Attractiveness-Based Airline Network Models with Embedded Spill and Recapture

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Abstract

In airline revenue management, the modeling of the spill and recapture effects is essential for an accurate estimation of the passenger flow and the revenue in a flight network. However, as most current approaches toward spill and recapture involve either non-linearity or a tremendous amount of additional variables, it is computationally intractable to apply those techniques to the classical network design and capacity planning models. We present a new framework that incorporates the spill and recapture effects, where the spill from an itinerary is recaptured by other itineraries based on their attractiveness. The presented framework distributes the accepted demand of an itinerary according to the currently available itineraries, without adding extra variables for the recaptured spill. Due to its compactness, we integrate the framework with the classical capacity planning and network design models. Our preliminary computational study shows an increase of 1.07% in profitability and a better utilization of the network capacity, on a medium-size North American airline provided by Sabre Airline Solutions. Our investigation leads to a holistic model that tackles the network design and capacity planning simultaneously with an accurate modeling of the spill and recapture effects. Furthermore, the presented framework for spill and recapture is versatile and can be easily applied to other disciplines such as the hospitality industry and product line design (PLD) problems.

Key words: spill and recapture; attractiveness; airline network design; airline capacity planning; fleet assignment; frequency and market selection; departure and block time selection

1 Introduction

Fleeting, the assignment of specific equipment types to all scheduled flights, is an essential part of network and capacity planning. Historically, optimization models for network planning largely focus on supply management, ensuring feasibility and cost-efficiency of a fleeting solution. However, due to increasing fuel prices, high labor cost and fickle customers, airlines' decisions regarding the composition of their flight network are progressively sensitive to the interaction between the supply and demand and thus require models that more closely approximate reality. On the demand management side, estimating the revenue from a network has long been one of the most critical areas in revenue management. As revenue estimation techniques are getting increasingly accurate, it is necessary to revise traditional network planning models with a more detailed representation of the supply and demand interaction. Many current network planning models ignore the dynamics in passenger choices with respect to different itineraries offered. Nevertheless, the changes in customers' preference under different scenarios can lead to a significant deviation from simplistic revenue estimation based on an unconstrainted demand forecast, because the excessive demand on a flight due to its capacity (spill) or due to a no-longer available itinerary would turn to other available itineraries (recapture). In this paper we discuss how to equip a broad selection of network planning models with a new spill and recapture paradigm in order to more precisely approximate reality.

In the airline industry, estimating the spill and recapture effects originated in revenue management, as incorporating correct demand information with various optimization models greatly improves the quality of demand management decisions. For instance, making upsell decisions, so that a passenger might purchase more expensive itineraries if cheaper options are not available, requires an accurate estimation of the probability that a passenger would turn to any open itinerary in the market. Since a flight network has a limited capacity, the excessive demand on an itinerary will be either lost or recaptured by other itineraries with free space in the same market. Modeling passenger choices without considering the spill and recapture effects dramatically underestimates the traffic in the flight network as well as the revenue captured.

Past approaches toward modeling spill and recapture include applying the discrete choice analysis by Ben-Akiva et al. 1985 [3] to select the best set of products offered. Gallego et al. 2004 [7] applied such a technique in revenue management and formulated the choice-based deterministic linear programming. Although the model has the ability to deal with spill and recapture, it introduces an exponential number of variables, because it needs to find the bestsubset of the itineraries to be offered. On the other hand, Rabetanety 2006 [12] derived an integrated model for airline schedule generation, also based on the discrete choice modeling concept, with mixed integer fractional programming methods. Even though the number of variables are greatly reduced in the Rabetanety's model, the problem becomes much harder to solve because of its nonlinear objective function and constraints.

Another notable effort in modeling spill and recapture lies in the work of formulating the itinerary-based fleet assignment model (FAM) by Barnhart et al. 2002 [2], where the rate of recapture is determined by the quantitative share index (QSI), the carrier's estimation of the relative attractiveness of an itinerary. One of the problems with this approach is that the rate of recapture stays constant regardless of the composition of the set of the itineraries available in the market, but in reality passengers constantly change their preference toward itineraries based on the options available in the market. Furthermore, because the traffic on an itinerary is represented by the sum of the demand recaptured by itself and other itineraries in the same market, the number of variables is essentially the number of all itineraries squared, significantly more than the number of variables in the original model that does not consider spill and recapture. Since in an airline network the number of itineraries is substantial, such characteristics of the model can make it impractical to solve.

Lately Dumas et al. 2008 [5] incorporated the spill and recapture effects in their model that approximates the expected passenger flow on the network, to improve the estimation of the revenue and cost of the given flight network. With a similar approach to QSI, spill coefficients are used to determine how much of the demand on an itinerary turns to other itineraries. Since the spill coefficients stay constant regardless of the itineraries available, the framework does not capture the dynamic reallocation of passengers with different sets of itineraries offered. In 2007, Budhiraja [4] presented a different methodology in modeling the spill and recapture effects by penalizing any spill or unmet capacity in the objective function. However, with the assumption of constant recapture rates and nonlinearity in both the objective function and constraints, the model can be difficult to solve and it also does not reflect the impact of the variation in the flight network on passenger choices.

The presented framework of spill and recapture is based on the attractiveness of the itineraries composed of not only the host carrier's network, but also the whole market. The attractiveness of each itinerary simultaneously determines the ratio at which the spilled passengers from an itinerary are recaptured by other itineraries. This ratio also automatically adjusts itself according to the itineraries available in the market. The presented framework has the advantage over other models on spill and recapture previously discussed, since it does not complicate the model with an excessive amount of variables and nonlinearity. Moreover the presented framework provides an intuitive and natural way to estimate the demand and spill split among itineraries.

FAM, traditionally a critical area in the airline planning process, is integrated with our new spill and recapture framework as the core model, since it closely interacts with both the supply and demand. Initially, FAM was modeled with independent leg demand in Abara 1989 [1]. However this model fails to recognize the network effect (Farkas 1995 [6]), the correlation among leg demands, because airlines' products are itineraries instead of flights and a large percentage of itineraries are multi-leg. In 1999, Jacobs et al. [10] proposed a formulation of FAM with itinerary-level demand to capture the network effect. Later the computational studies of Barnhart et al. 2002 [2] suggest that incorporating the network and recapture effects can significantly impact the revenue (with a range from around $34 million to $153 million in a year in their experiments). With such improvements in FAM modeling, we build our models based on the itinerary-based FAM. In addition, since both leg and itinerary demands are stochastic in nature, incorporating stochastic demands into FAM gives a better expected revenue, as shown in Jacobs et al. 1999

[10]. Nevertheless, integrating stochasticity in demand with the spill and recapture effects still remains an open topic. Therefore we treat the demand forecast as deterministic.

The purpose of this paper is to integrate the new spill and recapture framework with various network planning models, using the itinerary-based FAM as the core model. The demand management side of our models is related to the deterministic sales-based linear programming (SBLP) formulation by Gallego et al. 2010 [8], also based on the attractiveness of itineraries. Our preliminary computational experiments show that the optimal fleet assignment given by our reformulated FAM increases the profit by 7.58%, compared to the optimal solution from the traditional itinerary-based FAM, for a mid-size North American airline. In addition, the profit to revenue ratio increases by 1.07%, from 9.33% using the traditional FAM solution, to 10.40%. The improvement in profitability comes from a 0.34% decrease in revenue and a 1.46% decrease in cost, where the model maximizes the utilization of the network capacity with fewer aircraft in use. When we gradually lift the demand level, our model increasingly captures more revenue than the traditional model, whereas the cost in the solution increases only slightly more than the one from the traditional model. Such increase in profit benefits from the consideration of the spill and recapture effects.

The reminder of paper is organized as follows. First we give a conceptual example of the mechanism behind spill and recapture. Section 2 states the passenger mix problem incorporated with our spill and recapture framework. In section 3 we present different network planning models with spill and recapture. We formulate the itinerary-based fleet assignment model with spill and recapture. We also include some preliminary computational results and analyses of the proposed model versus the traditional model. This section also discusses various aspects of network design models such as the selection of markets, frequency, departure and block times. We present these aspects in a modularized way so that the model can be tailored to integrate only a subset of those aspects for flexibility.

2 Modeling Passenger Choices

To model the mechanism of spill and recapture, it is necessary to estimate the attractiveness of each itinerary in all markets. The attractiveness of an itinerary is usually quantified by the exponential utility e^U, where U is the linear combination of an itinerary's attributes such as its departure time, number of stops, total duration, the type of aircraft used, the operating airline's presence at the point of origin, and other influences on customers' preference.

The concept of our spill and recapture framework is derived from the classical multinomial logit model (MNL) introduced by Ben-Akiva et al. 1985 [3], where the probability that a customer chooses a product is determined by the ratio of the product's attractiveness to the total attractiveness of all products. This is equivalent to the probability of selecting the itinerary proportional to the highest utility. In the context of spill and recapture, this probability is automatically adjusted by considering only itineraries that are available. As demonstrated by Gallego et al. 2010 [8], the presented framework of spill and recapture differs from the classical MNL because the probability is estimated in a less optimistic way. Our model also considers the possibility that the spilled passengers will be lost or turn to the itineraries offered by other airlines. It derives the expected market share of each itinerary as the product of the market share times the unconstrained demand of the market where the itinerary resides. The market share of an itinerary is on the one hand a decision variable and on the other hand proportional to its attractiveness. Next we show an example revealing how the attractiveness of different itineraries splits the market demand among itineraries in the uncapacitated case.

Consider the following scenario of market IND $\rightarrow$ DSM (from Indianapolis to Des Moines) with estimated demand of 200 passengers. We assume there are three itineraries from the host airline: A, B and C.

Figure 1 shows a conceptual example listing all of the itineraries from IND to DSM. Apart from the set of the itineraries from other airlines that is shown as a cloud, itinerary A, B and C are from the host airline and are disjoint, since itinerary A is a non-stop itinerary and itineraries B and C have one stop at different stations. Since itinerary A is non-stop, it is naturally more attractive than one-stop itineraries B and C, and thus its attractiveness is higher, arbitrarily set to 5 for illustrative purpose (itinerary A is probably the most expensive, which drives its attractiveness lower but we nevertheless assume that it is the most attractive). As the departure time and duration of itineraries also affect customers' preference, itinerary C is less attractive than itinerary B with the first leg in itinerary C being a red-eye flight. Therefore the attractiveness of itinerary B and C is estimated to be 3 and 2, respectively.

Initially the itineraries are all uncapacitated, and the demand for each itinerary sums to the unconstrained total demand of the market from IND to DSM, *Table 1*. Since it is reasonable to expect more passengers on the more attrac-

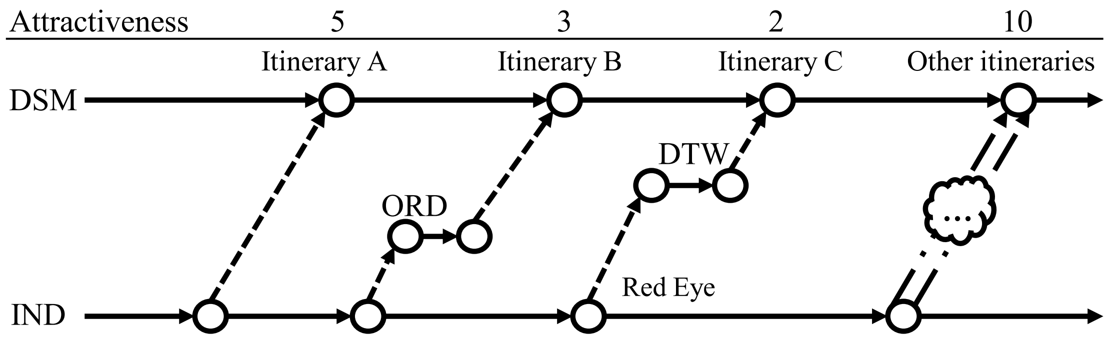

Figure 1: All itineraries from IND to DSM

tive itineraries, passengers are distributed proportionally to the attractiveness of each itinerary. With 200 passengers in this market, the difference between two itineraries' attractiveness naturally quantifies how many additional passengers would travel on the more attractive itinerary. Thus, according to *Table 1*, itinerary A receives 50 passengers, more than 30 passengers on itinerary B and 20 passengers on itinerary C, but fewer than 100 passengers on other itineraries combined. With this concept, next we present the mathematical formulation of the passenger mix model with spill and recapture.

	Itinerary A	Itinerary B	Itinerary C	Other itineraries
Attractiveness	5	3	2	10
Initial capacity	∞	∞	∞	∞
Market share	$\frac{5}{5+3+2+10} = \frac{1}{4}$	$\frac{3}{5+3+2+10} = \frac{3}{20}$	$\frac{2}{5+3+2+10} = \frac{1}{10}$	$\frac{10}{5+3+2+10} = \frac{1}{2}$
Demand	$\frac{1}{4} \times 200 = \mathbf{50}$	$\frac{3}{20} \times 200 = \mathbf{30}$	$\frac{1}{10} \times 200 = \mathbf{20}$	$\frac{1}{2} \times 200 = \mathbf{100}$

Table 1: An example of market IND-DSM in the uncapacitated case

2.1 Passenger Mix Model with Spill and Recapture: SBLP

Glover et al. 1982 [9] present the classic passenger mix model (PMM), where the objective is to find the optimal control of available itineraries, subject to both the limited capacity of the flights in an itinerary and the demand estimated for each itinerary. The passenger mix model maximizes the total revenue. Here for simplicity we assume that each itinerary is really a combination of the fare class and the itinerary as a sequence of flights. A fare class represents a particular fare for a trip, composed of a set of connecting flights. Each trip may have different fares due to different services or cabin classes. Therefore the demand has to be forecasted not only for a trip overall, but also for all of its fare classes. In the classical PMM, the accepted demand of an itinerary has to be less than or equal to the forecasted demand of the itinerary. More importantly, the total accepted demand of all of the itineraries that include a flight must be less than or equal to the number of seats available on the flight. However, with this formulation of the constraints in the classical PMM, the spills from an itinerary are considered lost, which usually leads to an underestimation of the revenue.

Gallego et al. 2010 [8] revise PMM to take into account the spill and recapture effects, based on the attractiveness of itineraries, also named sales-based linear programming (SBLP). We first present the mathematical formulation of SBLP and then illustrate the key concept of the model with an example. In the following notation, the host airline is the airline for which we maximize the revenue, and thus the other airlines refer to all of the airlines that compete with the host airline. Each market is an origin-destination pair, including all of the itineraries that begin from the market's origin and end at the market's destination, regardless of their number of stops. Hence the market share for an itinerary is just the ratio of how the itinerary splits the market demand while competing with other itineraries in the same market.

We consider the following sets

HA	the label of the host airline
OA	set of other airlines
M	set of all markets, indexed by m
L	set of all legs, indexed by l
$I_m^{HA}(l)$	set of all itineraries of the host airline, in market m, including leg l, $m \in M$, $l \in L$
$I_m^{OA}(l)$	set of all itineraries of the other airlines, in market m, including leg l, $m \in M$, $l \in L$,

and decision variables

s_i	market share taken by itinerary i in market m, $i \in I_m^{HA}$, $m \in M$
s_0^m	market share taken by all the itineraries of other airlines in market m, $m \in M$.

We also need parameters

Dem_m	total demand for market m, $m \in M$
$fare_i$	fare associated with itinerary i, $i \in I_m^{HA}$, $m \in M$
Cap_l	capacity of the equipment type assigned to leg l, $l \in L$
A_i	$= e^{U^i}$, attractiveness of itinerary i, where U^i is the utility of itinerary i, $i \in I_m^{HA}$, $m \in M$
A_0^m	$= \sum_{i \in I_m^{OA}} e^{U^i}$, attractiveness of all itineraries from other airlines, where U^i is the utility of itinerary i, $i \in I_m^{HA}$, $m \in M$.

The formulation of the passenger mix model with spill and recapture (RM-SBLP) reads:

maximize
$$\sum_{m \in M} (Dem_m \sum_{i \in I_m^{HA}} fare_i s_i) \tag{1}$$

subject to
$$\sum_{m \in M} (Dem_m \sum_{i \in I_m^{HA}(l)} s_i) \leq Cap_l \qquad \forall l \in L \tag{2}$$
$$s_0^m + \sum_{i \in I_m^{HA}} s_i = 1 \qquad \forall m \in M \tag{3}$$
$$A_0^m s_i \leq A_i s_0^m \qquad \forall i \in I_m^{HA}, m \in M \tag{4}$$
$$s \geq 0 \tag{5}$$

The objective function is the revenue captured from the demand accepted for all the itineraries from the host airline. Note that in (1), $Dem_m s_i$ is the accepted demand for itinerary i. Constraint (2) imposes that the total demand captured on all itineraries including leg l must not exceed the capacity of the aircraft assigned to leg l.

For spill and recapture, the set of variables s denotes the demand acceptance level of the competing itineraries in the market and forms the spill-recapturing constraints (3) and (4): market demand constraint (3) imposes that the total accepted demand of all itineraries in a particular market must be equal to the unconstrained demand estimation of that market; demand-splitting constraint (4) ensures that the spilled demand is recaptured in such a way that the probability of recapture is proportional to the attractiveness of the itinerary.

Note that if we replace constraints (3) and (4) with the following constraint (6), the model is the same as the classical PMM with traffic variable $t_i = Dem_m s_i$ for each itinerary $i \in I_m^{HA}$ and each market $m \in M$, where $DemEst_i$ is the expected value of the unconstrained demand for itinerary i.

$$Dem_m s_i \leq DemEst_i \qquad \forall i \in I_m^{HA}, m \in M \tag{6}$$

Note that we can reformulate constraint (4) as $s_i \leq (A_i/A_0^m) \cdot s_0^m$. In the uncapacitated case, the demand for a market should be split strictly according to the attractiveness of all the itineraries in the market. However with the capacity limit in constraint (2), some demand on the host itineraries might be lost. The inequality in constraint (4) hence gives room for the passengers leaving either the market or the host airline's itineraries. On the other hand, constraint (4) still stays feasible when an itinerary is closed ($s_i = 0$ and $0 \leq (A_i/A_0^m) \cdot s_0^m$), and constraint (3) guarantees that the market shares always sum up to one. Since the market share of the other airlines' itineraries, s_0^m, is always present in the model, the ratio A_i/A_0^m directly gives the upper limit of an itinerary's demand acceptance level without considering whether other itineraries are open or closed. Therefore the market shares for the host airline's itineraries are essentially adjusted automatically according to the itineraries left open.

Additionally, this formulation relieves the model from adding extra variables for the spilled passengers who leave the market. With the assumption of the unlimited capacity on other airlines, the variable s_0^m for market m, representing the market share of the itineraries from other airlines, takes any passenger spilled from the itineraries on the host airline, regardless of whether a passenger turns to the other airlines or is actually leaving the market. With such a formulation, SBLP provides an optimal control of the ticket sale for each itinerary on the host airline. Next we will show an example built upon *Figure 1* and *Table 1*.

	Itinerary A	Itinerary B	Itinerary C	Other itineraries
Attractiveness	5	3	2	10
Fare	\$100	\$200	\$300	–
Uncapacitated demand	50	30	20	100
Traditional demand	**35**	**30**	**20**	115
Traditional revenue	**\$3,500**	**\$6,000**	**\$6,000**	–
Leg capacity		IND-ORD: **40**	IND-DTW: 40	∞
	IND-DSM: **35**	ORD-DSM: 50	DTW-DSM: **30**	∞
Spill generated	$[50-35]^+ = 15$	$[30-40]^+ = 0$	$[20-30]^+ = 0$	–
Recapture rate	–	$\frac{3}{3+2+10} = \frac{1}{5}$	$\frac{2}{3+2+10} = \frac{2}{15}$	$\frac{10}{3+2+10} = \frac{2}{3}$
Spill recaptured	–	$\frac{1}{5} \times 15 = \mathbf{3}$	$\frac{2}{15} \times 15 = \mathbf{2}$	$\frac{2}{3} \times 15 = 10$
New demand	$50 - 15 + 0 = \mathbf{35}$	$30 - 0 + 3 = \mathbf{33}$	$20 - 0 + 2 = \mathbf{22}$	$100 - 0 + 10 = 110$
New revenue	**\$3,500**	**\$6,600**	**\$6,600**	–

Table 2: An example of spill and recapture for market IND-DSM with SBLP

Table 2 explains how the capacity on a flight invokes spill and recapture. The accepted demand of an itinerary is limited by the minimal capacity on one of the legs included in the itinerary. When this bottleneck capacity of the itinerary is lower than its uncapacitated market demand, a spill is generated and the spilled passengers will either turn to other itineraries available in the market or leave the market. In the example in *Table 2*, if the host carrier limits the capacity of itinerary A to 35, then the extra 15 passengers spilled from itinerary A in the uncapacitated case will go to other itineraries available in the market. The recapture rates of the spill to other itineraries, different from the market shares in *Table 1*, depend on the attractiveness of the itineraries left open. Contrary to a constant recapture rate, the formulation of constraints (2) and (3) ensures that each recapture rate automatically adjusts itself based on the viable options left, since the denominator of the recapture rate equals to the sum of the attractiveness of the open itineraries. If any excessive spill is further generated, it will be recaptured by the remaining itineraries with vacancies.

Compared to the traditional passenger mix model, SBLP has a more accurate representation of the revenue so that the solution identifies the most valuable itineraries in reality. For instance, in *Table 2* the actual benefit of offering itinerary A is more than \$3,500, with an additional revenue of \$1,200 from the spill toward itineraries B and C.

3 Network Models with Spill and Recapture

In general, two kinds of spill are generated when the composition of a flight network changes according to a fleeting decision - capacity spill and network spill. The capacity spill is the spill due to a capacity restriction imposed by different equipment types, whereas the network spill is created due to an elimination of a particular itinerary in the schedule design phase. Compared to the capacity spill, the network spill is more prominent and hence requires more accurate modeling to the spill and recapture effects, since all of the passengers who would have originally chosen a closed itinerary are spilled to other open itineraries. In the following discussion, we first introduce FAM with spill and recapture in order to address the capacity spill effect, and then various network design modules are integrated with our new FAM model to incorporate the network spill effect.

3.1 Fleet Assignment with Spill and Recapture: FAM-SBLP

An airline's fleeting decision is to make an optimal assignment of different equipment types to a set of flights with a fixed schedule. Originally, Abara 1989 [1] assumed independent leg-based demand and proposed the leg-based FAM, which minimizes the cost of assigning aircraft to flights subject to a set of network constraints. Although computationally hard, with the aid of various heuristics and the fast growing computing power, the leg-based FAM can be solved fairly easily nowadays.

The leg-based FAM assumes no dependency among the demand of different flights. However, since a significant portion of customers use multi-leg itineraries, leg demand is indeed dependent ("the network effects"). Itinerary-based FAM was proposed in order to capture network effects by incorporating itinerary-level demand estimation in Jacobs et al. 1999 [10]. As this approach significantly increases the problem size, itinerary-based FAM has many computational issues and takes hours to solve even with heuristics.

Since traditional itinerary-based FAM does not consider the spill of demand due to limited capacity, it fails to recognize the opportunity to gain revenue by recapturing spill. The solution is likely suboptimal and overly conservative because larger than needed aircraft are used to capture valuable demand. Several publications propose different methodologies to model the effect of spill and recapture, such as the QSI model used by Barnhart et al. 2002 [2] to calculate the recapture rate at which the spill from an itinerary would be accepted by other itineraries.

Due to the steadily rising demand for air transportation, incorporation of the effect of spill and recapture in operation management is critical to design a cost-effective and profitable capacity assignment. Using the framework of SBLP, FAM as a decision support system also takes passenger choices into account for a more accurate representation of the passenger flow. The following formulation aims at assigning a market share to each itinerary based on its attractiveness, so that the interaction between fleeting and demand management decisions can utilize the capacity efficiently to achieve maximum profit.

Let F be the set of all fleets, indexed by f, $cost_f$ the operations cost of equipment type f, $f \in F$, and let Cap_f be the capacity of equipment type f, $f \in F$.

In addition, we define a new set of decision variables:

$$x_{lf} = \begin{cases} 1 & \text{if fleet } f \text{ is assigned to flight } l, f \in F, l \in L, \\ 0 & \text{otherwise.} \end{cases}$$

The fleet assignment model with spill and recapture (FAM-SBLP) is formulated as follows.

$$\text{maximize} \quad \sum_{m \in M} (Dem_m \sum_{i \in I_m^{HA}} fare_i s_i) - \sum_{l \in L} \sum_{f \in F} cost_f x_{lf} \tag{1a}$$

subject to:

$$\text{Standard flow balance constraints} \quad (7)$$

$$\text{Standard aircraft count constraints} \quad (8)$$

$$\sum_{f \in F} x_{lf} = 1 \qquad l \in L \quad (9)$$

$$\sum_{m \in M} (Dem_m \sum_{i \in I_m^{HA}(l)} s_i) \leq \sum_{f \in F} Cap_f x_{lf} \qquad l \in L \quad \text{(2a)}$$

$$(3), (4), \text{and } (5)$$

$$x \in \{0, 1\} \quad (10)$$

The objective of FAM-SBLP (1a) subtracts the cost of assigning specific fleets to flights from the itinerary-level revenue estimation based on the demand captured in SBLP objective (1), hence maximizing the operating profit.

The three sets of network constraints (7)-(9) are standard inventory and assignment constraints in the traditional FAM (Abara 1989 [1]), ensuring the feasibility of any fleet assignment solution. Constraints (7) and (8) are standard network flow balance and aircraft count constraints. Flight coverage constraint (9) requires that each flight must be assigned to exactly one equipment type.

In addition to these constraints, we replace the right-hand side of the demand management constraint (2) with the right-hand side of constraint (2a), so that the capacity of a flight depends on the equipment type assigned to the flight in the solution. This allows the interaction between demand and supply to give not only an optimal fleet assignment, but also an optimal control policy for demand acceptance level.

In *Table 3*, which builds on the example in *Figure 1* and *Table 2*, originally the equipment type assigned to the leg from IND to ORD has a capacity of 40 passengers. However, changing the equipment type to a smaller fleet with a capacity of 27 passengers significantly reduces the cost of fleeting from \$6,000 to \$5,000 due to potential savings in fuel and crew cost. With spill and recapture we can evaluate different alternatives more accurately and thus achieve better profit. For instance, if we assign the smaller fleet to the leg from IND to ORD, 6 passengers are spilled from itinerary B. Since itinerary A and B already met their capacity limits, the spilled passengers can only turn to itinerary C or other itineraries. According to the attractiveness of itinerary B and other itineraries, 1 passenger is recaptured by itinerary C but the rest of them turn to other itineraries.

	Itinerary A	Itinerary B	Itinerary C	Other itineraries
Attractiveness	5	3	2	10
Fare	\$100	\$200	\$300	–
Initial capacity	35	40	30	∞
Initial demand	**35**	**33**	**22**	110
Initial revenue	**\$3,500**	**\$6,600**	**\$6,600**	–
Initial cost	–	(IND-DEN) **\$6,000**	–	–
New leg capacity	IND-DSM: 35	IND-ORD: **27** ORD-DSM: 50	IND-DTW: 40 DTW-DSM: 30	∞ ∞
Spill generated	–	$[33-27]^+ = 6$	–	–
Recapture rate	–	–	$\frac{2}{2+10} = \frac{1}{6}$	$\frac{10}{2+10} = \frac{5}{10}$
Spill recaptured	–	–	$\frac{1}{6} \times 6 = \mathbf{1}$	$\frac{5}{6} \times 6 = 6$
New demand	**35**	$33 - 6 + 0 = \mathbf{27}$	$22 - 0 + 1 = \mathbf{23}$	$110 - 0 + 5 = 115$
New revenue	**\$3,500**	**\$5,400**	**\$6,900**	–
New cost	–	(IND-DEN) **\$5,000**	–	–

Table 3: An example of spill and recapture for market IND-DSM with FAM-SBLP

With the more accurately-evaluated demand acceptance level, the net benefit of this assignment is \$100 in profit, \$1,000 for cost-saving plus \$300 for the spill recaptured, minus \$1,200 for the loss of revenue. However, without spill and recapture, the impact of this assignment is negative because the traditional model does not consider the $300 gained from the recaptured passenger on itinerary C. FAM-SBLP weighs the benefit of cost-saving versus the loss of demand considering the spill that can be recaptured. As fuel prices increase steadily nowadays, a small recapture of the spill might substantiate the cost-saving of assigning smaller aircraft to non-critical flights.

Next we present computational results of FAM-SBLP and a comparison with the traditional itinerary-based FAM.

3.1.1 Computational Results for FAM-SBLP

Table 4 shows the computational results of FAM-SBLP for a mid-size South American airline with a traditional hub and spoke flight network, operating both domestically and internationally. The airline has over 2,000 flights weekly and millions of itineraries. The range of capacity on its aircraft is between 60 and 250 seats. FAM-SBLP, solved at the weekly level, is tested by estimating the unconstrained demand at different levels. In order to compare the solutions from FAM-SBLP and itinerary-based FAM, we check the quality of itinerary-based FAM solutions under the FAM-SBLP model. For more reliable results, different scenarios are generated by randomly adjusting the fares based on the original values. These scenarios are further tested with various demand levels to conclude how demand influences the fleeting decisions with spill and recapture. The results in *Table 4* and *Table 5* are the average values across all scenarios associated with each demand level.

Demand Level	Profit	Revenue	Cost	Load Factor	Total Unused Aircraft
70%	4.24%	-0.27%	-0.81%	-0.28%	10.63%
100%	7.58%	-0.34%	-1.46%	-0.73%	45.02%
150%	2.04%	0.63%	-0.19%	1.23%	20.00%
230%	1.19%	0.92%	0.69%	2.34%	-3.55%

Table 4: Relative improvements of key performance indicators of FAM-SBLP over itinerary-based FAM

The optimal fleet assignment given by FAM-SBLP greatly increases the profit (by 7.58% for the original demand level, with 0.34% decrease in revenue and 1.46% decrease in cost), compared to an optimal solution from itinerary-based FAM. Profitability, defined as the ratio of profit to revenue, increases by 1.07%, from 9.33% under the traditional FAM solution to 10.40% under the solution in the new model. This increase results from a significant saving in cost as the FAM-SBLP solutions use fewer aircraft on average. When the demand level is increased, the improvement is mainly due to the additional revenue captured by considering the spill and recapture effects in FAM-SBLP. Especially when unconstrained demand is high, FAM-SBLP yields significant revenue increments.

Table 5 summarizes the changes in the fleet assignments of FAM-SBLP over itinerary-based FAM. The number of different fleet assignments in FAM-SBLP in comparison with itinerary-based FAM increases as the interaction between demand and seat capacity becomes stronger, where exploiting spill and recapture is beneficial. With high demand levels, negligible change occurs in the number of aircraft used, but the reassignment of aircraft is significant to encourage spill to flow into larger aircraft, as indicated by the number of different fleet assignments between FAM-SBLP and itinerary-based FAM. This difference gradually disappears when the two models seek more capacity for even higher demand levels.

Demand Level	Different Flight Assignments (in percentage)	Additional Number of Used Aircraft in Itinerary-based FAM over FAM-SBLP			
		Small	Medium	Medium-Large	Large
70%	6.9%	1.25	-0.25	0.25	2.00
100%	17.0%	3.75	-1.25	2.75	1.25
150%	25.4%	0.00	0.33	0.00	0.00
230%	20.5%	0.00	-0.25	0.00	0.00

Table 5: Difference in the characteristics of FAM-SBLP and Itinerary-based FAM solutions

In general, itinerary-based FAM gives a more conservative solution than FAM-SBLP, because more aircraft are used in the itinerary-based FAM solutions to capture valuable demand that cannot be recaptured as opposed to FAM-SBLP. The number of aircraft numbers are the average numbers across different scenarios.

3.2 Integrated Network Design: ISD-FAM-SBLP

FAM-SBLP can also be revised to facilitate network design. As the basis of a flight network, schedule design is one of the most critical areas in airline operations management. Correctly modeling passenger preference and choices on itineraries can guide the flight network toward critical flights and markets. Considering spill and recapture in schedule design models can greatly improve the robustness of the solution quality because any network spill generated from new or recovered flights can be recaptured. Next we present various models incorporating spill and recapture with different aspects of network design.

3.2.1 Frequency Selection

In network design, it is important to decide on which day of a week a flight should be operated due to the periodic nature of demand. For example, leisure travelers might prefer early afternoon flights during weekends, whereas business travelers tend to like early morning and evening flights on weekdays. The set of flights is split into two subsets, one for the fixed flights that the carrier mandates to be flown and the other one for the rest of the flights including optional flights that can be added and existing flights that can be canceled. The flight network here includes all flights that can be added to the service. For now we assume that possible newly-added flights use only already-served markets.

Let L^F be the set of mandatory legs that are not allowed to be canceled and let L^O be the set of optional legs that are allowed to be added or canceled. Frequency selection can be easily integrated with FAM-SBLP with a few additional constraints:

$$\sum_{f \in F} x_{lf} = 1 \qquad l \in L^F \tag{9a}$$

$$\sum_{f \in F} x_{lf} \leq 1 \qquad l \in L^O \tag{9b}$$

$$\sum_{f \in F} x_{lf} \geq s_i \qquad i \in I_m^{HA}(l), l \in L^O, m \in M. \tag{11}$$

Constraint (9) in FAM-SBLP is split into constraints (9a) and (9b), corresponding to the set of fixed and optional flight. Constraint (9a), similar to (9), imposes that the set of fixed flights must be flown. Constraint (9b) gives the flexibility to make decisions to add or cancel the optional flights. In addition, constraint (11) ensures that an itinerary is closed if any leg of the itinerary is canceled. Next we show an example based on *Figure 1* and *Table 3*, integrating frequency selection with spill and recapture.

In *Figure 2*, the host airline has the option to operate another direct red-eye flight from IND to DSM, A^1, the night before the current day time flight A^0 departs (the horizontal time axis is not up-to-scale). The original flight is mandated but the red-eye flight is optional, denoted by the thin dotted arrow. If the utility function that determines the attractiveness of an itinerary values appropriate departure time more than the number of stops, the optional red-eye flight in *Figure 2* would be less attractive than itinerary B, even though itinerary B has one more stop. Therefore for demonstration purposes, we evaluate itinerary A^1's attractiveness to be 2.4, higher than that of the one-stop red-eye itinerary C, but lower than the day time itinerary B. To make the example easier to follow, we update the original market demand from 200 passengers to 224 passengers and assume that the assigned capacity to the red-eye itinerary A^1 has capacity of 24 passengers, which is lower than the capacity of itinerary A^0, partially because the host carrier does not expect high passenger flow on this flight.

Table 6 starts with the initial scenario that the red-eye flight from IND to DSM is operated. Following the calculation shown in the example in *Table 3*, itinerary A^1's capacity of 24 passengers is filled up, but no capacity spill is generated. In addition to the profit from the case in *Table 3*, the operating profit increases by \$600 from the \$3,600 revenue from the 24 passengers on itinerary A^1, minus the \$3,000 operating cost for the morning flight.

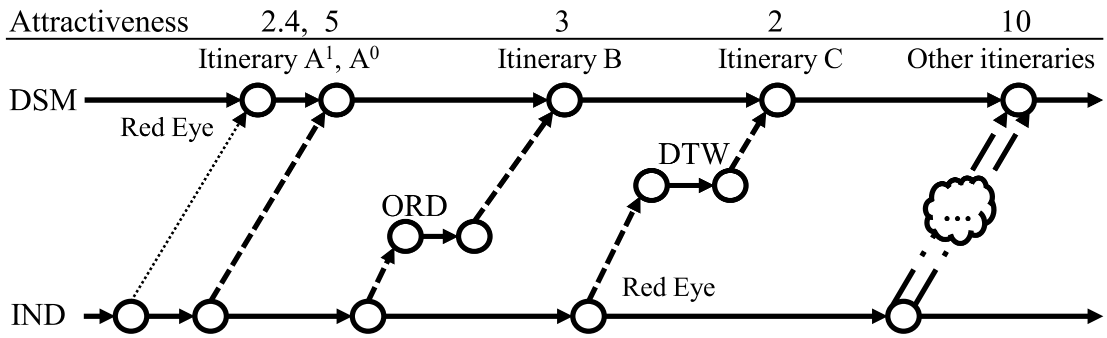

Figure 2: All itineraries from IND to DSM including an optional red-eye itinerary A^1

	Itinerary A^0	Itinerary A^1	Itinerary B	Itinerary C	Other itineraries
Attractiveness	5	2.4	3	2	10
Fare	\$100	\$150	\$200	\$300	–
Initial capacity	35	24	27	30	∞
Initial demand	**35**	**24**	**27**	**23**	115
Initial revenue	**\$3,500**	**\$3,600**	**\$5,400**	**\$6,900**	–
Additional cost	–	**\$3,000**	–	–	–
Operated	Yes	**No**	Yes	Yes	–
Spill generated	–	**24**	–	–	–
Recapture rate	–	–	–	$\frac{2}{2+10} = \frac{1}{6}$	$\frac{10}{2+10} = \frac{5}{10}$
Spill recaptured	–	–	–	$\frac{1}{6} \times 24 = \mathbf{4}$	$\frac{5}{6} \times 24 = 20$
New demand	**35**	**0**	**27**	$23 - 0 + 4 = \mathbf{27}$	$115 - 0 + 20 = 135$
New revenue	**\$3,500**	**\$0**	**\$5,400**	**\$8,100**	–
New cost	–	**\$0**	–	–	–

Table 6: An example of spill and recapture for market IND-DSM with frequency selection

The second part of *Table 6* shows the scenario when the red-eye flight from IND to DSM is not operated. The 24 passengers on itinerary A^1 are spilled to itinerary C and the itineraries from other airlines, since itineraries A^0 and B are already full. With the spill and recapture framework, itinerary C takes 4 passengers from the spill, bringing in an additional net profit of \$1,200 from the recaptured spill. This scenario has a net increment in profit of \$600 over the initial scenario where the red-eye flight is operated.

With the traditional frequency selection model, however, removing the red-eye flight from IND and DSM results in \$600 less profit over the initial scenario where itinerary A^1 is offered, since the spilled passengers are not recaptured. Hence in this example with spill and recapture, the net profit of \$1,200 from 4 passengers spilled from itinerary A^1 well justifies the cost-saving approach, against the loss in profit of \$600 from the remaining 20 passengers.

Table 6 clearly illustrates how spill and recapture with attractiveness of itineraries can lead the model to select a better alternative, because considering spill and recapture significantly improves the accuracy of the model.

3.2.2 Market Selection

Market selection is slightly more complicated than frequency selection because usually a fixed cost is invoked by entering a new market. Similar to the approach used in frequency selection, we define a set of potential markets to enter to determine whether these markets benefit the whole flight network. Market selection is not a standalone feature though it is presented separately. Instead, market selection is built on top of frequency selection to incorporate the cost of entering new markets, since marking all of the itineraries in a market as optional naturally determines if a market should be opened.

Let M^O be the set of optional markets that are allowed to be opened or closed and let *MKT* be the maximum number of new markets to enter. We need to also include the following decision variables z to indicate if a potential market is selected:

$$z_m = \begin{cases} 1 & \text{if market } m \text{ is opened}, m \in M^O, \\ 0 & \text{otherwise.} \end{cases}$$

Market selection can be easily integrated with the previous models by adding the following constraints:

$$z_m \geq s_i \qquad i \in I_m^{HA}, m \in M^O \tag{12}$$

$$\sum_{m \in M^O} z_m \leq \mathit{MKT} \tag{13}$$

$$z \in \{0, 1\}. \tag{14}$$

Constraint (12) imposes that market m is considered open if any itinerary for that market accepts passengers in the solution. Constraint (13) limits the number of markets allowed to be opened. Note that constraint (12) marks a market to be open only if an itinerary for this market is available, instead of an operated flight for the market. This approach gives the airline the flexibility to offer itineraries that include flights between two stations without entering the market between these two stations, as long as the airline does not offer any itinerary solely consisting of such flights.

With an additional term denoting the fixed cost for all open markets in the objective function (1a), the cost of gate acquisitions, marketing, and fees for the new markets can be correctly represented with market selection variables z. As market selection is quite similar to frequency selection, the example from *Figure 2* and *Table 4* illustrates the same concept of how the model compares and screens various alternatives.

Similar to market selection, codeshare selection includes additional set of codeshared flights as decision variables. With the set of all possible itineraries regenerated, each codeshared itinerary is turned on or off depending on whether the codeshared flights belonging to the itinerary are operated. The codeshared flights are operated by a partner airline, but the codeshared itineraries generated are marketed both through the host airline as well as by the partner airline. The host airline usually takes a percentage of the revenue from the codeshared segments on an itinerary. In such a case, accurate modeling of spill and recapture is even more important, because the opportunity cost of a spill from a codeshared itinerary lies within a larger network.

Since codeshare agreements are often quite complicated, many additional business constraints are required. For example, only a certain number of groups of flights can be codeshared due to the limited number of flight numbers. Some airlines also impose consistency constraints, where all of the flights in the same flight group must be codeshared together. Additionally, the symmetry constraint mandates that if a codeshare flight from station A to station B is open, then its counterpart, from station B to station A, also needs to be codeshared.

3.2.3 Departure Time Selection

In addition to frequency selection and market selection, flights can be retimed to generate different flight schedules and different itineraries as in the work of Lohatepanont et al. 2004 [11]. To allow retiming of the flights, each fleet assignment variable is decomposed into a set of binary variables representing the retimed copies for the flight and all equipment types as *Figure 3* illustrates.

Figure 3 has only the flights from IND retimed a few minutes earlier than the original departure times, highlighted by the arrows in thin dotted lines. Different flights can have different retiming rules. For instance, a flight from IND to ORD can be retimed to be 20 minutes earlier, denoted in the figure by the longer gap between the two flight copies from IND to ORD. The set of itineraries also needs to be updated so that all possible and feasible itineraries are regenerated, based on the extra flight copies added in the model. The attractiveness of each new itinerary also needs to be assessed based on its characteristics such as its total duration and number of stops. In our example, this itinerary overbuilding step forms itineraries A_0, A_1, B_0, B_1, C_0 and C_1.

We define $C(l)$ as the set of retimed copies of leg l, $l \in L$. Additional decision variables are added to the model as follows:

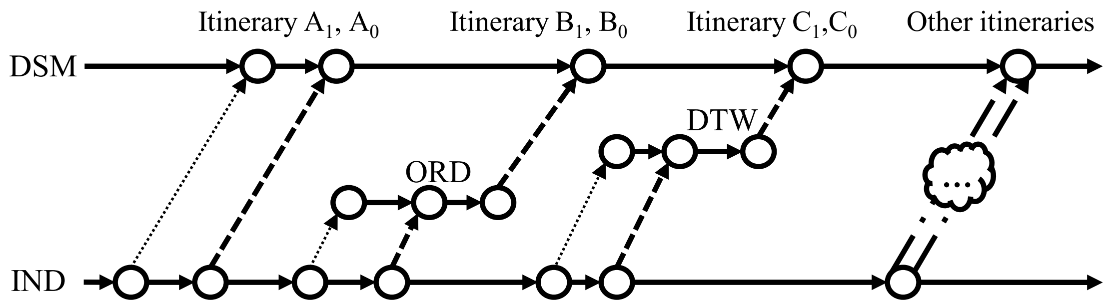

Figure 3: Creating copies with different departure times for all flights from IND

$$x_{lf}^c = \begin{cases} 1 & \text{if fleet } f \text{ is assigned to flight } l, \text{and retimed copy } c \text{ is selected}, c \in C(l), l \in L, f \in F, \\ 0 & \text{otherwise.} \end{cases}$$

With these new decision variables, the original fleet assignment variables in FAM-SBLP are decomposed into copies and they are related by

$$x_{lf} = \sum_{c \in C(l)} x_{lf}^c \qquad l \in L, f \in F. \tag{15}$$

Since the flight copies of different departure times are included, the set of itineraries on the host airline also has to be updated with all possible and feasible combinations of the retimed flight copies and optional flights. Departure time selection is quite similar to frequency selection because the same flight with different departure times can be considered as a set of optional flights, except that exactly one flight copy from this set must be operated. Therefore the example from *Figure 2* and *Table 4* gives similar intuition behind departure time selection with spill and recapture.

3.2.4 Block Time Selection

Service metrics are crucial for airlines to maintain and improve their market share, due to the high level of competition in the industry. One of the key service quality measures for airlines is their on time performance (OTP). To preserve goodwill, airlines design their schedules carefully so that the impact of block and connection times on the chance of a flight delay or cancellation is minimal. Approaching block time selection with the spill and recapture framework can achieve a better balance between high service quality and operational profits.

In order to integrate block time design with fleet assignment and schedule design, additional copies are created for all flight activities (departure and arrival) to represent flights with different departure and block times. An example of such an expanded network is provided in *Figure 4*.

In *Figure 4* we use the example in *Figure 3* as the original flight network. If all the flights from IND can be operated with either the original block time or 10 minutes more than the original block time, then new copies for all the flight activities are created so that with the same departure time, the copies arrive 10 minutes later at the destination.

Sohoni et al. 2011 [16] propose several models with the focus on designing robust schedules so that potential disruptions in the network have minimal impact on the network and service quality. Service level for each flight is defined as the probability that passengers from the respective flight have enough time to connect to subsequent flights. Network service level (NSL), γ_{NSL}, represents the minimum service level across all flights in the flight network. On the other hand, flight service level (FSL) γ_{FSL} is the probability that any flight is not delayed in accordance to the acceptable OTP measure for flight delays, set by the Department of Transportation. This value δ is typically 15 minutes after the scheduled arrival time.

The connection set for leg l and copy c, $Con_l^c \subseteq C \times L$, includes all flights that can connect from c, where $c \in C(l)$ and $l \in L$. Furthermore, we denote by W_l^c the unpredictable block time of leg l and copy c, $l \in L$, $c \in C$.

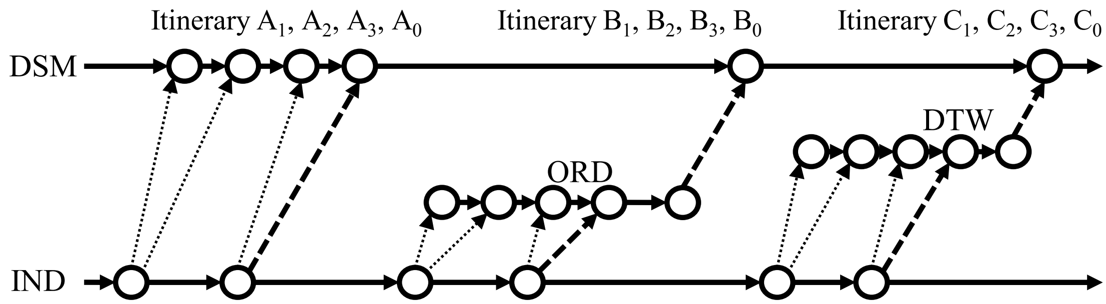

Figure 4: Creating copies with different departure and block times for all flights from IND

Random quantity W_l^c follows a given probability distribution of the block time for a specific copy c of leg l, since the probability of various block times may depend on the departure time as well as the market that the flight is in.

The constraints formulated below integrate FAM-SBLP with block time selection so that both service level requirements are imposed.

$$\Pr[W_l^c \leq \text{Time required to connect to flight copy } (e,k) \text{ from flight copy } (c,l)] \geq \gamma_{NSL}$$
$$(e,k) \in Con_l^c, c \in C(l), l \in L \tag{16}$$
$$\Pr[W_l^c \leq \delta + \text{Block time of flight copy } (c,l)] \geq \gamma_{FSL}$$
$$c \in C(l), l \in L \tag{17}$$

Constraint (16) maintains NSL so that the probability that passengers from flight l can connect to any follow-on flight k in the connection set for l is above the threshold. Meanwhile, constraint (17) guarantees that FSL for any flight is at least γ_{FSL} to keep up the on-time performance measure. The objective function (1a) can be modified to take into account the penalty from the deviation of the incumbent (preferred) schedule, as well as the total variable cost associated with the block time of each flight, including fuel, crew time and other fees.

If the model does not consider adding and canceling flights, constraint (16) and (17) can be linearized using standard statistical techniques. With frequency and departure time selection, however, the right-hand side of constraint (17) needs to be modified so that the FSL requirement is only necessary when a flight is operated. The right-hand side of constraints (16) also requires a modification to impose its validity only when both flight l and its possible connection flight k are present in the solution. In such cases, constraint (16) is generally non-linear and requires Benders' decomposition to solve the whole model efficiently. The related methodology is given in detail in Sohoni et al. 2011 [16].

4 Future Studies

With FAM-SBLP as the core model, all modules in Section 3.2 can be integrated seamlessly to provide a complete model ISD-FAM-SBLP (see Appendix) that targets various aspects in network planning simultaneously. For a particular area in network planning, a subset of these modules can be grouped together to tailor an appropriate network design model. As FAM-SBLP preserves the basic structure of the traditional FAM, it can be further expanded with many common models in other areas such as maintenance and crew scheduling. Due to its great flexibility and intuitive incorporation of the spill and recapture effects, ISD-FAM-SBLP provides the airline industry with an all-purpose integrated network planning model that is viable in the future. Given an accurate modeling of revenue, the general ISD-FAM-SBLP framework is capable of making centralized optimization decisions simultaneously with respect to all aspects of airline schedule design and capacity planning. With such a versatile framework governing all major aspects of network planning, the focus on airline network planning now shifts to the optimization side. How to efficiently tackle ISD-FAM-SBLP becomes an immediate task for future studies.

Although our preliminary computational studies show promising results for this new framework of spill and recapture with various network planning models, the computational studies also suggest that FAM-SBLP is much harder to solve than the traditional itinerary-based FAM. Furthermore, the significant increase in the number of nonzero coefficients in the constraint matrix makes even the root node relaxation of FAM-SBLP much harder to solve than the one from the traditional FAM. We suspect that, with larger airlines, the complete ISD-FAM-SBLP model with spill and recapture is very challenging to solve, due to the additional flight copies created. With such complexity of our holistic optimization models, novel methodologies and further computational analyses are required to approach problems with our new framework for spill and recapture.

In addition, our computational experiments indicate that a large number of fleet assignment variables (around 80%) turn out to be binary at the root node relaxation. Such characteristics could be used as a base towards new heuristics. Sherali et al. 2005 [14] and 2010 [15] conducted a polyhedral analysis on the traditional itinerary-based FAM and integrated schedule design with FAM. With a lot of similarity between our models and the models that their work is based on, it is likely that analogous techniques could be applied to either FAM-SBLP alone or the complete ISD-FAM-SBLP model.

The spill and recapture effects are common phenomena in any competitive industry as customers' choices heavily affect the strategy of suppliers. For instance, in the retail business, product line design (PLD) tries to determine the best subset of products to offer considering potential consequences such as market expansion and cannibalization. Schön 2010 [13] approached PLD with a multinomial logit choice model and fractional programming techniques. Since PLD is very similar to itinerary selection in FAM-SBLP (with itineraries being the products in the airline industry [12]), the presented framework of spill and recapture can be applied to PLD and give a more accurate and efficient evaluation of the interaction between supply and demand. The framework of SBLP could also be used in the hospitality industry since it highly resembles the airline industry, with perishable goods such as hotel rooms and cruise trips. As many airlines are integrating hospitality services into their product line, it is beneficial to consider passenger choices with a bigger picture in mind. Modeling spill and recapture at a higher level could achieve a globally optimized result and inspire suppliers with fresh strategies from a different perspective.

5 Appendix: Complete ISD-FAM-SBLP

The following additional sets are introduced.

N_f	set of nodes in equipment type f's network, $f \in F$, indexed by n
CS_f	set of arcs or legs passing forward in time through a counting time line in equipment type f's network, $f \in F$
$O(n)$	set of outgoing legs at node n, $n \in N_f$, $f \in F$
$I(n)$	set of incoming legs at node n, $n \in N_f$, $f \in F$
$o(n)$	outgoing ground arc at node n, $n \in N_f$, $f \in F$
$i(n)$	incoming ground arc at node n, $n \in N_f$, $f \in F$

In order to keep track of the aircraft, let the decision variable y_g be the number of aircraft on ground arc g in equipment type f's network, $g \in CS_f$, $f \in F$.

New parameters are required as follows.

$Avail_f$	number of total available equipments of equipment type f, $f \in F$
$\hat{d}_l$	original departure time for leg l, $l \in L$
d_l^c	departure time of leg l and copy c, $c \in C(l)$, $l \in L$
t_l^c	block time of leg l and copy c, $c \in C(l)$, $l \in L$
m_{lk}	minimum passenger connecting time between leg l and k, $l \in L$, $k \in L$
p_l	the penalty for deviating from the preferred departure time of leg l, $l \in L$
$Mcost_m$	cost of entering market m, $m \in M^O$
$Fcost_f$	fixed cost of using aircraft of type f, $f \in F$
$Vcost_l$	per time unit cost incurred for leg l, $l \in L$, which includes the costs for crew pay, fuel consumption, etc.

The full model reads as follows.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{m \in M} (Dem_m \sum_{i \in I_m^{HA}} fare_i s_i) - \sum_{l \in L} \sum_{f \in F} Fcost_f \sum_{c \in C(l)} x_{lf}^c \\
& - \sum_{m \in M^O} MCost_m z_m \\
& - \sum_{l \in L} Vcost_l \sum_{c \in C(l)} (t_l^c \sum_{f \in F} x_{lf}^c) \\
& - \sum_{l \in L} p_l \, | \sum_{c \in C(l)} (d_l^c \sum_{f \in F} x_{lf}^c) - \hat{d}_l \sum_{f \in F} \sum_{c \in C(l)} x_{lf}^c | \qquad (18)
\end{aligned}
$$

subject to

$$\sum_{m\in M}(Dem_m \sum_{i\in I_m^{HA}(l)} s_i) \leq \sum_{f\in F} Cap_f \sum_{c\in C(l)} x_{lf}^c \qquad l \in L \tag{19}$$

$$s_0^m + \sum_{i\in I_m^{HA}} s_i = 1 \qquad m \in M \tag{20}$$

$$A_0^m s_i \leq A_i s_0^m \qquad i \in I_m^{HA}, m \in M \tag{21}$$

$$\sum_{l\in O(n)}\sum_{c\in C(l)} x_{lf}^c - \sum_{l\in I(n)}\sum_{c\in C(l)} x_{lf}^c + y_{o(n)} - y_{i(n)} = 0 \qquad n \in N_f, f \in F \tag{22}$$

$$\sum_{l\in CS_f}\sum_{c\in C(l)} x_{lf}^c + \sum_{g\in CS_f} y_g \leq Avail_f \qquad f \in F \tag{23}$$

$$\sum_{f\in F}\sum_{c\in C(l)} x_{lf}^c = 1 \qquad l \in L^F \tag{24}$$

$$\sum_{f\in F}\sum_{c\in C(l)} x_{lf}^c \leq 1 \qquad l \in L^O \tag{25}$$

$$\sum_{f\in F}\sum_{c\in C(l)} x_{lf}^c \geq s_i \qquad i \in I_m^{HA}(l), l \in L^O, m \in M \tag{26}$$

$$z_m \geq \ s_i \qquad i \in I_m^{HA}, m \in M^O \tag{27}$$

$$\sum_{m\in M^O} z_m \leq MKT \tag{28}$$

$$\Pr[W_l^c \leq d_k^e \sum_{f\in F} x_{kf}^e - d_l^c \sum_{f\in F} x_{lf}^c - m_{lk}] \geq \gamma_{NSL}(\sum_{f\in F} x_{kf}^e + \sum_{f\in F} x_{lf}^c - 1) \qquad (e,k) \in Con_l^c, c \in C(l), l \in L \tag{29}$$

$$t_l^c \sum_{f\in F} x_{lf}^c + \delta \geq \Pr^{-1}[\gamma_{FSL}] \sum_{f\in F} x_{lf}^c \qquad c \in C(l), l \in L \tag{30}$$

$$x \in \{0,1\} \tag{31}$$

$$z \in \{0,1\} \tag{32}$$

$$y \geq 0 \tag{33}$$

$$s \geq 0 \tag{34}$$

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