. For $j \leq i$, since $M(i,W) \leq i$ by definition, it follows that

$$-1 \leq \frac{\partial}{\partial d_j}\frac{\partial S^1_{i,t}(y)}{\partial y} = E[\frac{\partial}{\partial d_j}\frac{\partial}{\partial y}S^1_{M(i,W),t+1}(y-D)] \leq 0.$$

Note that we interchanged integrals and derivatives on several occasions. □

Lemma 7. *In sequential systems, for all i we have*

$$\frac{\partial}{\partial d_i}\frac{\partial f_{i,t}(y)}{\partial y} \geq 0,$$

and for $i > j \geq 1$,

$$\frac{\partial}{\partial d_j}\frac{\partial f_{i,t}(y)}{\partial y} \leq 0.$$

Proof. From Lemma 6 for all $j \leq i$ we obtain

$$-1 \leq E[\frac{\partial}{\partial d_j}\frac{\partial}{\partial y}S^1_{M(i,W),t+1}(y-D)] \leq 0.$$

If $j = i$, we have

$$\frac{\partial}{\partial d_i}\frac{\partial f_{i,t}(y)}{\partial y} = 1 + E[\frac{\partial}{\partial d_i}\frac{\partial}{\partial y}S^1_{M(i,W),t+1}(y-D)] \geq 0,$$

and, otherwise if $j < i$, we have

$$\frac{\partial}{\partial d_j}\frac{\partial f_{i,t}(y)}{\partial y} = E[\frac{\partial}{\partial d_j}\frac{\partial}{\partial y}S^1_{M(i,W),t+1}(y-D)] \leq 0.$$

This establishes the proof. □

Proof of Theorem 4. First it is obvious that changes in d_i do not affect $f_{j,t}$ for $j = 1, 2, ...i$ for all t since $M(k, W) \leq k$ for all k. From Lemma 7, we have $\frac{\partial}{\partial d_i}\frac{\partial f_{i,t}(y)}{\partial y} \geq 0$, and this implies $\frac{\partial y^*_{i,t}}{\partial d_i} \leq 0$ for all i. Similarly, $\frac{\partial}{\partial d_j}\frac{\partial f_{i,t}(y)}{\partial y} \leq 0$ implies $\frac{\partial y^*_{i,t}}{\partial d_j} \geq 0$, for $j < i$ by Lemma 8, which is in the appendix. The proof is complete. □

6 A Numerical Study of Sequential Systems

In this section we present a numerical study focusing on two aspects. In the first part, we present numerical results when system parameters vary for a fixed $\mathbf{W}$. Second, we present a numerical analysis for different sets of movement patterns $\mathbf{W}$. We consider a three installation system, which includes a supplier, an intermediate installation, and a manufacturer with random movement patterns and demand. For three installation systems, there is a total of 5 different movement patterns, i.e., w_1, w_2, $\cdots$, w_5, as depicted in Figure 6. By combining all or some of these movement patterns and by assigning probability distribution to each of these patterns, different sets of movement patterns, i.e., $\mathbf{W}_1$, $\mathbf{W}_2$, $\cdots$, $\mathbf{W}_6$ can be developed as shown in Figure 6. We first present the results with $\mathbf{W}_1$ in the first part to study the dynamics of optimal policies as system parameters vary, and then we present the results with all sets of movement patterns with fixed system parameters to study the effect of different movement patterns. Practical meanings of different sets of movement patterns are discussed later.

Movement patterns	Different sets of movement patterns: Independent probability ($\mathbf{W}_1$)	Frequent error at int. ins. ($\mathbf{W}_2$)	Frequent error at supplier ($\mathbf{W}_3$)	Frequent system down ($\mathbf{W}_4$)	Fast delivery is possible ($\mathbf{W}_5$)	Combination of all ($\mathbf{W}_6$)
Normal pattern (w_1)	p^2	60%	60%	60%	60%	40%
Down at int. ins. (w_2)	$p(1-p)$	40%				15%
Down at supplier (w_3)	$(1-p)p$		40%			15%
Down at Both (w_4)	$(1-p)^2$			40%		15%
Fast Delivery (w_5)	0				40%	15%

Figure 6: Different sets of movement patterns

6.1 Variation of system parameters

In this section, we present a numerical analysis of nine carefully designed representative cases in addition to the base case of the three installation system to study the dynamics of optimal policies as system parameters vary.

Time of delivery is a geometric random variable with delivery probability p at each time period. Formally, on order at an installation has probability p to be delivered to the next installation in a single time period. Demand follows the symmetric triangular or uniform distribution in range $(0, D)$. In our numerical experiments D is fixed at 100 for simplicity. The per unit expediting cost is d_2 for expediting from the supplier and d_1 for expediting from the intermediate installation. The inventory and backlogging costs are c_1, c_2, respectively per unit. These parameters are stationary. A single simulation run has 5,000 time periods, and 50 independent runs are tested in each case to get less than 2% of the double side confidence interval length with respect to the total cost per period. An optimal policy is computed by conducting such a simulation with every combination of z in the multiple of 10 and y_1 and y_2 in the multiple of 5 from -1,000 to 1,000. It is based on the assumption that the manager in practice will keep the base stock levels as simple numbers with reasonable precisions.

As summarized in Figure 7, the nine cases have variations in the system parameters from the base case, namely Case 0. Case 1 has a higher variability in the delivery lead time due to lower p; Case 2 has a lower variability in the delivery lead time due to higher p; Case 3 has a higher variability in demand due to the uniform demand distribution instead of triangular; Case 4 has higher inventory and backlogging costs; Case 5 has the reversed cost structure, where the inventory cost is higher than the backlogging cost; Case 6 has higher expediting costs; Case 7 has higher expediting costs from the supplier (d_2) representing a stronger sequential system case; Case 8 has a higher expediting cost from the intermediate installation (d_1) representing a non-sequential system case; Case 9 has equal expediting costs from both installations ($d_2 = d_1$) representing a stronger non-sequential system case.

Case No.	p	demand dist.	d_2	d_1	c_1	c_2
0. Base case	0.5	triangular	2	1	1	2
1. Higher variability in delivery	0.2	triangular	2	1	1	2
2. Lower variability in delivery	0.8	triangular	2	1	1	2
3. Higher variability in demand	0.5	uniform	2	1	1	2
4. Higher inventory and backlogging costs	0.5	triangular	2	1	2	4
5. Reversed cost structure (inv/backlogging)	0.5	triangular	2	1	2	1
6. Higher expediting costs	0.5	triangular	4	2	1	2
7. Stronger sequential system case	0.5	triangular	4	1	1	2
8. Non-sequential system case	0.5	triangular	2	1.5	1	2
9. Stronger non-sequential system case	0.5	triangular.	2	2	1	2

Figure 7: Nine representative cases for the numerical study

For each case, base stock policies are applied and optimal base stock levels are numerically obtained for both of the situations with or without expediting. In particular, without expediting, only the base stock level for regular order z is obtained along with the total logistics cost, which is the sum of expediting, inventory, and backlogging costs. With expediting, base stock levels for expediting, y_2 and y_1, are also obtained. Figure 8 summarizes the results of the numerical study, where T.C. stands for the total cost per period and C.I. stands for the 95% confidence interval length with respect to the total cost per period (both sides).

In each case, if expediting is optimally used, the total cost reduction is significant (at least 20% in these cases) and the resultant base stock levels z are smaller than the ones without expediting suggesting that the system becomes leaner with expediting and achieves higher service rates. In

Case	T.C.(w/o exp)	C.I.	z	T.C.(w/ exp)	C.I.	z	y_1	y_2	Cost saving
0	123	1.1%	270	67	0.6%	210	50	50	46%
1	365	1.9%	650	91	0.5%	290	55	55	75%
2	61	0.8%	190	48	0.7%	170	40	40	21%
3	134	1.1%	270	80	0.6%	210	50	50	41%
4	251	1.1%	290	98	0.5%	180	55	55	61%
5	101	0.9%	190	71	0.6%	170	30	30	30%
6	123	1.1%	270	92	0.8%	240	40	40	25%
7	123	1.1%	270	77	1.0%	220	45	30	37%
8	123	1.1%	270	75	0.6%	210	40	55	39%
9	123	1.1%	270	83	0.6%	220	35	55	33%

Figure 8: Results of the numerical analysis

Case 0 (the base case), optimal expediting could save 46% of total logistics costs. Due to the linear cost structure of the base case ($d_2 = 2d_1$), it is expected to observe $y_2 = y_1$. In Case 1 (higher variability in delivery), if there is no expediting, the total cost increases significantly (about three times from 123 to 365) above the base case due to increased variability with much higher base stock level z. However, through expediting, the system is much more effectively hedging the risk from increased variability. The cost increase from the base case is only about 35% (from 67 to 91) and the corresponding reduction in the total cost is 75%, much higher than 46% of the base case. It is expected that the value of visibility through expediting should be higher with higher variability in the delivery lead time, which is confirmed by this numerical case. Also, the expediting base stock levels y_1 and y_2 are higher than in the base case, suggesting more expediting is necessary with increased variability in the delivery lead time. In Case 2 (lower variability in delivery), the total cost is reduced from 123 in the base case to 61 even without expediting due to the decreased variability in delivery. Despite of this, the introduction of expediting saves 21% of the total cost, although this is lower than 46% in the base case. Also, the base stock levels are all lower than in the base case meaning that the system needs less safety stock due to decreased variability in delivery. In Case 3 (higher variability in demand), increased variability in demand leads to higher total cost and does not affect the base stock levels of optimal policies for both with and without expediting. The cost saving is 41%, slightly less than the base case. In Case 4 (higher inventory and backlogging costs), as expected twice higher inventory and backlogging costs than in the base case lead to about double the total cost without expediting. With expediting, however, the total cost increase above the base case is only 46% (from 67 to 98), and the total cost saving is about 61% higher than 46% of the base case. More interestingly, the base stock level for regular ordering z is lower than in the base case while expediting base stock levels y_1 and y_2 are higher, which suggests that it is optimal to place leaner regular orders but to use more expediting when facing higher inventory and backlogging costs. In Case 5 (reversed cost structure), because inventory cost is higher than in the base case while backlogging cost is lower, the optimal policy is minimizing inventory pile up at the manufacturer and hence less expediting is present than in the base case, therefore all of the base stock levels are lower than in the base case for both with and without expediting. By the same reasoning, the total cost reduction percentage is also lower than in the base case. In Case 6 (higher expediting costs), because the expediting cost is higher, the optimal policy places more regular orders and reduces the amount of expediting. Therefore, the base stock level for regular ordering z is higher than in the base case but expediting base stock levels y_1 and y_2 are lower. Also, the total cost saving percentage is smaller due to more expensive expediting. In Case 7 (the stronger sequential system case), higher expediting cost d_2 makes the stronger sequential system, and it results in higher y_1 than y_2 as predicted by theory, suggesting more expediting from closer intermediate installation (y_1) than the farther suppler (y_2). Directional movements of the base stock levels are due to the increase in d_2 as discussed in Case 6. In Case 8 (the non-sequential system case), the system is non sequential, so it is hard to guarantee that the base stock policy is an optimal policy. Nevertheless, expediting with numerically chosen optimal base stock levels results in 39% of the total cost saving, slightly less than 46% of the base case. An interesting observation is that now y_2 is higher than y_1 suggesting more expediting from the supplier that is relatively cheaper on the per unit cost of expediting. In Case 9 (the stronger non-sequential system case), d_1 further apart from d_2, results in reduced y_1 and increased z with respect to the base case, suggesting more regular ordering and less expediting from the more expensive intermediate installation. As the system becomes increasingly non-sequential, the gap between y_1 and y_2 also increases.

Next in Figure 9, we provide the results of the sensitivity analysis on parameter p with respect to the base case, case 1, and case 2. As probability increases we find that the total cost saving also reduces gradually. Also, a deviation of 0.05 in p from the base case, where $p = 0.5$, results in about 5% deviation in total cost saving.

p	T.C.(w/o exp)	C.I.	z	T.C.(w/ exp)	C.I.	z	y_1	y_2	Cost saving
0.2 (Case 1)	365	1.9%	650	91	0.5%	290	55	55	75%
0.4	165	1.3%	330	74	0.5%	220	55	55	55%
0.45	142	1.3%	290	71	0.5%	210	55	55	50%
0.5 (Case 0)	123	1.1%	270	67	0.6%	210	50	50	46%
0.55	109	1.2%	250	63	0.5%	200	50	50	42%
0.6	96	1.1%	230	60	0.8%	190	45	45	38%
0.8 (Case 2)	61	0.8%	190	48	0.7%	170	40	40	21%

Figure 9: Results of the sensitivity analysis

6.2 Variation of movement patterns

In this section, we provide the numerical results with different sets of movement patterns based on Figure 6. The simulation setting (e.g., number of runs, finding optimal base stocks) is identical to the previous section. System parameters (i.e., demand dist., d_1, d_2, c_1, c_2) are identical with the base case in the previous section. Different sets of movement patterns were carefully chosen to have practical meanings. First, $\mathbf{W}_1$ captures where each delivery as an independent event with a similar delivery probability. Set of movement pattern $\mathbf{W}_2$ is where the operations at the intermediate installation is relatively unreliable causing frequent delays. The next set, $\mathbf{W}_3$ is where significant production and operation variability exists at the supplier. Set $\mathbf{W}_4$ is the case of the supplier and intermediate installations linked through a common system (e.g., IT, labor union) causing simultaneous failures. Set $\mathbf{W}_5$ is where supplier uses frequently expedited transportation at no additional charge due to extra space or as a value-added customer service. Lastly, $\mathbf{W}_4$ combines situations from W_2 to W_4. Figure 10 summarizes the numerical results for each set.

Without expediting, $\mathbf{W}_1$ has the highest total cost since it has the highest variability in the delivery lead time. Set $\mathbf{W}_4$ has the second highest total cost without expediting since it has relatively higher variability. Since inventory cost is lower than the backlogging cost, set $\mathbf{W}_5$ has the lowest total cost, in which unexpected fast delivery is possibly resulting in more inventory

Set	T.C.(w/o exp)	C.I.	z	T.C.(w/ exp)	C.I.	z	y_1	y_2	Cost saving
$\mathbf{W}_1$	123	1.1%	270	67	0.6%	210	50	50	46%
$\mathbf{W}_2$	72	1.1%	190	49	0.7%	170	40	40	32%
$\mathbf{W}_3$	72	1.0%	190	55	0.7%	180	35	35	24%
$\mathbf{W}_4$	98	1.3%	220	60	0.7%	200	45	45	38%
$\mathbf{W}_5$	46	0.4%	150	43	0.6%	140	35	35	7%
$\mathbf{W}_6$	79	0.8%	200	57	0.7%	170	45	45	28%

Figure 10: Results of the numerical analysis on different sets of movement patterns

stocking than backlogging.

Expediting saves the total costs for all sets of movement patterns. Except $\mathbf{W}_5$, where expediting yields only a 7% increase due to the similar nature between expediting and the random movement, expediting reduces more than 20% of the total cost. Expediting also reduces the regular ordering base stock levels for all sets which results in a leaner system.

7 Conclusion

In this paper, we consider an optimal policy for expediting and regular ordering of a stochastic lead time model with multiple intermediate installations. Since in general the model exhibits complex and nonintuitive policies, we confine our interest to a class of systems defined by conditions on expediting cost and movement patterns of regular orders. We call such systems sequential since outstanding orders, including expediting, do not cross in time. For sequential systems, the optimal policy for regular ordering is the base stock policy with respect to the inventory position, and the optimal policy for expediting from an installation is the base stock policy with respect to the echelon stock of the downstream installations.

The numerical study suggests that expediting optimally results in a significant reduction in the total logistics cost, and the system is operated leaner due to the decreased optimal base stock level for regular ordering while achieving a higher service level due to expediting. As variability in lead time, inventory cost, and backlogging costs increase, the impact of expediting and the total cost saving also increase.

Song and Zipkin (1996), who considered a non-expediting system, found that the optimal regular ordering policy does not require any state variable information, and that the only relevant information is the inventory position and the lead time distribution. Our results indicate that the optimal regular ordering policy and expediting require the real time state information, since expediting and regular ordering have to be considered concurrently. In other words, the stochastic movement of regular orders in our model requires new information systems to capture state information to enable optimal expediting decision making in real time.

There are multiple tracking systems that can be used for this purpose. 2D bar code, GPS, and RFID are typical examples of such tracking technologies. In determining the most suitable type of a tracking system, several criteria should be considered at the same time. First, the speed to capture the state of the system should be fast enough relatively to the length of the time period. Second, initial investments and operating costs should not exceed the total cost saving that can be achieved through expediting. Third, the information accuracy should be high enough not to send false signals to the decision maker.

RFID is a good candidate due to its relatively low deployment and maintenance costs. Tags are currently below 10 cents and reader costs range in a few thousand U.S. dollars. With tags attached on units of goods (e.g., pallets or cases) and readers installed at each installation, real-time location information of outstanding orders is available to the manufacturer. Minimal labor is an additional benefit and accuracy is getting improved everyday. However, the final decision of the suitable tracking system should be made according to the business case based on a detailed cost benefit analysis as suggested by the proposed model.

For expediting to be optimal, it is also important to estimate the distribution of the stochastic lead time, which can be accomplished by observing movements of orders for multiple time periods by leveraging the order tracking system. It may be possible to estimate the initial distribution based on business insight and constantly updating it for some initial time horizon. However, if a time period is a week or even longer, then it could take significant time to estimate a reliable distribution, which may be sufficient to threat business profitability, for example, in the electronics industry. Therefore, an important consideration is also how to estimate the distribution faster and more accurately. The choice of the tracking system should reflect this consideration. One additional benefit of RFID is that it provides abundance of real-time information for better estimation of the delivery lead time distribution and movement patterns since RFID tags can store extra information such as the dwell time in each installation.

Without expediting, according to the result by Song and Zipkin (1996), adding visibility does not bring additional value to inventory control. Therefore, we need to actively use new information to unveil additional benefits, and this should be done through quantitative analysis as Lee and Özer (2007) also assert.

Appendix

Proof of Lemma 1. The statement clearly holds when $i = j$. By Assumption 3, for $i > j$ we have

$$\begin{aligned} d_i - d_{i-1} &\geq E[d_{M(i,W)} - d_{M(i-1,W)}] \\ d_{i-1} - d_{i-2} &\geq E[d_{M(i-1,W)} - d_{M(i-2,W)}] \\ &\vdots \\ d_{j+1} - d_j &\geq E[d_{M(j+1,W)} - d_{M(j,W)}]. \end{aligned}$$

By summing the above inequalities we obtain $d_i - d_j \geq E[d_{M(i,W)} - d_{M(j,W)}]$. Assumption 1 ensures $M(i,W) \geq M(j,W)$ for $i \geq j$, thus setting $i = M(i,W)$ and $j = M(j,W)$ and taking expectation results in

$$E[d_{M(i,W)} - d_{M(j,W)}] \geq E[d_{M(M(i,W),W')} - d_{M(M(j,W),W')}] = E[d_{M^2(i,W)} - d_{M^2(j,W)}].$$

Therefore,

$$d_i - d_j \geq E[d_{M(i,W)} - d_{M(j,W)}] \geq E[d_{M^2(i,W)} - d_{M^2(j,W)}].$$

Note that $M^n(i,W) \geq M^n(j,W)$ for every n, which follows from Assumption 1 and the definition of M^n. By applying the above relation repeatedly, we obtain

$$d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}],$$

which completes the proof. $\square$

Proof of Lemma 2. Part (a): We have $\{w : M^n(i,w) = 0\} \subseteq \{w : M^{n+1}(i,w) = 0\}$ since an order can stay at installation 0 for one time period. From Assumption 2 it follows

$$1 = Prob[\cup_{n=1}^{\infty}\{w : M^n(i,w) = 0\}] = \lim_{n\to\infty} Prob[M^n(i,W) = 0].$$

Part (b): Clearly $\sum_k Prob[M^n(i,W) = k] = 1$ and by taking the limit we get

$$\sum_k \lim_{n\to\infty} Prob[M^n(i,W) = k] = 1,$$

or equivalently

$$1 = \sum_{k\neq 0} \lim_{n\to\infty} Prob[M^n(i,W) = k] + \lim_{n\to\infty} Prob[M^n(i,W) = 0].$$

Since $\lim_{n\to\infty} Prob[M^n(i,W) = 0] = 1$ by part (a), we conclude $\sum_{k\neq 0}\lim_{n\to\infty} Prob[M^n(i,W) = k] = 0$. $\square$

Proof of Lemma 4. We first fix y and minimize over x as a function of y, then minimize over y. We obtain

$$\begin{aligned}
\min_{b\leq x\leq y}\{f_1(x) + f_2(y)\} &= \min_{b\leq y}\{\{\min_{b\leq x\leq y} f_1(x)\} + f_2(y)\} \\
&= \min_{b\leq y}\{a_1 + g_1(b) + h_1(y) + f_2(y)\} \qquad (12) \\
&= a_1 + g_1(b) + \min_{b\leq y}\{h_1(y) + f_2(y)\},
\end{aligned}$$

where, in (12), we use Lemma 3. $\square$

Proof of Theorem 2. We first provide the following preliminary results. For a convex function $f : \mathbb{R} \to \mathbb{R}$, let $\partial f(x)$ be its subdifferential at x, which is a set. For two sets S_1 and S_2, we denote $S_1 \leq S_2$ if there exists $s_2 \in S_2$ such that $s_1 \leq s_2$ for any $s_1 \in S_1$, and there exists $s_1 \in S_1$ such that $s_1 \leq s_2$ for any $s_2 \in S_2$. We provide useful lemmas below.

Lemma 8. *Let f_1 and f_2 be convex functions. If $\partial f_1(x) \leq \partial f_2(x)$ for all $x \in \mathbb{R}$, then*

$$\arg\min_x f_1(x) \geq \arg\min_x f_2(x).$$

Proof. Let $u \in \arg\min_x f_1(x)$. Then $0 \in \partial f_1(u)$. It follows that $0 \leq x$ for every $x \in \partial f_2(u)$. This implies that the minimum of f_2 is smaller than or equal to u, which completes the proof. □

Lemma 9. *Let f_1 and f_2 be convex functions, and let g_1 and g_2 be their penalty functions as in Lemma 3. If $\partial f_1(x) \leq \partial f_2(x)$, then $\partial g_1(x) \leq \partial g_2(x)$.*

Proof. We have $\partial g_1(x) = \partial f_1(x)$ if $x \geq \max\arg\min f_1$, and 0 otherwise. Similarly, we have $\partial g_2(x) = \partial f_2(x)$ if $x \geq \max\arg\min f_2$, and 0 otherwise. From Lemma 8, we have $\arg\min f_2 \leq \arg\min f_1$. For $x < \arg\min f_2$, we have $\partial g_1(x) = \partial g_2(x) = \{0\}$. On the other hand, for $\arg\min f_2 \leq x < \arg\min f_1$, we have $\partial g_1(x) = \{0\}$ and $\partial f_2(x) = \partial g_2(x) \geq \{0\}$ since $\partial f_2(x)$ is non-decreasing, therefore $\partial g_1(x) \leq \partial g_2(x)$. For $x \geq \arg\min f_1$, we have $\partial g_1(x) = \partial f_1(x) \leq \partial f_2(x) = \partial g_2(x)$ by definition. This completes the proof. □

Lemma 10. *Let f_1, f_2, $\tilde{f}_1$, and $\tilde{f}_2$ be convex functions. If $\partial f_1(x) \leq \partial f_2(x)$ and $\partial \tilde{f}_1(x) \leq \partial \tilde{f}_2(x)$, then $\partial\{f_1 + \tilde{f}_1\}(x) \leq \partial\{f_2 + \tilde{f}_2\}(x)$.*

Proof. For any real number $u \in \partial\{f_1 + \tilde{f}_1\}(x)$, there exist $v \in \partial f_1(x)$ and $w \in \partial \tilde{f}_1(x)$ such that $u = v + w$. Likewise, for any $u' \in \partial\{f_2 + \tilde{f}_2\}(x)$, there exist $v' \in \partial f_2(x)$ and $w' \in \partial \tilde{f}_2(x)$ such that $u' = v' + w'$. Since $\partial f_1(x) \leq \partial f_2(x)$, we have $v \leq v'$ and from $\partial \tilde{f}_1(x) \leq \partial \tilde{f}_2$ we obtain $w \leq w'$. It thus follows $u = v + w \leq v' + w' = u'$. This completes the proof. □

Lemma 11. *Let f_1 and f_2 be convex functions, and let $F_1(x) = E[f_1(x - D)]$ and $F_2(x) = E[f_2(x - D)]$. If $\partial f_1(x) \leq \partial f_2(x)$ for every x, then $\partial F_1(x) \leq \partial F_2(x)$ for every x.*

Proof. Let x be fixed and C be the compact support of demand D. The 'convex' mean value theorem asserts that for every $d \in C$, and $1 \geq h \geq 0$, there exists $x - d < z < \alpha < x - d + h$ such that

$$\partial^+ f_i(z) \leq \frac{f_i(x - d + h) - f_i(x - d)}{h} \leq \partial^- f_i(\alpha),$$

where ∂^-, ∂^+ are the left, right derivatives, respectively. Due to convexity and the fact that $\{\gamma | x - d \leq \gamma \leq x - d + 1, d \in C\}$ is compact, it follows that there exists $M(x) \geq 0$ such that $\partial^+ f_i(z) \geq M(x)$, $\delta^- f_i(\alpha) \leq M(x)$ for every z, α. This implies that

$$|\frac{f_i(x - d + h) - f_i(x - d)}{h}| \leq M(x),$$

for each $0 < h \leq 1$ and d. We also have $\int M(x) dD = M(x)$. This implies that we can use the dominated convergence theorem which implies

$$\lim_{h \to 0, h \geq 0} \int \frac{f_i(x - d + h) - f_i(x - d)}{h} dD(d) = \int \lim_{h \to 0, h \geq 0} \frac{f_i(x - d + h) - f_i(x - d)}{h} dD(d).$$

The statement now follows from the basic definitions, the assumption $\partial f_1(\cdot) \leq \partial f_2(\cdot)$, and by applying similar arguments to $[x - h, x]$. This completes the proof. □

We prove Theorem 2 by induction on t that $\partial f_{i,t}(y) \leq \partial f_{i+1,t}(y)$ for every y and i. For the base case ($t = T$), we have $\partial f_{i,T}(y) \leq \partial f_{i+1,T}(y)$ for all i because $f_{i,T}(y) = d_i y + L(y)$ and d_i is nondecreasing in i by Proposition 1. In the induction step, for a fixed $t + 1 \leq T$, we assume that $\partial f_{i,t+1}(y) \leq \partial f_{i+1,t+1}(y)$ for all i and y. We have

$$\begin{aligned}
f_{i,t}(y) &= d_i y + L(y) + E[S^1_{M(i,W),t+1}(y - D)] \\
&= d_i y + L(y) + E[g_{M(i,W),t+1}(y - D) - d_{M(i,W)}(y - D)] \\
&= (d_i - E[d_{M(i,W)}])y + L(y) + E[g_{M(i,W),t+1}(y - D)] + E[d_{M(i,W)}]\, E[D], \text{ and} \\
f_{i+1,t}(y) &= (d_{i+1} - E[d_{M(i+1,W)}])y + L(y) + E[g_{M(i+1,W),t+1}(y - D)] + E[d_{M(i+1,W)}]\, E[D].
\end{aligned}$$

Note that $\partial[(d_i - E[d_{M(i,W)}])y] \leq \partial[(d_{i+1} - E[d_{M(i+1,W)}])y]$ by Assumption 3, and $\partial\, E[g_{M(i,W),t+1}(y - D)] \leq \partial\, E[g_{M(i+1,W),t+1}(y - D)]$ for all i since the induction assumption $\partial f_{M(i,W),t+1}(y - D) \leq \partial f_{M(i+1,W),t+1}(y - D)$ is equivalent to $\partial g_{M(i,W),t+1}(y - D) \leq \partial g_{M(i+1,W),t+1}(y - D)$. Therefore we get $\partial f_{i,k}(y) \leq \partial f_{i+1,k}(y)$ for all i. The proof is thus completed. □

Proof of Lemma 5. The proof is by induction on t. In the base case $t = T$ we have $g_{i,T}(x) +$

$S^2_{M(i,w),T}(x) = g_{i,T}(x) + h_{M(i,w),T}(x) - L(x)$. Consider the following two cases.

(Case 1) If $x \leq y^*_{M(i,w),T}$, then $g_{M(i,w),T}(x) = 0$, thus $g_{i,T}(x) + h_{M(i,w),T}(x) - L(x) = g_{i,T}(x) + f_{M(i,w),T}(x) - a_{M(i,w),T} - L(x) = g_{i,T}(x) + d_{M(i,w)}x - a_{M(i,w),T}$ is convex.

(Case 2) If $x \geq y^*_{i,T}$, then $h_{i,T}(x) = 0$, thus $g_{i,T}(x) + h_{M(i,w),T}(x) - L(x) = f_{i,T}(x) - a_{i,T} + h_{M(i,w),T}(x) - L(x) = d_i x - a_{i,T} + h_{M(i,w),T}(x)$ is convex.

From Theorem 2 it follows $y^*_{i,T} \leq y^*_{M(i,w),T}$ since $i \geq M(i,w)$. If $y^*_{i,T} < y^*_{M(i,w),T}$, then $g_{i,T}(x) + S^2_{M(i,w),T}(x)$ is globally convex because it is convex on two partially overlapping intervals, which are $x \leq y^*_{M(i,w),T}$ and $x \geq y^*_{i,T}$. When $y^*_{i,T} = y^*_{M(i,w),T} = y^*$, then by Proposition 1, we have

$$\partial\{g_{i,T}(y^*) + d_{M(i,w)}y^* - a_{M(i,w),T}\} \leq \partial\{d_i y^* - a_{i,T} + h_{M(i,w),T}(y^*)\}.$$

Since $g_{i,T}(x)$ is nondecreasing and $h_{M(i,w),T}(x)$ is nonincreasing, we obtain $\partial h_{M(i,w),T}(x) \leq \partial g_{i,T}(x)$, hence global convexity. This completes the base case.

Now let us assume that $g_{i,t+1}(x) + S^2_{M(i,w),t+1}(x)$ is convex for all $w \in \mathbf{W}$ and all i, and for some $t+1 \leq T$. We need to prove that $g_{i,t}(x) + S^2_{M(i,w),t}(x) = g_{i,t}(x) + h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x-D)]$ is convex for any $w \in \mathbf{W}$ and for all i. Again consider the following two cases.

(Case 1) If $x \leq y^*_{M(i,w),t}$, then $g_{M(i,w),t}(x) = 0$. Thus

$$\begin{aligned}
&g_{i,t}(x) + h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x-D)] \\
&\quad = g_{i,t}(x) + f_{M(i,w),t}(x) - a_{M(i,w),t} - L(x) + E[S^2_{M^2(i,w),t+1}(x-D)] \\
&\quad = g_{i,t}(x) + d_{M(i,w)}x - a_{M(i,w),t} + E[S^1_{M^2(i,w),t+1}(x-D)] + E[S^2_{M^2(i,w),t+1}(x-D)] \\
&\quad = g_{i,t}(x) + d_{M(i,w)}x - a_{M(i,w),t} - S^0_{M^2(i,w),t+1}
\end{aligned}$$

is convex.

(Case 2) If $x \geq y^*_{i,t}$, then $h_{i,t}(x) = 0$. Thus

$$\begin{aligned}
&g_{i,t}(x) + h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] \\
&\quad = f_{i,t}(x) - a_{i,t} + h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] \\
&\quad = d_i x - a_{i,t} + h_{M(i,w),t}(x) + E[S^1_{M(i,w),t+1}(x - D)] + E[S^2_{M^2(i,w),t+1}(x - D)] \\
&\quad = d_i x - a_{i,t} + h_{M(i,w),t}(x) + E[g_{M(i,w),t+1}(x - D) - d_{M(i,w)}(x - D) \\
&\qquad + S^2_{M^2(i,w),t+1}(x - D)]
\end{aligned}$$

is convex since $g_{M(i,w),t+1}(x - D) + S^2_{M^2(i,w),t+1}(x - D)$ is convex by the induction hypothesis.

Now we apply a similar logic as in the base case. From Theorem 2 we obtain $y^*_{i,t} \leq y^*_{M(i,w),t}$ since $i \geq M(i,w)$. If $y^*_{i,t} < y^*_{M(i,w),t}$, then $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ is globally convex because it is convex for two partially overlapping intervals, which are $x \leq y^*_{M(i,w),t}$ and $x \geq y^*_{i,t}$. If $y^*_{i,t} = y^*_{M(i,w),t} = y^*$, then

$$g_{i,t}(x) + S^2_{M(i,w),t}(x) = \begin{cases} h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] & x \leq y^* \\ g_{i,t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] & x \geq y^*. \end{cases}$$

Since $g_{i,t}(x)$ is nondecreasing and $h_{M(i,w),t}(x)$ is nonincreasing, we have $\partial h_{M(i,w),t}(x) \leq \partial g_{i,t}(x)$, which means global convexity of $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ when $y^*_{i,t} = y^*_{M(i,w),t}$. This completes the proof. □

The remainder of the proof of Theorem 3 and Proposition 3. We show part (c) at time period t by assuming parts (a), (b), and (d) hold on and after time period t and part (c) holds on and after time period $t + 1$. We compare two states $(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}})$ and $(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$.

For convenience in the remainder of the proof, let A denote $(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}})$ and let B denote $(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$. Also, let A^+ and B^+ denote the next states of A and B under the respective optimal control (they depend on the underlying realization but we do not show this dependency). Let w be the realized value of W at the current time period and let j denote $M(i,w)$. Finally, let A_j^+ and B_j^+ denote the next states of A and B under respective optimal control given w at the beginning of the next time period. We consider three cases.

Case 1 If $y_{i,t}^* \leq x^{i-1}$, then no expediting is necessary. If $j > 0$, then the two states in the next time period $t+1$ are $A_j^+ = (x^{i-1} - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)}, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$ and $B_j^+ = (x^{i-1} + e - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)} - e, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$, where u is the regular ordering quantity, which is the same for both states. For $j > 0$, the induction hypothesis implies

$$J_{t+1}(A_j^+) - J_{t+1}(B_j^+) = S_{j,t+1}^0 + S_{j,t+1}^1(x^{i-1} - D) + S_{j,t+1}^2(x^{i-1} + e - D). \tag{13}$$

On the other hand, if $j = 0$, then the two states at time period $t+1$ are the same and they are $A_0^+ = B_0^+ = (x^{N(0,w)} - D, x^{N(1,w)} - x^{N(0,w)}, \cdots, x^{N(M(\bar{K},w)-1,w)} - x^{N(M(\bar{K},w)-2,w)}, x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$. Since $S_{0,t+1}^0 = S_{0,t+1}^1(x^{i-1} - D) = S_{0,t+1}^2(x^{i-1} + e - D) = 0$ by definition, (13) still holds. Using (13) we get

$$\begin{aligned}
&E[J_{t+1}(A^+) - J_{t+1}(B^+)] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A^+) - J_{t+1}(B^+)|M(i,W) = j\}] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A_j^+) - J_{t+1}(B_j^+)\}] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{S_{j,t+1}^0 + S_{j,t+1}^1(x^{i-1} - D) + S_{j,t+1}^2(x^{i-1} + e - D)\}] \\
&\quad = E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(x^{i-1} - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].
\end{aligned}$$

No expediting implies $J_t(A) = L(x^{i-1}) + \min_{z \geq x^{\bar{K}}}\{c(z - x^{\bar{K}}) + E[J_{t+1}(A^+)]\}$, and $J_t(B) = L(x^{i-1} + e) + \min_{z \geq x^{\bar{K}}}\{c(z - x^{\bar{K}}) + E[J_{t+1}(B^+)]\}$. Since the minimizations in the above equations have the same optimal control with respect to regular ordering, $J_t(A) - J_t(B) = L(x^{i-1}) - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(x^{i-1} - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)]$.

Because $y^*_{i,t} \leq x^{i-1}$, we have $h_{i,t}(x^{i-1}) = 0$ and $h_{i,t}(x^{i-1} + e) = 0$. Therefore,

$$\begin{aligned}
&L(x^{i-1}) - L(x^{i-1} + e) + E[S^0_{M(i,W),t+1} + S^1_{M(i,W),t+1}(x^{i-1} - D) \\
&\quad + S^2_{M(i,W),t+1}(x^{i-1} + e - D)] \\
&= d_i x^{i-1} + L(x^{i-1}) + E[S^1_{M(i,W),t+1}(x^{i-1} - D)] - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)] \\
&= f_{i,t}(x^{i-1}) - d_i x^{i-1} - L(x^{i-1} + e) + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)] \\
&= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)] \\
&= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)].
\end{aligned}$$

Case 2 If $x^{i-1} < y^*_{i,t} \leq x^{i-1} + e$, then expediting $y^*_{i,t} - x^{i-1}$ from installation i is optimal in state A and no expediting is optimal in state B. We have

$$\begin{aligned}
A_j^+ = (&y^*_{i,t} - D, \bar{0}^{j-1}, x^{N(j,w)} - y^*_{i,t}, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, \\
&x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)}), \\
B_j^+ = (&x^{i-1} + e - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)} - e, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, \\
&x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})
\end{aligned}$$

for $j > 0$, and

$$\begin{aligned}
A_0^+ = B_0^+ = (&x^{N(0,w)} - D, x^{N(1,w)} - x^{N(0,w)}, \cdots, x^{N(M(\bar{K},w)-1,w)} - x^{N(M(\bar{K},w)-2,w)}, \\
&x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})
\end{aligned}$$

for $j = 0$. From the induction hypothesis, $J_{t+1}(A_j^+) - J_{t+1}(B_j^+) = S_{j,t+1}^0 + S_{j,t+1}^1(y_{i,t}^* - D) + S_{j,t+1}^2(x^{i-1} + e - D)$ for $j \geq 0$, and therefore

$$
\begin{aligned}
&E[J_{t+1}(A^+) - J_{t+1}(B^+)] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A^+) - J_{t+1}(B^+)|M(i,W) = j\}] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A_j^+) - J_{t+1}(B_j^+)\}] \\
&\quad = E[\sum_j Prob[M(i,W) = j]\{S_{j,t+1}^0 + S_{j,t+1}^1(y_{i,t}^* - D) + S_{j,t+1}^2(x^{i-1} + e - D)\}] \\
&\quad = E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(y_{i,t}^* - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].
\end{aligned}
$$

We have $J_t(A) = d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} + \min_{z \geq x^{\bar{K}}}\{c(z - x^{\bar{K}}) + E[J_{t+1}(A^+)]\}$, and $J_t(B) = L(x^{i-1} + e) + \min_{z \geq x^{\bar{K}}}\{c(z - x^{\bar{K}}) + E[J_{t+1}(B^+)]\}$. Therefore, $J_t(A) - J_t(B) = d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(y_{i,t}^* - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)]$. Because $x^{i-1} < y_{i,t}^* \leq x^{i-1} + e$, we have $g_{i,t}(x^{i-1}) = 0$ and $h_{i,t}(x^{i-1} + e) = 0$, and

$$
\begin{aligned}
&d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\qquad + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(y_{i,t}^* - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)]. \\
&\quad = f_{i,t}(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&\quad = a_{i,t} - d_i x^{i-1} - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&\quad = a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,T}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\qquad + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].
\end{aligned}
$$

Case 3 If $y_{i,t}^* > x^{i-1} + e$, then we expedite $\min(v_i, y_{i,t}^* - x^{i-1})$ from installation i in state A and $\min(v_i, y_{i,t}^* - x^{i-1} - e)$ from installation i in state B. Therefore, in the next time period, states A^+ and B^+ are the same and the only cost difference between $J_t(A)$ and $J_t(B)$ is $d_i e = d_i(x^{i-1} + e) - d_i x^{i-1}$. Thus, $J_t(A) - J_t(B) = d_i(x^{i-1} + e) - d_i x^{i-1}$.

Because $y_{i,t}^* > x^{i-1} + e$, we have $g_{i,t}(x^{i-1}) = 0$ and $g_{i,t}(x^{i-1} + e) = 0$. Note that $S_{j,t+1}^0 +$

$S^1_{j,t+1}(x) + S^2_{j,t+1}(x) = 0$, or $S^1_{j,t+1}(x) = -S^0_{j,t+1} - S^2_{j,t+1}(x)$. We conclude that

$$\begin{aligned}
&d_i(x^{i-1}+e) - d_i x^{i-1} \\
&\quad = a_{i,t} - a_{i,t} + g_{i,t}(x^{i-1}) - g_{i,t}(x^{i-1}+e) + h_{i,t}(x^{i-1}+e) - h_{i,t}(x^{i-1}+e) \\
&\qquad + d_i(x^{i-1}+e) - d_i x^{i-1} \\
&\quad = a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}+e) - f_{i,t}(x^{i-1}+e) + d_i(x^{i-1}+e) - d_i x^{i-1} \\
&\quad = a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}+e) - d_i(x^{i-1}+e) - L(x^{i-1}+e) \\
&\qquad - E[S^1_{M(i,W),t+1}(x^{i-1}+e-D)] + d_i(x^{i-1}+e) - d_i x^{i-1} \\
&\quad = a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}+e) - d_i x^{i-1} - L(x^{i-1}+e) \\
&\qquad - E[S^1_{M(i,W),t+1}(x^{i-1}+e-D)] \\
&\quad = a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}+e) - d_i x^{i-1} - L(x^{i-1}+e) \\
&\qquad + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1}+e-D)].
\end{aligned}$$

Finally, Cases 1, 2, and 3 can be summarized as

$$\begin{aligned}
&J_t(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}}) - J_t(x^{i-1}+e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}}) \\
&= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}+e) - d_i x^{i-1} - L(x^{i-1}+e) \\
&\quad + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1}+e-D)] \\
&= S^0_{i,t} + S^1_{i,t}(x^{i-1}) + S^2_{i,t}(x^{i-1}+e).
\end{aligned}$$

Therefore, part (c) is proved, and this completes the induction step of the entire proof. □

Glossary of notation

- $|\cdot|$: the number of elements in a set
- $A \setminus B$: set difference $\{x : x \in A \text{ and } x \notin B\}$
- $\bar{K}$: installation index number of the supplier; there are total $\bar{K}+1$ installation in the system including the supplier, intermediate installations, and the manufacturer

- T: the planning horizon
- v_i: the amount of inventory at installation i for $0 \le i \le \bar{K}$ and
- $(v_0, v_1, v_2, \cdots, v_{\bar{K}})$: the state vector. Based on the current state of the system from the
- d_i: per unit expediting cost d_i from installation i to the manufacturer
- $L(x)$: $E[r(x - D)]$, where $r(\cdot)$ is a convex holding/backlogging cost function,
- x^i: echelon stock; the sum of the inventory from installation 0 to installation i: $x^i = \sum_{j=0}^{i} v_j$,
- $\bar{0}^i$: a vector containing i zeros, or $(0, 0, \cdots, 0)$
- u: regular ordering amount
- e_i: expediting amount from installation i
- w: a movement pattern or a vector of movement patterns with an appropriate length
- $\mathbf{W}$: set of of all movement patterns, i.e., $\mathbf{W} = \{w_1, w_2, w_3, \cdots\}$
- W: an exogenous random variable or a vector with an appropriate length with known distribution that selects a movement pattern in $\mathbf{W}$
- $M(i, w)$: a function that represents the destination of regular movement originally at installation i based on the realized movement pattern w
- $N(j, w)$: maximum indexed installation among the installations that delivers its orders to installations indexed less than or equal to j depending on the realized movement pattern w; $N(j, W)$ is corresponding random variable
- $Q^i(W)$ denote $N(M(\bar{K}, W) - i, W)$. $Q^i(W)$ is a random variable representing the maximum indexed installation among the installations that the regular movement will deliver its orders to i-th downstream installation of the installation to where the supplier ($\bar{K}$) will deliver its orders by regular movement

- $M^n(i,W)$: the n-period random movement function that represents the location (an installation) after n regular movements of the outstanding orders at installation i, where W is an n-dimensional random vector; The dimension of W can always be inferred from the underlying usage. Formally, $M^1(i,W) = M(i,W)$ where W is a random variable, and $M^n(i,W) = M(M^{n-1}(i,W'),W'')$ where $W = (W',W'')$ is a vector of length n, W' is a vector of length $n-1$, and W'' is a random variable.
- NS: the next state
- J_t: the cost-to-go at the beginning of time period t
- $J_t^j(\cdot)$: the optimal cost-to-go that can be achieved by a restricted control space, in which expediting from installations $j+1, j+2, \cdots, \bar{K}$ in time period t is not allowed

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