Lifting for Mixed Integer Programs with Variable Upper Bounds

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July 14, 2013

Abstract

We investigate the convex hull of the set defined by a single inequality with continuous and binary variables, which are additionally related by variable upper bound constraints. First we elaborate on general sequence dependent lifting for this set and present a dynamic program for calculating lifting coefficients. Then we study variable fixings of this set to knapsack covers and to the single binary variable polytope. We explicitly give lifting coefficients of continuous variables when lifting the knapsack cover inequality. We provide two new families of facet-defining inequalities for the single binary variable polytope and we prove that combined with the trivial inequalities they give a full description of this polytope.
Keywords: Mixed integer programming, Polyhedral theory

1 Introduction

Many optimization problems arising from a variety of applications are formulated as mixed integer programs. In many of these applications *variable upper bound constraints* are already present, e.g. the facility location problem (see e.g. Aardal (1998)), the lot-sizing problem (see e.g. Salomon (1991)), and the network design problems (see e.g. Bienstock and Günlük (1996)). Even if these constraints are not present, they can be generated by preprocessing, Savelsbergh (1994). A successful approach for solving problems of this type is branch-and-cut, Nemhauser and Wolsey (1988), which requires generating valid inequalities for the underlying polyhedron. Surveys for recent techniques in mixed integer programs are provided in Richard (2011), Atamtürk (2004) and Atamtürk (2005). In this paper we study the polyhedron $\check{S}$ associated with the set consisting of a single inequality involving both continuous and binary variables and variable upper bounds that additionally link continuous and binary variables. Set $\check{S}$ is described by

$$\begin{aligned} \sum_{i\in N}\check{a}_i\check{x}_i + \sum_{i\in N}\check{b}_i\check{y}_i &\le \check{d} \\ 0 \le \check{x}_i &\le \check{u}_i + \check{v}_i\check{y}_i \qquad i\in N \\ \check{x} \ge 0, \check{y} &\text{ binary,} \end{aligned}$$

where $\check{a}_i, \check{b}_i, \check{u}_i, \check{v}_i \in \mathbb{Q}$ for every $i \in N$ and $\check{d} \in \mathbb{Q}$. By defining new variables $x_i = |\check{a}_i|\check{x}_i$, $y_i = \check{y}_i$ if $\check{b}_i \geq 0$, and $y_i = 1 - \check{y}_i$ if $\check{b}_i < 0$, $\check{S}$ is equivalent to set S given by

$$\begin{aligned}
\sum_{i \in N_1^+} x_i - \sum_{i \in N_1^-} x_i + \sum_{i \in N_2^+} b_i y_i + \sum_{i \in N_2^-} b_i y_i \leq d \\
0 \leq x_i \leq u_i + v_i y_i \qquad i \in N_2^+ \\
0 \leq x_i \leq u_i - v_i y_i \qquad i \in N_2^- \\
y \text{ binary},
\end{aligned}$$

where $b_i = |\check{b}_i|$, $d = \check{d} - \sum_{b_i < 0} \check{b}_i$,

$$\begin{array}{ll}
u_i = |\check{a}_i|\check{u}_i & \text{if } \check{b}_i \geq 0, \\
u_i = |\check{a}_i|(\check{u}_i + \check{v}_i) & \text{if } \check{b}_i < 0, \\
v_i = |\check{a}_i|\check{v}_i & \text{if } (\check{b}_i \geq 0 \text{ and } |\check{a}_i|\check{v}_i \geq 0), \text{ or if } (\check{b}_i < 0 \text{ and } |\check{a}_i|\check{v}_i < 0), \\
v_i = -|\check{a}_i|\check{v}_i & \text{if } (\check{b}_i \geq 0 \text{ and } |\check{a}_i|\check{v}_i < 0), \text{ or if } (\check{b}_i < 0 \text{ and } |\check{a}_i|\check{v}_i \geq 0).
\end{array}$$

The defined parameters satisfy $b_i \geq 0$ and $v_i \geq 0$ for every $i \in N = N_1^+ \cup N_1^- = N_2^+ \cup N_2^-$. Let P be the convex hull of S. We say that variable i has a *zero constant bound* if $u_i = 0$ and it has a *positive constant bound* otherwise.

To avoid trivial cases, we make the following assumption.

Assumption 1. u_i's and v_i's satisfy

1. $u_i - v_i \geq 0$ for $i \in N_2^-$,
2. $u_i \geq 0$ for $i \in N_2^+$,
3. $u_i + v_i > 0$ for $i \in N_2^+$,
4. $u_i > 0$ for $i \in N_2^-$.

Note that Assumption 1 is necessary for full dimensionality of P. Shebalov and Klabjan (2006) give sufficient and necessary conditions for full dimensionality of P.

The basic special case not involving binary variables in the constraint, i.e. $b_i = 0$ for every $i \in N$, and the seminal study on the topic is the work by Padberg *et al.* (1985), which is extended and enhanced in Van Roy and Wolsey (1986), Goemans (1989), Gu *et al.* (1999), and Atamtürk *et al.* (2001). They all build on the notion of a cover. Richard *et al.* (2003a,b) studied a similar polyhedron, where $v_i = 0$ for all $i \in N$. This is clearly a relaxation of S. However, S has more structure, which is embedded with the variable upper bound constraints and it is heavily exploited in our work. Another special case of our polyhedron is studied by Miller *et al.* (2003) in the context of multi-item lot-sizing. Their case corresponds to $N_1^- = N_2^- = \emptyset$, $u_i = 0, v_i = K - b_i$ for every $i \in N$, where K is a constant. Cimren (2010) studied the polyhedron with $N_1^- = N_2^- = \emptyset$ and $u_i = 0$ for every $i \in N$. Atamtürk and Günlük (2007) studied the problem with $N_1^- = N_2^- = \emptyset$ and $b_i = u_i = 0$ for every $i \in N$ but their constraint has an additional integer variable in the constraint. Atamtürk *et al.* (2001) study the problem with no binary variables in the constraint ($b_i = 0$ for all $i \in N$) but uses more general variable upper bounds. The polyhedron considered by Marchand and Wolsey (1999) can be obtained from our polyhedron if $v_i = 0$ for all $i \in N$ and $u_i = 0$ for all $i \in N$ but one. Their paper also shows that their model is a relaxation of the standard single node fixed charge flow model. By using the same technique, it can be seen that it is also a relaxation of our model. Agra and Constantino (2006) studied the polyhedron with $N_1^- = \emptyset$ and $b_i = 0$ for all $i \in N$ but the variable bound is defined as $Ly_i \leq x_i \leq Uy_i$, where L and U are positive constants and y_i's are integer. Shebalov and Klabjan (2006) study S. They develop a flow cover type inequality, which is valid when $N_1^- = \emptyset$. They lift it into a valid inequality for P by using sequence independent lifting.

The present work differs from Shebalov and Klabjan (2006) as we study sequence dependent lifting and lifting of knapsack covers, whereas Shebalov and Klabjan (2006) studies sequence independent lifting and lifting of flow cover inequalities. Sequence independent lifting requires completely different proof techniques than those used in the current paper and also the resulting valid inequalities are very different. We also present a full description of convex hull for the single binary polytope. The main contribution of this paper is that we give different sets of lifted inequalities from Shebalov and Klabjan (2006) for S by using completely different techniques and we provide the full description of a single binary variable polytope which is an interesting result on its own.

In this work we focus on sequence dependent lifting. In Section 2 we present two optimization problems for computing the lifting coefficients. We also develop a dynamic program for computing lifting coefficients of binary variables. Unfortunately the optimization problem for computing lifting coefficients for continuous variables is a nonlinear mixed integer program and therefore very hard to solve. Section 3 first gives the knapsack cover inequality, which is facet-defining if all variables outside of the cover are fixed at zero, and it discusses both sequence independent and dependent lifting of these inequalities. For sequence dependent lifting we explicitly obtain lifting coefficients for continuous variables if these variables are lifted first. Note that this elevates the problem of solving the nonlinear mixed integer program for computing the lifting coefficients of continuous variables. In Section 4 we consider the single binary variable polytope obtained from S by fixing all but one binary variables. We derive a full description of this polytope by presenting two new families of facet-defining inequalities. Lifting coefficients of binary variables can be computed by dynamic programming. Since continuous variables are not fixed, as with the knapsack cover inequalities, we do not need to solve the nonlinear mixed integer programs for computing the lifting coefficients of continuous variables.

2 Sequence dependent lifting

In this section we first give a brief overview of sequence dependent lifting for P. As is typically the case, different lifting orders can yield different inequalities. In the remainder of the section we focus on the underlying optimization problem for computing lifting coefficients of binary variables. We also show that in many circumstances it is easy to obtain the lifting coefficients for the continuous variables in N_1^+.

Whenever we do not need to distinguish between N_2^+ and N_2^-, we write $\pm v_k$, since they are handled similarly. Let $L_0 \subseteq N$ and $L_1 \subseteq N$ be the set of binary variables that are fixed at 0 and 1, respectively. The actual value is denoted by $\bar{y}_i$, i.e. $\bar{y}_i = 0$ for $i \in L_0$ and $\bar{y}_i = 1$ for $i \in L_1$. Let $L_l \subseteq N$ be the set of continuous variables fixed at 0 and the corresponding value is denoted by $\bar{x}_i = 0$. The set $L_u \subseteq L_0 \cup L_1$ corresponds to the continuous variables that are fixed at their upper bounds. If $i \in L_u$, then we define $\bar{x}_i = u_i \pm v_i \bar{y}_i$.

The set of non fixed continuous variables is denoted by $C^x = N \setminus (L_l \cup L_u)$ and the set of non fixed binary variables is denoted by $C^y = N \setminus (L_0 \cup L_1)$. Let us define $\delta_i = 1$ for $i \in N_1^+$, and $\delta_i = -1$ for $i \in N_1^-$. The resulting polyhedron is

$$P^0 = \left\{ \begin{array}{lll} (x, y) \in \mathbb{R}^{|C^x|} \times \mathbb{R}^{|C^y|} : & \sum_{i \in C^x} \delta_i x_i + \sum_{i \in C^y} b_i y_i \leq d - \sum_{i \in L_u} \delta_i \bar{x}_i - \sum_{i \in L_1} b_i & \\ & 0 \leq x_i \leq u_i \pm v_i y_i, \qquad i \in C^x \cap C^y & \\ & 0 \leq x_i \leq u_i \pm v_i \bar{y}_i, \qquad i \in C^x \setminus C^y & \\ & \text{y binary} & \end{array} \right\}. \tag{1}$$

We denote the convex hull of P^0 by P^C. Let

$$0 \leq \alpha_0 - \sum_{i \in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i \tag{2}$$

be a valid inequality for P^0. The goal is to construct a valid inequality for P of the form

$$0 \leq \alpha_0 - \sum_{i \in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i \in L_l \cup L_u} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in L_0 \cup L_1} \beta_i (y_i - \bar{y}_i). \tag{3}$$

We lift variables one by one. For example, the lifting sequence $\{2^x, 3^y, 1^x, 3^x, \cdots\}$ encodes that we first lift x_2, then y_3, next x_1, followed by x_3 and so forth. Observe that not all lifting orders are possible, e.g. if $u_k = 0$ and y_k is fixed at 0, then we cannot lift x_k before y_k.

2.1 Optimization problems for computing the lifting coefficients

Let first consider lifting x_k. Let $I_x \subseteq L_l \cup L_u$ and $I_y \subseteq L_0 \cup L_1$ be the index sets for x and y, respectively, that have been lifted prior to x_k. Let us define

$$Q_k = \left\{ \begin{array}{lll} (x,y): & \delta_k x_k + \sum_{i\in C^x \cup I_x} \delta_i x_i + \sum_{i \in C^y \cup I_y} b_i y_i \le d' & \\ & 0 \le x_i \le u_i \pm v_i y_i, & i \in \big(C^x \cap C^y\big) \cup \big(I_x \cap I_y\big) \\ & 0 \le x_i \le u_i \pm v_i \bar{y}_i, & i \in \big(C^x \setminus C^y\big) \cup \big(I_x \setminus I_y\big) \\ & 0 \le x_k \le u_k \pm v_k y_k, & \text{for } k \in I_y, \text{ or} \\ & 0 \le x_k \le u_k \pm v_k \bar{y}_k, & \text{for } k \notin I_y \\ & y \text{ binary} & \end{array} \right\}. \tag{4}$$

where $d' = d - \sum_{i\in L_u \setminus (I_x \cup \{k\})} \delta_i \bar{x}_i - \sum_{i \in L_1 \setminus I_y} b_i$. If $k \in L_l$, then

$$\alpha_k = \min \left\{ \Big(\alpha_0 - \sum_{i\in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i\in I_x} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) \Big) / x_k : (x,y) \in Q_k, x_k > 0 \right\}.$$

If $k \in L_u$, then

$$\alpha_k = \max \left\{ \Big(\sum_{i\in C^x} \alpha_i x_i + \sum_{i \in C^y} \beta_i y_i + \sum_{i\in I_x} \alpha_i (x_i - \bar{x}_i) + \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) - \alpha_0 \Big) / (\bar{x}_k - x_k) : (x,y) \in Q_k, x_k < \bar{x}_k \right\}.$$

Note that both values are attainable.

The objective functions are nonlinear and we do not know how to reformulate it as a linear optimization problem with the same structure. We present two approaches to overcome this difficulty. In the first one, described in Section 3.3, we exploit the special structure of (2) to explicitly derive αs if the continuous variables are lifted first. An alternative approach, discussed in Section 4, is to include all continuous variables in C^x.

Consider now lifting y_k, where I_x and I_y are defined in the same way as before. Let $d' = d - \sum_{i\in L_u \setminus I_x} \delta_i \bar{x}_i - \sum_{i \in L_1 \setminus (I_y \cup \{k\})} b_i$. If $k \in L_0$, then β_k is defined by

$$\begin{array}{rll} \beta_k = & \min & \Big(\alpha_0 - \sum_{i\in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i\in I_x} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) \Big) \\ & \text{s.t.} & \sum_{i\in C^x \cup I_x} \delta_i x_i + \sum_{i \in C^y \cup I_y} b_i y_i \le d' - b_k \\ & & 0 \le x_i \le u_i \pm v_i y_i, \quad i \in \big(C^x \cap C^y\big) \cup \big(I_x \cap I_y\big) \\ & & 0 \le x_i \le u_i \pm v_i \bar{y}_i, \quad i \in \big(C^x \setminus C^y\big) \cup \big(I_x \setminus I_y\big) \\ & & 0 \le x_k \le u_k \pm v_k, \quad \text{if } k \in I_x \\ & & y \text{ binary.} \end{array}$$

If $k \in L_1$, then β_k is defined by

$$\begin{array}{rll} \beta_k = & \max & \Big(\sum_{i\in C^x} \alpha_i x_i + \sum_{i \in C^y} \beta_i y_i + \sum_{i\in I_x} \alpha_i (x_i - \bar{x}_i) + \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) - \alpha_0 \Big) \\ & \text{s.t.} & \sum_{i\in C^x \cup I_x} \delta_i x_i + \sum_{i \in C^y \cup I_y} b_i y_i \le d' \\ & & 0 \le x_i \le u_i \pm v_i y_i, \quad i \in \big(C^x \cap C^y\big) \cup \big(I_x \cap I_y\big) \\ & & 0 \le x_i \le u_i \pm v_i \bar{y}_i, \quad i \in \big(C^x \setminus C^y\big) \cup \big(I_x \setminus I_y\big) \\ & & 0 \le x_k \le u_k, \quad \text{if } k \in I_x \\ & & y \text{ binary.} \end{array}$$

These two optimization problems, with the appropriate choice of M, $\hat{a}$, $\hat{c}$, $\hat{b}$, $\hat{d}$, $\hat{u}$, and $\hat{v}$ can be transformed into the following general optimization problem that defines βs.

$$\max \quad \sum_{i=1}^{M} \hat{a}_i x_i + \sum_{i=1}^{M} \hat{b}_i y_i \tag{5a}$$

$$s.t. \quad -\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{M} x_i + \sum_{i=1}^{M} \hat{c}_i y_i \leq \hat{d} \tag{5b}$$

$$0 \leq x_i \leq \hat{u}_i + \hat{v}_i y_i \qquad i = 1, 2, \dots, M \tag{5c}$$

$$y \text{ binary}, \tag{5d}$$

where $\hat{a}_i \leq 0$ for $i = 1, \dots, k$ and $\hat{a}_i > 0$ for $i = k+1, \dots, M$, and $\hat{c} \geq 0, \hat{u} \geq 0$. Note that $\hat{v}_i$, and $\hat{d}$ can be negative and we allow $\hat{v}$s to be 0. Without loss of generality we assume that $\hat{a}, \hat{b}, \hat{c}, \hat{u}, \hat{v}$, and $\hat{d}$ are integer. In the next section we develop a dynamic program, which solves this program, and thus computes βs.

2.2 A dynamic program for computing the lifting coefficients of binary variables

The presence of continuous variables makes a non trivial task of developing a dynamic program. Without loss of generality we assume $\hat{a}_1 \leq \hat{a}_2 \leq \cdots \leq \hat{a}_k \leq 0 < \hat{a}_{k+1} \leq \cdots \leq \hat{a}_M$. We parameterize (5) with respect to the number of variables and the right hand side. For each integer $n, 1 \leq n \leq M$ let

$$\begin{aligned} f_n(\tilde{d}) = \quad & \max \quad \sum_{i=1}^{n} \hat{a}_i x_i + \sum_{i=1}^{n} \hat{b}_i y_i \\ & \text{s.t.} \quad \sum_{i=1}^{n} \delta_i x_i + \sum_{i=1}^{n} \hat{c}_i y_i \leq \tilde{d} \\ & \qquad 0 \leq x_i \leq \hat{u}_i + \hat{v}_i y_i \qquad i = 1, 2, \dots, n \\ & \qquad y \text{ binary}, \end{aligned}$$

where $\delta_i = -1$ if $i \leq k$ and $\delta_i = 1$ if $i \geq k+1$. Note that $f_M(\tilde{d})$ gives a solution to (5). Let

$$\omega_1 = -\sum_{i=1}^{k} (\hat{u}_i + \hat{v}_i^+), \qquad \omega_2 = \sum_{i=k+1}^{M} (\hat{u}_i + \hat{v}_i^+ + \hat{c}_i) + \sum_{i=1}^{k} \hat{c}_i \, ,$$

where $s^+ = \max\{0, s\}$. In addition, we define $\Omega = \{\omega_1, \omega_1 + 1, \dots, \omega_2\}$. It is easy to see that it suffices to define f_n on $[\omega_1, \omega_2]$. The dynamic program will actually show that it suffices to consider $\tilde{d} \in \Omega$.

For $k+1 \leq i \leq M$, we define $t(i) \in \{1, \dots, k+1\}$ as

$$t(i) = \begin{cases} 1 & |\hat{a}_j| \leq \hat{a}_i \text{ for all } 1 \leq j \leq k, \\ k+1 & |\hat{a}_k| > \hat{a}_i, \\ s & |\hat{a}_j| \leq \hat{a}_i \text{ for all } j \geq s \text{ and } |\hat{a}_{s-1}| > \hat{a}_i \, . \end{cases}$$

From definition it follows that $t(i)$ is the index where $\hat{a}_i$ fits in the order $|\hat{a}_k| \leq |\hat{a}_{k-1}| \leq \cdots \leq |\hat{a}_1|$.

For any integers $1 \leq s \leq k, k \leq p \leq M, 1 \leq j \leq s, k+1 \leq l \leq M$ and $\tilde{d} \in \Omega$ we define

$$\begin{aligned} g_{pj}^{ls}(\tilde{d}) = \quad & \max \quad \sum_{i=s}^{k} (\hat{a}_i + \hat{a}_l) \hat{v}_i y_i + \sum_{i=j}^{p} (\hat{b}_i - \hat{a}_l \hat{c}_i) y_i \\ & \text{s.t.} \quad \tilde{d} - \hat{u}_l + \sum_{i=s}^{k} \hat{u}_i \leq \sum_{i=j}^{p} \hat{c}_i y_i - \sum_{i=s}^{k} \hat{v} y_i \leq \tilde{d} + \sum_{i=s}^{k} \hat{u}_i \\ & \qquad y \text{ binary} \end{aligned}$$

and

$$
\begin{aligned}
\tilde{g}^{ls}_{pj}(\tilde{d}) = \quad & \max \quad \sum_{i=s}^{k}(\hat{a}_i + \hat{a}_l)\hat{v}_i y_i + \sum_{i=j}^{p}(\hat{b}_i - \hat{a}_l\hat{c}_i)y_i \\
& \text{s.t.} \quad \tilde{d} - \hat{c}_l - \hat{v}_l - \hat{u}_l + \sum_{i=s}^{k}\hat{u}_i \le \sum_{i=j}^{p}\hat{c}_i y_i - \sum_{i=s}^{k}\hat{v} y_i \le \tilde{d} - \hat{c}_l + \sum_{i=s}^{k}\hat{u}_i \\
& \qquad\ \ y \text{ binary.}
\end{aligned}
$$

For any integers $1 \le l \le k$ and $1 \le p \le k$ and $\tilde{d} \in \Omega$ we define

$$
\begin{aligned}
h^l_p(\tilde{d}) = \quad & \max \quad \sum_{i=1}^{p}(\hat{b}_i + \hat{a}_l\hat{c}_i)y_i \\
& \text{s.t.} \quad \tilde{d} \le \sum_{i=1}^{p}\hat{c}_i y_i \le \tilde{d} + \hat{u}_l \\
& \qquad\ \ y \text{ binary}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}^l_p(\tilde{d}) = \quad & \max \quad \sum_{i=1}^{p}(\hat{b}_i + \hat{a}_l\hat{c}_i)y_i \\
& \text{s.t.} \quad \tilde{d} + \hat{c}_l \le \sum_{i=1}^{p}\hat{c}_i y_i \le \tilde{d} + \hat{u}_l + \hat{c}_l + \hat{v}_l \\
& \qquad\ \ y \text{ binary.}
\end{aligned}
$$

Whenever the underlying feasible region is empty, we define the corresponding function value to be $-\infty$. We show how to use dynamic programming to calculate these functions in Appendix A. First we give recursive relationships for f_n.

Theorem 1. For any $n = 1, 2, \ldots, k$ and any $\tilde{d} \in \Omega$ we have

$$
f_n(\tilde{d}) = \max \left\{ \begin{array}{l}
f_{n-1}(\tilde{d}), \\
f_{n-1}(\tilde{d} - \hat{c}_n) + \hat{b}_n, \\
f_{n-1}(\tilde{d} + \hat{u}_n) + \hat{a}_n\hat{u}_n, \\
f_{n-1}(\tilde{d} + (\hat{u}_n + \hat{v}_n) - \hat{c}_n) + \hat{a}_n(\hat{u}_n + \hat{v}_n) + \hat{b}_n, \\
h^n_{n-1}(\tilde{d}) - \hat{a}_n\tilde{d}, \\
\tilde{h}^n_{n-1}(\tilde{d}) - \hat{b}_n - \hat{a}(\tilde{d} - \hat{c}_n)
\end{array} \right\}. \tag{6}
$$

For any $n = k+1, k+2, \ldots, M$ and any $\tilde{d} \in \Omega$ we have

$$
f_n(\tilde{d}) = \max \left\{ \begin{array}{l}
f_{n-1}(\tilde{d}), \\
f_{n-1}(\tilde{d} - \hat{c}_n) + \hat{b}_n, \\
f_{n-1}(\tilde{d} - \hat{u}_n) + \hat{a}_n\hat{u}_n, \\
f_{n-1}(\tilde{d} - (\hat{u}_n + \hat{v}_n) - \hat{c}_n) + \hat{a}_n(\hat{u}_n + \hat{v}_n) + \hat{b}_n, \\
g^{n,t(n)}_{n-1,1}(\tilde{d}) + \sum_{i=t(n)}^{k}\hat{a}_i\hat{u}_i + \hat{a}_n\left(\tilde{d} - \sum_{i=t(n)}^{k}\hat{u}_i\right), \\
\tilde{g}^{n,t(n)}_{n-1,1}(\tilde{d}) + \sum_{i=t(n)}^{k}\hat{a}_i\hat{u}_i + \hat{a}_n\left(\tilde{d} - c_n - \sum_{i=t(n)}^{k}\hat{u}_i\right) + \hat{b}_n
\end{array} \right\}. \tag{7}
$$

For the proof of Theorem 1 and the computation of $g, \tilde{g}, h, \tilde{h}$ by dynamic programming, see Appendix A and Appendix B. We conclude this section by discussing the complexity of the presented algorithm. Let

$$
\xi = \max_{i=1,\ldots,M}\{\hat{c}_i, \hat{u}_i, |\hat{v}_i|\}
$$

and note that $-\omega_1 = \mathcal{O}(M\xi), \omega_2 = \mathcal{O}(M\xi)$. The running time of the algorithm is $\mathcal{O}(M^4(\omega_2 - \omega_1)) = \mathcal{O}(M^5\xi)$. The algorithm is clearly pseudo-polynomial. If $k = 0$, i.e., $a_i \ge 0$ for every i, then we can use $\omega_1 = 0, \omega_2 = \hat{d}$ and in turn the running time is $\mathcal{O}(M^4\hat{d})$.

2.3 Lifting coefficients for continuous variables in N_1^+ projected to 0

The next theorem shows that under some mild conditions the lifting coefficients of the continuous variables in N_1^+, which are projected to 0, are 0 regardless of the lifting order. A similar result for the case $v_i = 0$ for each $i \in N$ is given by Richard *et al.* (2003a).

Theorem 2. Let (2) be valid facet defining inequality for P^0, which is not a multiple of (1). Let us assume that αs and βs in (3) are obtained by using the optimization problems from Section 2.1. Then $\alpha_k = 0$ for every $k \in N_1^+ \cap L_l$.

Proof. Let I_x, I_y be defined as in Section 2.1 and next we lift $k \in N_1^+ \cap L_l$. We need

$$0 \leq \alpha_0 - \sum_{i \in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i \in I_x} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) - \alpha_k x_k \tag{8}$$

to be valid for Q_k with $\delta_k = 1$. Let $\hat{Q}_k$ be obtained from Q_k by fixing $x_k = 0$. Thus

$$0 \leq \alpha_0 - \sum_{i \in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i \in I_x} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) \tag{9}$$

is valid for $\hat{Q}_k$. Since all αs and βs are either (i) from the optimization problem Section 2.1, or (ii) a facet defining inequality in (2), it is easy to see that (9) is a valid minimal inequality for $\hat{Q}_k$.

Assume first that $u_k > 0$ if $k \in C^y \cup I_y$ or assume that $k \notin C^y \cup I_y$. Note that in the latter case $u_k \pm v_k \bar{y}_k > 0$. Consider $(\hat{x}, \hat{y}) \in \hat{Q}_k$, which satisfies (9) at equality, and $\sum_{i \in C^x \cup I_x} \delta_i \hat{x}_i + \sum_{i \in C^y \cup I_y} b_i \hat{y}_i < d'$. Such $(\hat{x}, \hat{y})$ exists, since (2) is not a multiple of (1) and (9) is minimal. Let $(\tilde{x}, \tilde{y})$ be the vector obtained from $(\bar{x}, \bar{y})$ by appending ε to x_k, where $\varepsilon > 0$ is small enough in order to make $(\tilde{x}, \tilde{y}) \in Q_k$. If $\alpha_k > 0$, then $(\tilde{x}, \tilde{y}) \in Q_k$, but it violates (8). Therefore $\alpha_k \leq 0$. Since we select α_k minimizing (8), we obtain (9) with $\alpha_k = 0$.

Assume now that $u_k = 0$ and $k \in C^y \cup I_y$. Let

$$\begin{aligned} h_k(z) = \quad & \max \quad \alpha_k x_k \\ & \text{s.t.} \quad -x_k = z \\ & \qquad\quad 0 \leq x_k \leq v_k y_k \end{aligned}$$

and

$$f_k(z) = \min \left\{ \left(\alpha_0 - \sum_{i \in C^x} \alpha_i x_i - \sum_{i \in C^y} \beta_i y_i - \sum_{i \in I_x} \alpha_i (x_i - \bar{x}_i) - \sum_{i \in I_y} \beta_i (y_i - \bar{y}_i) \right) : (x, y) \in \hat{Q}_k \right\}.$$

Validity of (8) for Q_k is equivalent to $h_k(z) \leq f_k(z)$ for all z. We need to consider this inequality only for $y_k = 1$, since $y_k = 0$ implies $x_k = 0$ and therefore by assumption we have validity. In this case $h_k(z) = -\alpha_k z$ for $z \in [-v_k, 0]$, $f_k(0) = 0$, and since $f_k(z)$ is nondecreasing $f_k(z) \leq 0$ for $z \in [-v_k, 0]$. Therefore $\alpha_k \leq 0$, and we know that $\alpha_k = 0$ yields a valid inequality. Thus also in this case $\alpha_k = 0$. $\square$

We use this theorem later with respect to the knapsack cover inequalities in Section 3.3.

3 Lifting of knapsack covers

In this section we introduce the knapsack cover inequality, which has the same form as the standard cover inequality for the knapsack problem, see e.g. Nemhauser and Wolsey (1988). We then develop sequence independent and dependent lifting procedures for these inequalities.

3.1 Knapsack cover inequality

Consider $C \subset N_1^+$ such that $\lambda = \sum_{i\in C} b_i - d > 0$ and $\sum_{i\in C\setminus\{j\}} b_i \le d$ for any $j \in C$. We first generalize the knapsack cover inequalities.

Theorem 3. *The knapsack cover inequality*

$$\sum_{i\in C} y_i \le |C| - 1 \tag{10}$$

is facet-defining for P^0 with $C^x = C^y = C, L_u = L_1 = \emptyset$, i.e. all variables not in C are projected to 0, if and only if there exists a $j \in C$ with $\sum_{i\in C\setminus\{j\}} b_i < d$, and either $u_j > 0$ or there exists $l \in C \setminus \{j\}$ with $\sum_{i\in C\setminus\{l\}} b_i < d$.

Proof. We denote the convex hull of P^0 by P^C. It is easy to see that (10) is valid for P^C.

To prove that (10) is a facet-defining inequality under the stated conditions, we construct $2|C|$ affinely independent vectors in P^C satisfying (10) at equality. By assumption there exists $j \in C$ such that $\sum_{i\in C\setminus\{j\}} b_i < d$. Without loss of generality assume that $j = 1$ and $C = \{1, \dots, |C|\}$.

First, let us consider the case when $u_1 > 0$ and consider vectors

$$\begin{array}{rcl}
z_1 & = & (\varepsilon \quad 0 \quad \dots \quad 0 \quad 0 \qquad 0 \quad 1 \quad \dots \quad 1 \quad 1) \\
 & & (0 \quad \varepsilon \quad \dots \quad 0 \quad 0 \qquad 0 \quad 1 \quad \dots \quad 1 \quad 1) \\
 & & (\vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \qquad \vdots \quad \ddots \quad \ddots \quad \vdots \quad \vdots) \\
 & & (0 \quad 0 \quad \dots \quad \varepsilon \quad 0 \qquad 0 \quad 1 \quad \iddots \quad 1 \quad 1) \\
 & & (0 \quad 0 \quad \dots \quad 0 \quad \varepsilon \qquad 0 \quad 1 \quad \dots \quad 1 \quad 1) \\
 & & (0 \quad 0 \quad \dots \quad 0 \quad 0 \qquad 1 \quad 1 \quad \dots \quad 1 \quad 0) \\
 & & (0 \quad 0 \quad \dots \quad 0 \quad 0 \qquad 0 \quad 1 \quad \dots \quad 1 \quad 1) \\
 & & (\vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \qquad \vdots \quad \ddots \quad \ddots \quad \vdots \quad \vdots) \\
 & & (0 \quad 0 \quad \dots \quad 0 \quad 0 \qquad 1 \quad 1 \quad \iddots \quad 1 \quad 1) \\
z_{2|C|} & = & (\underbrace{0 \quad 0 \quad \dots \quad 0 \quad 0}_{x} \qquad \underbrace{1 \quad 1 \quad \dots \quad 0 \quad 1}_{y}),
\end{array}$$

where $\varepsilon = \min\left\{u_1, d - \sum_{i\in C\setminus\{1\}} b_i, \min_{i\ge 2}\{u_i \pm v_i\}\right\} > 0$. These vectors are feasible, affinely independent and satisfy (10) at equality. Therefore (10) is a facet-defining inequality.

Next, let us consider the case when $u_1 = 0$. By assumption, there exists $l \in C\setminus\{1\}$ such that $\sum_{i\in C\setminus\{l\}} b_i < d$. In this case consider the same vectors, except that instead of z_1 we take $(\epsilon e_1, \mathbf{1} - e_l)$, where $\mathbf{1} = (1, \dots, 1)$ and $\varepsilon = \min\left\{d - \sum_{i\in C\setminus\{1\}} b_i, d - \sum_{i\in C\setminus\{l\}} b_i, \min_{i\ge 1}\{u_i \pm v_i\}\right\} > 0$.

Finally, we show that (10) is not facet-defining if neither of the conditions stated in the theorem hold. Consider a feasible point $(\hat{x}, \hat{y})$ such that it satisfies (10) at equality. If $\sum_{i\in C\setminus\{j\}} b_i = d$ for all $j \in C$, then $\sum_{i\in C} b_i \hat{y}_i = d$, and therefore $\sum_{i\in C} \hat{x}_i = 0$. Thus in this case $\bar{x}_j = 0$ for any $j \in C$. This shows that there exists a $j \in C$ such that $d - \sum_{i\in C\setminus\{j\}} b_i > 0$. If there exists a unique $k \in C$ such that $\sum_{i\in C\setminus\{k\}} b_i < d$ and $u_k = 0$, then $\hat{x}_k = 0$. In both cases (10) implies $x_k = 0$, hence (10) is not facet-defining inequality. $\square$

Example. Let P be given by

$$\begin{aligned}
x_1 + {} & x_2 + x_3 + x_4 + x_5 - x_6 - x_7 - x_8 + \\
& 4y_1 + 4y_2 + y_3 + 6y_4 + 2y_5 + 9y_6 + 7y_7 + 2y_8 \le 12 \\
& 0 \le x_1 \le 3 + 2y_1 \qquad 0 \le x_4 \le 3 - 2y_4 \qquad 0 \le x_7 \le 5 - 4y_7 \\
& 0 \le x_2 \le 5 - y_2 \qquad 0 \le x_5 \le 5 + 2y_5 \qquad 0 \le x_8 \le 3 - y_8. \\
& 0 \le x_3 \le 4 - 3y_3 \qquad 0 \le x_6 \le 2 + y_6
\end{aligned}$$

Thus $N_1^+ = \{1,2,3,4,5\}$, $N_1^- = \{6,7,8\}$, $N_2^+ = \{1,5,6\}$, and $N_2^- = \{2,3,4,7,8\}$. Consider $C = \{1,2,4,5\}$, which is a cover, since $\sum_{i\in C} b_i - d = 4+4+6+2-12 = 4 > 0$. Note that $\sum_{i\in C\setminus\{4\}} b_i = 10 < 12$ and $u_4 > 0$. Therefore, by Theorem 3

$$y_1 + y_2 + y_4 + y_5 \le 3 \tag{11}$$

is a facet-defining inequality for P^C. □

Since (10) is valid only for P^C, we need to lift it. We first consider sequence independent lifting and then we elaborate on sequence dependent lifting.

3.2 Sequence independent lifting

One of the techniques used to construct valid inequalities for a given polyhedron is sequence independent lifting, see Wolsey (1977) and Gu *et al.* (2000). In sequence independent lifting, variables (x_i, y_i) are lifted simultaneously and the lifting order does not matter. In order to lift pairs simultaneously, we must impose $C^x = C^y$, which for knapsack cover inequalities equals to C, and we assume that $L_u = L_1 = \emptyset$.

We start with (2), which is valid for P^0. We need to choose (α_i, β_i) for every $i \in N \setminus C$ such that the lifted inequality (3) is valid for P. In order to do so, we introduce the functions

$$h_i(z) = \max\left\{\alpha_i(x-\bar{x}_i) + \beta_i(y-\bar{y}_i) : \delta_i(x-\bar{x}_i) + b_i y = z, 0 \le x \le u_i \pm v_i y, y \text{ binary}\right\},$$

and

$$f(z) = \min\left\{\alpha_0 - \sum_{i\in C}(\alpha_i x_i + \beta_i y_i) : \sum_{i\in C}(\delta_i x_i + b_i y_i) \le d - z, 0 \le x_i \le u_i \pm v_i y_i, i \in C, y \text{ binary}\right\}.$$

The following theorem from Gu *et al.* (1999) provides a way to obtain the lifting coefficients α_i and β_i from functions $h_i(z)$ and $f(z)$ when $f(z)$ is superadditive. A function f is *superadditive* on Z if $f(z_1) + f(z_2) \le f(z_1 + z_2)$ for all z_1, z_2, $z_1 + z_2 \in Z$.

Theorem 4. Assume that (2) is valid for P^0 and that (α_i, β_i) are chosen in such a way that $h_i(z) \le f(z)$ for any z where both functions are defined, and any $i \in L_l = L_0$. Assume also that $f(z)$ is superadditive. Then (3) is valid for P. □

If f is not superadditive, Gu *et al.* (1999) prove that it is sufficient to find a superadditive function g such that $g(z) \le f(z)$ for all z, and use the inequality $h_i(z) \le g(z)$ to find values for (α_i, β_i). There might be many functions that satisfy these conditions. To obtain the strongest inequality we choose a *non-dominated* $g(z)$, which means that there exist no $g', g' \neq g$ such that g' is superadditive and $g(z) \le g'(z) \le f(z)$ for every z. In addition, the strongest inequalities are obtained by choosing (α_i, β_i) in such a way that $h_i(z) = g(z)$ for at least two distinct z values (see Gu *et al.* (1999) for details).

3.2.1 The lifting function

The lifting function $f(z), z \le d$ for the knapsack cover inequality is given by

$$\begin{aligned}
f(z) \quad = \quad \min \quad & \Big(|C| - 1 - \sum_{i\in C} y_i\Big) \\
\text{s.t.} \quad & \sum_{i\in C} x_i + \sum_{i\in C} b_i y_i \le d - z \\
& 0 \le x_i \le u_i \pm v_i y_i \qquad i \in C \\
& y \text{ binary.}
\end{aligned}$$

Since variables x are not present in the objective function we eliminate them from the problem and thus obtain the pure integer knapsack case. Gu *et al.* (2000) completely characterize $f(z)$ when $z \ge 0$. To express

$f(z)$ in a closed form, let $|C| = r$, and let us reorder the variables in C so that $b_1 \geq b_2 \geq \ldots \geq b_r$. Let also $\mu_i = d - \sum_{i+1 \leq j \leq r} b_i$ for $i = 0, \ldots, r$. Note that $\mu_0 = -\lambda$. Then

$$f(z) = \begin{cases} -1 & z \leq \mu_0 \\ j & \mu_j < z \leq \mu_{j+1}, \quad j = 0, \ldots, r-1. \end{cases}$$

Function $f(z)$ is not superadditive. To construct a superadditive valid lifting function $g(z)$ we use the function used by Gu *et al.* (2000) and extend it for negative z. For $z \leq 0$ we apply the idea developed in Shebalov and Klabjan (2006), i.e. we repeat $g(z)$ for $z \geq 0$, shifting it down by $f(d)$. Thus

$$g(z) = \begin{cases} j - (\mu_j + \rho_j - z)/\rho_1 & \mu_j \leq z < \mu_j + \rho_j, \quad j = 1, \ldots, r-1 \\ j & \mu_j + \rho_j \leq z < \mu_{j+1}, \quad j = 1, \ldots, r-1 \\ g(z + \mu_{r-1} + \rho_{r-1}) - (r-1) & \mu_j - \mu_{r-1} \leq z < \mu_{j+1} - \mu_{r-1} - \rho_{r-1}, \quad j = 1, \ldots, r-1 \\ & \text{or } -\mu_{r-1} - 2\rho_{r-1} \leq z < \mu_1 - \mu_{r-1} - \rho_{r-1} \\ j - r + \frac{z - \mu_j + \mu_{r-1} + \rho_{r-1}}{\rho_1} & \mu_j - \mu_{r-1} - \rho_{r-1} \leq z < \mu_j - \mu_{r-1}, \quad j = 1, \ldots, r-1 \\ g\left(z + t(\mu_{r-1} + \rho_{r-1})\right) - (r-1)t & -\mu_{r-1} - 2\rho_{r-1} - t(\mu_{r-1} + \rho_{r-1}) \leq z < -\rho_{r-1} - t(\mu_{r-1} + \rho_{r-1}) \\ & \text{for } t = 1, 2, 3, \ldots, \end{cases}$$

where $\rho_j = \max\{0, b_{j+1} - b_1 + \lambda\}$ for $j = 0, \ldots, r-1$, see Figure 1. It is proven in Gu *et al.* (2000) that $g(z)$ is a superadditive valid lifting function for $f(z)$ for all $z \geq 0$ and that is nondominated and maximal for $z \geq 0$. For negative z we have $f(z) \geq -1$, and therefore $g(z) \leq f(z)$ for all $z \leq d$. Function $g(z)$ is superadditive for all z, since it is constructed identically to $g(z)$ in Shebalov and Klabjan (2006).

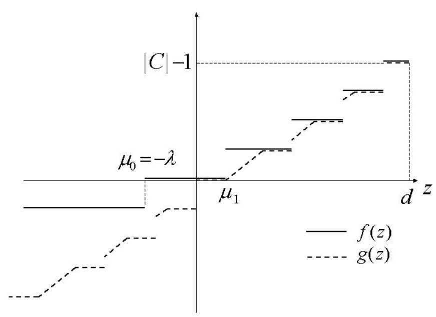

Figure 1: $f(z)$ and $g(z)$ for knapsack cover inequality in the case $\rho_{r-1} = 0$

3.2.2 The lifted knapsack cover inequality

We now present the inequality obtained by sequence independent lifting of (10).

Theorem 5. If

a) $\{i \in N_1^- : u_i > 0\} = \emptyset$ or

b) $b_2 = \ldots = b_r$, and $\{i \in N_1^- : u_i > 0\} \neq \emptyset$,

then the *lifted knapsack cover inequality*

$$\sum_{i \in C} y_i + \sum_{\substack{i \in N_1^+ \setminus C \\ u_i > 0}} g(b_i) y_i - \frac{1}{\rho_1} \sum_{\substack{i \in N_1^- \setminus C \\ u_i > 0}} x_i + \sum_{\substack{i \in N_1^- \setminus C \\ u_i > 0}} g(b_i) y_i + \sum_{\substack{i \in N \setminus C \\ u_i = 0}} (\alpha_i x_i + \beta_i y_i) \leq |C| - 1\,, \tag{12}$$

where $(\alpha_i, \beta_i) \in J_i$, is a valid inequality for P. The lifting sets J_i are defined in the online appendix. [1] $\square$

[1] http://www.klabjan.dynresmanagement.com/articles/Sequence dependent lifting online appendix.pdf

Proof. To obtain lifting coefficients we need to consider cases $u_i > 0$ and $u_i = 0$ separately.

Consider first the lifting coefficients for variables with positive constant bound. Shebalov and Klabjan (2006) show that for $u_i > 0$, $h_i(z)$ is given by

$$h_i(z) = \begin{cases} \max\left(\varphi_{\alpha_i}^{[0,u_i]}, \psi_{\alpha_i}^{[b_i,u_i+v_i+b_i]}\right) & \text{for } i \in N_1^+ \cap N_2^+ \\ \max\left(\varphi_{\alpha_i}^{[0,u_i]}, \psi_{\alpha_i}^{[b_i,u_i-v_i+b_i]}\right) & \text{for } i \in N_1^+ \cap N_2^- \\ \max\left(\varphi_{-\alpha_i}^{[-u_i,0]}, \psi_{-\alpha_i}^{[b_i-u_i-v_i,b_i]}\right) & \text{for } i \in N_1^- \cap N_2^+ \\ \max\left(\varphi_{-\alpha_i}^{[-u_i,0]}, \psi_{-\alpha_i}^{[b_i-u_i+v_i,b_i]}\right) & \text{for } i \in N_1^- \cap N_2^-, \end{cases}$$

where

$$\varphi_{\alpha_i}^I(z) = \begin{cases} \alpha_i z & z \in I \\ -\infty & z \notin I \end{cases}$$

$$\psi_{\alpha_i}^I(z) = \begin{cases} \alpha_i(z - b_i) + \beta_i & z \in I \\ -\infty & z \notin I. \end{cases}$$

To obtain lifting coefficients (α_i, β_i) we consider three cases.

1. If $i \in N_1^+$ (see Figure 2), then $\alpha_i z \le h_i(z) \le g(z) = 0$ for $0 \le z \le \mu_1$, and therefore $\alpha_i = 0$. We obtain the value of β_i from the condition $h_i(z) = \beta_i \le g(z)$, which has to be satisfied for $b_i \le z \le u_i \pm v_i + b_i$. We obtain $\beta_i = g(b_i)$, which corresponds to the second term in (12).

2. If $i \in N_1^-$ and there exist $i_1, i_2 \in C$ such that $i_1 \neq 1$, $i_2 \neq 1$ and $b_{i_1} \neq b_{i_2}$, then $-\alpha_i z \le h_i(z) \le g(z) = -1$ for $-\delta \le z < 0$, where $\delta > 0$ is small enough. This behavior of $g(z)$ around 0 follows from the definition of i_1, i_2, see also Figure 2. Therefore lifting is not possible in this case and we require (a) and (b) in Theorem 5.

3. If $i \in N_1^-$ and $b_2 = \ldots = b_r$, then lifting is possible, since $g(z)$ is continuous at $z = 0$ (see Figure 3). In this case $\alpha_i = -1/\rho_1$ and $\beta_i = g(b_i)$, which corresponds to the third and fourth terms in (12).

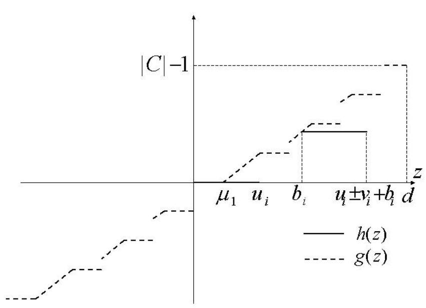

Figure 2: $g(z)$ and $h(z)$ for $i \in N_1^+$ and $u_i > 0$

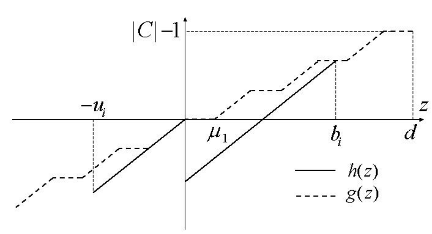

Figure 3: $g(z)$ and $h(z)$ for $i \in N_1^-$, $u_i > 0$ and $b_2 = b_3 = \ldots = b_r$

We next study the lifting coefficients for variables with zero constant bound. These coefficients correspond to the fifth term in (12). In this case

$$h(z) = \begin{cases} 0 & z = 0 \\ \alpha_i(z - b_i) + \beta_i & z \in [b_i, b_i + v_i] \end{cases}$$

for $i \in N_1^+$ and

$$h(z) = \begin{cases} 0 & z = 0 \\ -\alpha_i(z - b_i) + \beta_i & z \in [b_i - v_i, b_i] \end{cases}$$

for $i \in N_1^-$, see Figure 4 and Figure 5.

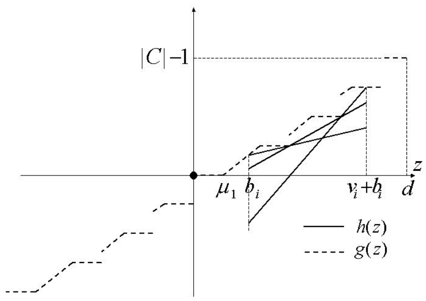

Figure 4: $g(z)$ and $h(z)$ for $i \in N_1^+$, $u_i = 0$

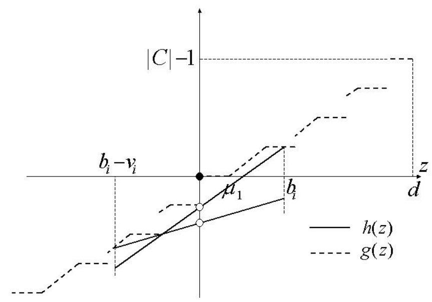

Figure 5: $g(z)$ and $h(z)$ for $i \in N_1^-$, $u_i = 0$

Similar to Shebalov and Klabjan (2006), there are several possible optimal values for (α_i, β_i). The derivation of J_i is omitted since it follows closely Shebalov and Klabjan (2006).

$\square$

3.3 Sequence dependent lifting

In this section we use sequence dependent lifting to lift (10). We argued in Section 2.1 that computing αs is very difficult due to integrality and nonlinearity, and for this reason we do not consider an arbitrary lifting sequence. We assume that first the continuous variables are lifted and then all of the binary variables. The order within these two sets is arbitrary. For continuous variables we are able to explicitly derive αs while for binary variables the dynamic program from Section 2.2 needs to be employed.

3.3.1 Lifting of continuous variables

We first derive $\alpha_i, i \in N_1^+$. Since (10) is not a multiple of $\sum_{i\in C} x_i + \sum_{i \in C} b_i y_i \leq d$ and is a facet defining inequality, we can apply Theorem 2. Hence, we conclude $\alpha_i = 0$ for every $i \in N_1^+$.

Now we explicitly derive α_i for $i \in N_1^-$. For ease of exposition we assume $N_1^- = \{1, \dots, |N_1^-|\}$ and that this is the lifting order within N_1^-. Let $j \geq 1$ be the index such that $\sum_{1 \leq i \leq j-1} u_i < \lambda \leq \sum_{1 \leq i \leq j} u_i$ (we define $\sum_{1\leq i \leq 0} u_i = 0$ and $\sum_{1 \leq i \leq N_1^-} u_i = \infty$). For $k \in N_1^-$ let

$$\begin{array}{rll} h_k(z) = & \max & \alpha_k x_k \\ & \text{s.t.} & -x_k = z \\ & & 0 \leq x_k \leq u_k \end{array}$$

and

$$\begin{array}{rllcll} f_k(z) = & \min & \left(|C| - 1 - \sum_{i \in C} y_i + \sum_{i=1}^{k-1} \alpha_i x_i\right) & = & \min & \left(|C| - 1 - \sum_{i\in C} y_i + \sum_{i=1}^{k-1} \alpha_i x_i\right) \\ & \text{s.t.} & \sum_{i\in C} x_i - \sum_{i=1}^{k-1} x_i + \sum_{i\in C} b_i y_i \leq d - z & & \text{s.t.} & -\sum_{i=1}^{k-1} x_i + \sum_{i \in C} b_i y_i \leq d - z \\ & & 0 \leq x_i \leq u_i \pm v_i y_i \qquad i \in C & & & 0 \leq x_i \leq u_i \quad i = 1, \dots, k-1 \\ & & 0 \leq x_i \leq u_i \quad i = 1, \dots, k-1 & & & y \text{ binary.} \\ & & y \text{ binary} & & & \end{array}$$

We obtain the lifting coefficient α_i by induction.

1. $\alpha_1 = 0$ if $j > 1$, and $\alpha_1 = -\frac{1}{\lambda}$ if $j = 1$.

   We start from (10), which is facet-defining for P^C, and consider

$$\alpha_1 x_1 + \sum_{i \in C} y_i \le |C| - 1, \tag{13}$$

   which has to be valid for

$$\begin{array}{ll} \sum_{i \in C} x_i - x_1 + \sum_{i \in C} b_i y_i \le d & \\ 0 \le x_i \le u_i \pm v_i y_i & i \in C \\ 0 \le x_1 \le u_1 & \\ y \text{ binary.} & \end{array}$$

   If (13) is to be valid, then $h_1(z) \le f_1(z)$. In this case $h_1(z) = -\alpha_1 z$ for $z \in [-u_1, 0]$, and $f_1(z) = |C| - 1 - \max\left\{\sum_{i \in C} y_i : \sum_{i \in C} b_i y_i \le d - z,\ y \text{ binary}\right\}$, which is -1 for $z \le -\lambda$ and 0 for $-\lambda < z \le 0$, see Figure 6. If $\lambda > u_1$, then the largest $h_1(z) = 0$, and therefore $\alpha_1 = 0$. If $\lambda \le u_1$, then $\alpha_1 = -\frac{1}{\lambda}$.

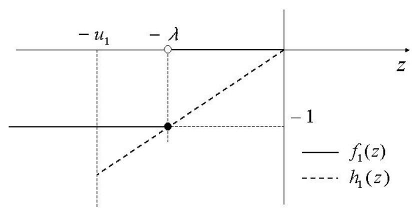

Figure 6: $h_1(z)$ and $f_1(z)$ if $\lambda \le u_1$

2. If $\alpha_1 = \ldots = \alpha_{k-1} = 0$, then $\alpha_k = 0$ for $k < j$.

   In this case we want

$$\alpha_k x_k + \sum_{i \in C} y_i \le |C| - 1$$

   to be valid for

$$\begin{array}{ll} \sum_{i \in C} x_i - \sum_{i=1}^{k} x_i + \sum_{i \in C} b_i y_i \le d & \\ 0 \le x_i \le u_i \pm v_i y_i & i \in C \\ 0 \le x_i \le u_i & i = 1, \ldots, k \\ y \text{ binary.} & \end{array}$$

   As before we have $h_k(z) = -\alpha_k z$ for $z \in [-u_k, 0]$, and since $\alpha_i = 0$ for all $i = 1, \ldots, k-1$, variables x_i do not affect the value of $f_k(z)$, therefore we eliminate them from the problem by setting them to their upper bounds. As a result we have

$$\begin{array}{llll} f_k(z) = & |C| - 1 & - & \max \quad \sum_{i \in C} y_i \\ & & & \text{s.t.} \quad \sum_{i \in C} b_i y_i \le d + \sum_{i=j}^{k-1} u_i - z \\ & & & \qquad y \text{ binary.} \end{array}$$

For $-u_k \le z \le 0$ we have

$$d + \sum_{i=1}^{k-1} u_i - z \le \sum_{i \in C} b_i - \lambda + \sum_{i=1}^{k} u_i < \sum_{i \in C} b_i \,,$$

where the last inequality follows from $k < j$. Therefore at an optimal solution $y_i = 0$ for at least one $i \in C$. We conclude that $f_k(z) \ge 0$ and thus $\alpha_k = 0$.

3. $\alpha_j = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$.

   Again we obtain α_j from the inequality $h_j(z) \le f_j(z)$, which in this case gives

$$-\alpha_j z \le f_j(z) = \begin{cases} 0 & -\lambda + \sum_{i=1}^{j-1} u_i < z \le 0 \\ -1 & -\infty < z \le -\lambda + \sum_{i=1}^{j-1} u_i. \end{cases}$$

   The expression for $f_j(z)$ is obtained by using the similar argument as in Case 1. Thus $\alpha_j = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$. Note that this case is consistent with Case 1, since if $j = 1$, then $\sum_{i=1}^{j-1} u_i = 0$.

4. If $\alpha_j = \alpha_{j+1} = \ldots = \alpha_{k-1} = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$, then $\alpha_k = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$.

   We want

$$\alpha_k x_k + \sum_{i=j}^{k-1} \alpha_i x_i + \sum_{i \in C} y_i \le |C| - 1$$

   to be valid for

$$\begin{array}{ll} \sum_{i \in C} x_i - \sum_{i=1}^{k} x_i + \sum_{i \in C} b_i y_i \le d & \\ 0 \le x_i \le u_i \pm v_i y_i & i \in C \\ 0 \le x_i \le u_i & i = 1, \ldots, k \\ y \text{ binary,} & \end{array}$$

   where we used that $\alpha_i = 0$ for $i \le j - 1$. We have, after fixing $x_i = u_i$ for $i = 1, \ldots, j-1$,

$$\begin{array}{llll} f_k(z) = & |C| - 1 & - & \max \quad \left(\sum_{i \in C} y_i + \alpha_j \sum_{i=j}^{k-1} x_i \right) \\ & & & \text{s.t.} \quad -\sum_{i=j}^{k-1} x_i + \sum_{i \in C} b_i y_i \le d + \sum_{i=1}^{j-1} u_i - z \\ & & & \qquad 0 \le x_i \le u_i \pm v_i y_i \qquad\qquad i \in C \\ & & & \qquad 0 \le x_i \le u_i \qquad\qquad i = j, \ldots, k-1 \\ & & & \qquad y \text{ binary.} \end{array}$$

   We consider several ranges for z.

   (a) For $-u_k \le z \le -\lambda$, we have $\sum_{i \in C} b_i = d + \lambda \le d - z \le d - z + \sum_{i=1}^{j-1} u_i$. Therefore $x_i = 0$ for $j \le i \le k-1$ and $y_i = 1$ for every $i \in C$ is a feasible solution to f_k. It gives an objective value of -1 and therefore $f_k(z) \ge -1$.

   (b) Let now $-\lambda \le z \le 0$. If $\sum_{i=j}^{k-1} x_i \ge \lambda - \sum_{i=1}^{j-1} u_i + z$, then $-\sum_{i=j}^{k-1} x_i + \sum_{i \in C} b_i \le -z + \sum_{i=1}^{j-1} u_i + d$ and therefore at the optimal solution $y_i = 1$ for any $i \in C$. Therefore $f_k(z) = -1 - \alpha_j\big(\lambda - \sum_{i=1}^{j-1} u_i + z\big)$.

(c) If $\sum_{i=j}^{k-1} x_i < \lambda - \sum_{i=1}^{j-1} u_i + z$, then $-\sum_{i=j}^{k-1} x_i + \sum_{i\in C} b_i > -z + \sum_{i=1}^{j-1} u_i + d$ and therefore in an optimal solution at least one $y_i = 0, i \in C$. Therefore $f_k(z) \geq 0$.

Thus the condition $h_k(z) \leq f_k(z)$ gives $-\alpha_k z \leq -1 + \frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}\left(\lambda - \sum_{i=1}^{j-1} u_i\right) + \frac{1}{\lambda - \sum_{i=1}^{j-1} u_i} z$, and therefore $\alpha_k = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$.

To summarize, $\alpha_k = 0$ for $k \leq j-1$ and $\alpha_k = -\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i}$ for $k \geq j$.

3.3.2 Lifting of binary variables

We are not able to obtain a closed form expression for the lifting coefficients of binary variables. We show how to compute βs by the dynamic program from Section 2.2 if all continuous variables are lifted first. Let us assume that we start with (10) and y_i for $i \in I_y$ have already been lifted. Then

$$-\frac{1}{\lambda - \sum_{i=1}^{j-1} u_i} \sum_{i=j}^{|N_1^-|} x_i + \sum_{i\in C} y_i + \sum_{i \in I_y} \beta_i y_i \leq |C| - 1$$

is valid when $y_i = 0$ for every $i \in N \setminus I_y$. From Section 2.1, $\beta_k, k \in N \setminus I_y$ is determined by

$$\begin{array}{lll}
\beta_k = & \min & \left(|C| - 1 + \frac{1}{\lambda - \sum_{i=1}^{j-1} u_i} \sum_{i=j}^{|N_1^-|} x_i - \sum_{i\in C} y_i - \sum_{i\in I_y} \beta_i y_i\right) \\
& \text{s.t.} & -\sum_{i\in N_1^-} x_i + \sum_{i \in C\cup I_y} b_i y_i \leq d - b_k \\
& & 0 \leq x_i \leq u_i \pm v_i y_i \qquad\qquad i \in I_y \\
& & 0 \leq x_i \leq u_i \qquad\qquad i \in N_1^- \setminus (I_y \cup \{k\}) \\
& & 0 \leq x_k \leq u_k \pm v_k \qquad\qquad \text{if } k \in N_1^- \\
& & y \text{ binary.}
\end{array}$$

This problem can be easily transformed to (5), and therefore it can be solved by the dynamic programming algorithm presented in Theorem 1.

Example (continued). Here we show the procedure of lifting (11). First we lift the continuous variables x_3, x_6, x_7, and x_8, and then the binary variables in the order y_3, y_6, y_7, and y_8. Variable x_3 has zero lifting coefficient by Theorem 2. Since $\lambda = 4$ we have $j = 2$, and $\alpha_6 = 0$, $\alpha_7 = \alpha_8 = -\frac{1}{4-2} = -\frac{1}{2}$. Value β_3 is defined by the following optimization problem

$$\begin{array}{lll}
\beta_3 = & \min & 3 + \frac{1}{2}x_7 + \frac{1}{2}x_8 - y_1 - y_2 - y_4 - y_5 \\
& \text{s.t} & 4y_1 + 4y_2 + 6y_4 + 2y_5 - x_6 - x_7 - x_8 \leq 11 \\
& & 0 \leq x_7 \leq 5 \\
& & 0 \leq x_8 \leq 3 \\
& & y \text{ binary,}
\end{array}$$

which has a solution $\beta_3 = 0$. Repeating this procedure in the selected order we obtain $\beta_6 = 1$, $\beta_7 = 1$ and $\beta_8 = 0$. Thus the lifted inequality

$$y_1 + y_2 + y_4 + y_5 + y_6 + y_7 - \frac{1}{2}x_7 - \frac{1}{2}x_8 \leq 3$$

is valid for P.

If order x_7, x_8, x_6, x_3 is selected, then $j = 1$, and hence $\alpha_7 = \alpha_8 = \alpha_6 = \frac{1}{4}, \alpha_3 = 0$. □

4 Single binary variable polytope

As discussed in Section 2, we do not know an effective procedure to compute the lifting coefficients of continuous variables. One possible approach is not to fix them. In order to be able to obtain a valid inequality (2), all but one binary variables are projected. We give facets of the underlying polytope and we show that they completely describe the underlying convex hull.

For ease of exposition, we assume that index 1 is the only non fixed binary variable. Thus $C^x = N$ and $C^y = \{1\}$. The resulting projection P^0 reads

$$\begin{aligned}
&\sum_{i \in N_1^+} x_i - \sum_{i \in N_1^-} x_i + b_1 y_1 \le d - \sum_{i \in L_1} b_i \\
&0 \le x_1 \le u_1 \pm v_1 y_1 \\
&0 \le x_i \le u_i \pm v_i \bar{y}_i \qquad i \in N \setminus \{1\} \\
&y_1 \text{ binary.}
\end{aligned}$$

If we denote $b'_1 = b_1$, $d' = d - \sum_{i \in L_u^y} b_i$, $u'_1 = u_1$, $v'_1 = v_1$, $u'_i = u_i \pm v_i \bar{y}_i$ for $i \in N \setminus \{1\}$, then the above set is equivalent to

$$\sum_{i \in N_1^+} x_i - \sum_{i \in N_1^-} x_i + b'_1 y_1 \le d' \tag{14a}$$

$$0 \le x_1 \le u'_1 \pm v'_1 y_1 \tag{14b}$$

$$0 \le x_i \le u'_i \qquad i \in N \setminus \{1\} \tag{14c}$$

$$y_1 \text{ binary.} \tag{14d}$$

Let P^1 be the convex hull of the set described by (14a)-(14d). We are only interested in the case when P^1 is full-dimensional. Hence we assume that $u'_i > 0$ for $i \ne 1$, $u'_1 + v'_1 > 0$ if $1 \in N_2^+$ and either $u'_1 > 0$ or $u'_1 - v'_1 > 0$ if $1 \in N_2^-$. In addition, it is easy to see that for full-dimensionality we also need $\sum_{i \in N_1^+} u'_i > d'$, and $-\sum_{i \in N_1^-} u'_i + b'_1 < d'$ if $1 \in N_1^+$, and $-\sum_{i \in N_1^-} u'_i \mp v'_1 + b'_1 < d'$ if $1 \in N_1^-$.

If $v'_1 = 0$, then the resulting polytope has been studied in Atamtürk *et al.* (2001), who also give a complete polyhedral description. Magnanti *et al.* (1993) studied the same polytope with $v'_1 = 0$, $d' = 0$, $b'_1 < 0$ and y_1 integer, in the context of network design problems. The main result of this section is to identify a family of facets of P^1 and to show that they completely describe this polytope. Throughout this section we consider the following example.

Example. Let P be given by

$$\begin{aligned}
&x_1 + x_2 + x_3 - x_4 - x_5 + 5y_1 + 2y_2 + 3y_3 + 2y_4 + 4y_5 \le 14 \\
&\quad 0 \le x_1 \le 3 - y_1 \qquad\qquad 0 \le x_4 \le 2 + y_4 \\
&\quad 0 \le x_2 \le 2 + 2y_2 \qquad\qquad 0 \le x_5 \le 5 - 3y_5 \\
&\quad 0 \le x_3 \le 4 - 3y_3 \\
&\quad y \text{ binary.}
\end{aligned}$$

Thus $N_1^+ = \{1, 2, 3\}$ and $N_1^- = \{4, 5\}$. We fix y_2 and y_3 to 0, and y_4 and y_5 to 1. Then the resulting polytope P^1 is described by

$$\begin{aligned}
&x_1 + x_2 + x_3 - x_4 - x_5 + 5y_1 \le 8 \\
&\quad 0 \le x_1 \le 3 - y_1 \qquad\qquad 0 \le x_4 \le 3 \\
&\quad 0 \le x_2 \le 2 \qquad\qquad 0 \le x_5 \le 2 \\
&\quad 0 \le x_3 \le 4 \\
&\quad y_1 \text{ binary,}
\end{aligned}$$

and thus $1 \in N_1^+ \cap N_2^-$. $\square$

4.1 Facet-defining inequalities

Consider $C = C^+ \cup C^-$, where $C^+ \subseteq N_1^+$, $C^+ \neq \emptyset$ and $C^- \subseteq N_1^-$. We define

$$\lambda = \sum_{i \in C^+} u_i' - \sum_{i \in N_1^- \setminus C^-} u_i' + b_1' - d'.$$

The following theorem gives facet-defining inequalities of P^1.

Theorem 6. If $1 \in \left(N_1^- \setminus C^-\right) \cup C^+$ and $\mp v_1' < \lambda < b_1'$, or $1 \in \left(N_1^+ \setminus C^+\right) \cup C^-$ and $0 < \lambda < b_1'$, then

$$\sum_{i \in C^+} x_i - \sum_{i \in C^-} x_i + (\lambda \mp \delta v_1') y_1 \leq \sum_{i \in C^+} u_i' \tag{15}$$

is facet-defining for P^1. Here $\delta = 1$ if $1 \in N_1^- \setminus C^-$ and $\delta = 0$ otherwise.

Proof. First we show that (15) is valid. Let $(\tilde{x}, \tilde{y}_1)$ be a vector satisfying (14a)-(14d). If $\tilde{y}_1 = 0$, then (15) is equivalent to $\sum_{i \in C^+} \tilde{x}_i - \sum_{i \in C^-} \tilde{x}_i \leq \sum_{i \in C^+} u_i'$, which follows from (14c). If $\tilde{y}_1 = 1$, then

$$\begin{aligned} \sum_{i \in C^+} \tilde{x}_i - \sum_{i \in C^-} \tilde{x}_i + \sum_{i \in C^+} u_i' - \sum_{i \in N_1^- \setminus C^-} u_i' \mp \delta v_1' + b_1' - d' &\leq \sum_{i \in C^+} \tilde{x}_i - \sum_{i \in C^-} \tilde{x}_i - \sum_{i \in N_1^- \setminus C^-} \tilde{x}_i + b_1' - d' + \sum_{i \in C^+} u_i' \\ &\leq \sum_{i \in N_1^+} \tilde{x}_i - \sum_{i \in N_1^-} \tilde{x}_i + b_1' - d' + \sum_{i \in C^+} u_i' \leq \sum_{i \in C^+} u_i'. \end{aligned}$$

Since the first term equals to the left hand side of (15), this shows the claim.

Next we show that (15) is facet-defining. By assumption P^1 is full-dimensional, we must have a feasible solution with $y_1 = 1$. Hence,

$$-\sum_{i \in N_1^-} u_i' + b_1' - d' < 0 \text{ if } 1 \in N_1^+ \quad \text{and} \quad -\sum_{i \in N_1^-} u_i' \mp v_1' + b_1' - d' < 0 \text{ if } 1 \in N_1^-. \tag{16}$$

If $1 \in \left(N_1^- \setminus C^-\right) \cup C^+$, then by assumption $\lambda > \mp v_1'$ and hence by definition of λ we have

$$\sum_{i \in C^+} u_i' - \sum_{i \in N_1^- \setminus C^-} u_i' + b_1' - d' \pm v_1' > 0. \tag{17}$$

If $1 \in \left(N_1^+ \setminus C^+\right) \cup C^-$, then by assumption $\lambda > 0$ and again by definition of λ we have

$$\sum_{i \in C^+} u_i' - \sum_{i \in N_1^- \setminus C^-} u_i' + b_1' - d' > 0. \tag{18}$$

Inequalities (16)-(18) guarantee that the polytope described by

$$\sum_{i \in C^+} x_i - \sum_{i \in C^-} x_i = \sum_{i \in N_1^- \setminus C^-} u_i' \pm \delta v_1' - b_1' + d' \tag{19a}$$

$$0 \leq x_1 \leq u_1' \pm v_1' \qquad \text{if } 1 \in C^+ \cup C^- \tag{19b}$$

$$0 \leq x_i \leq u_i' \qquad i \in C^+ \cup C^-,\ i \neq 1 \tag{19c}$$

has dimension $|C| - 1$. Therefore there exist $k = |C|$ affinely independent vectors $\bar{x}^i$ which satisfy (19). Let

$$\varepsilon = \min_{i \in N \setminus C} \left\{ u_i', d' - \sum_{i \in C^+} u_i' + \sum_{i \in N_1^- \setminus C^-} u_i' \mp \delta v_1' \right\}.$$

Note that $\varepsilon > 0$, since $\lambda < b'_1$. Consider vectors

$$
\begin{array}{rccccccl}
z_1 = & (\varepsilon \ \dots \ 0 & u'_{p+1} \quad \dots \quad u'_q & u'_{q+1} \dots u'_r & 0 \ \dots \ 0 & 0 &) \\
& \vdots \quad \ddots \quad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots & \vdots & \\
& (0 \ \dots \ \varepsilon & u'_{p+1} \quad \dots \quad u'_q & u'_{q+1} \dots u'_r & 0 \ \dots \ 0 & 0 &) \\
& (0 \ \dots \ 0 & u'_{p+1} - \varepsilon \dots \quad u'_q & u'_{q+1} \dots u'_r & 0 \ \dots \ 0 & 0 &) \\
& \vdots \qquad \vdots & \vdots \quad \ddots \quad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots & \vdots & \\
& (0 \ \dots \ 0 & u'_{p+1} \quad \dots \quad u'_q - \varepsilon & u'_{q+1} \dots u'_r & 0 \ \dots \ 0 & 0 &) \\
& (0 \ \dots \ 0 & u'_{p+1} \quad \dots \quad u'_q & \tilde{x}^1_1 \dots\dots & \dots\dots \tilde{x}^1_k & 1 &) \\
& \vdots \qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \quad \ddots \quad \vdots & \vdots \qquad \vdots & \vdots & \\
& \vdots \qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad \vdots & \vdots \quad \ddots \quad \vdots & \vdots & \\
& (0 \ \dots \ 0 & u'_{p+1} \quad \dots \quad u'_q & \tilde{x}^k_1 \dots\dots & \dots\dots \tilde{x}^k_k & 1 &) \\
z_{n+1} = & (\underbrace{0 \ \dots \ 0}_{N_1^+ \setminus C^+} & \underbrace{u'_{p+1} \quad \dots \quad u'_q}_{N_1^- \setminus C^-} & \underbrace{u'_{q+1} \dots u'_r}_{C^+} & \underbrace{0 \ \dots \ 0}_{C^-} & \underbrace{0}_{y_1} &).
\end{array}
$$

Vectors $\tilde{x}^1, \dots, \tilde{x}^k$ are affinely independent, hence $\tilde{x}^2 - \tilde{x}^1, \dots, \tilde{x}^k - \tilde{x}^1$ are linearly independent. Thus $(\tilde{x}^2 - \tilde{u}) - (\tilde{x}^1 - \tilde{u}), \dots, (\tilde{x}^k - \tilde{u}) - (\tilde{x}^1 - \tilde{u})$ are linearly independent for an arbitrary vector $\tilde{u}$. Therefore $\tilde{x}^1 - \tilde{u}, \dots, \tilde{x}^k - \tilde{u}$ are affinely independent and in turn $(\tilde{x}^1 - \tilde{u}, 1), \dots, (\tilde{x}^k - \tilde{u}, 1)$ are linearly independent. Finally, $\hat{z}_i = z_i - z_{n+1}$ for $1 \le i \le n$ are linearly independent, and hence z_i for $1 \le i \le n+1$ are affinely independent and they all satisfy (15) at equality. Therefore (15) is facet-defining. □

Example (continued). There are 9 subsets C that satisfy conditions of Theorem 6. Together with the resulting facets they are given in Table 1. □

C^+	C^-	λ	Inequality
{1,3}	{4}	2	$x_1 + x_3 - x_4 + 2y_1 \le 7$
{2,3}	{4}	1	$x_2 + x_3 - x_4 + y_1 \le 6$
{3}	{4,5}	1	$x_3 - x_4 - x_5 + y_1 \le 4$
{1,2,3}	{4}	4	$x_1 + x_2 + x_3 - x_4 + 4y_1 \le 9$
{1,2,3}	{5}	3	$x_1 + x_2 + x_3 - x_5 + 3y_1 \le 9$
{1,2}	{4,5}	2	$x_1 + x_2 - x_4 - x_5 + 2y_1 \le 5$
{1,3}	{4,5}	4	$x_1 + x_3 - x_4 - x_5 + 4y_1 \le 7$
{2,3}	{4,5}	3	$x_2 + x_3 - x_4 - x_5 + 3y_1 \le 6$

Table 1: Facet-defining inequalities of P^1

Note that Theorem 6 does not introduce a single inequality for $1 \in C^+ \cap N_2^-$ and $v'_1 \ge b'_1$. We discuss this case separately, since the resulting inequalities have a different structure. Let

$$\mu = d' - \sum_{i \in C^+} u'_i + v'_1 + \sum_{i \in N_1^- \setminus C^-} u'_i.$$

Theorem 7. If $1 \in C^+ \cap N_2^-$ and $b'_1 < \mu < v'_1$, then

$$\sum_{i \in C^+} x_i - \sum_{i \in C^-} x_i + \mu y_1 \le d' + \sum_{i \in N_1^- \setminus C^-} u'_i \tag{20}$$

is facet-defining for P^1.

Proof. First we show that (20) is valid for P^1. Let $(\tilde{x}, \tilde{y}_1)$ be a vector satisfying (14a)-(14d). If $\tilde{y}_1 = 0$, then

$$\sum_{i\in C^+} \tilde{x}_i - \sum_{i\in C^-} \tilde{x}_i \le \sum_{i\in N^+} \tilde{x}_i - \sum_{i\in N^-} \tilde{x}_i + \sum_{i\in N_1^-\setminus C^-} u'_i \le d' + \sum_{i\in N_1^-\setminus C^-} u'_i.$$

If $\tilde{y}_1 = 1$, then

$$\begin{aligned}\sum_{i\in C^+} \tilde{x}_i - \sum_{i\in C^-} \tilde{x}_i + d' - \sum_{i\in C^+} u'_i + v'_1 + \sum_{i\in N_1^-\setminus C^-} u'_i \le& \sum_{C^+\setminus\{1\}} (\tilde{x}_i - u'_i) + (\tilde{x}_1 - u'_1 + v'_1) - \sum_{i\in C^-} \tilde{x}_i + d' + \sum_{i\in N_1^-\setminus C^-} u'_i \\ \le& \, d' + \sum_{i\in N_1^-\setminus C^-} u'_i \,.\end{aligned}$$

Since the first term equals to the left hand side of (20), this shows the claim.

Next we show that (20) is facet-defining. By assumption P^1 is full-dimensional, and therefore

$$-\sum_{i\in N_1^-} u'_i - d' < 0. \tag{21}$$

By assumption $\mu < v'_1$, and therefore by definition of μ we have

$$\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i - d' > 0. \tag{22}$$

Inequalities (21) and (22) guarantee that the polytope described by

$$\sum_{i\in C^+} x_i - \sum_{i\in C^-} x_i = \sum_{i\in N_1^-\setminus C^-} u'_i + d' \tag{23a}$$

$$0 \le x_i \le u'_i \qquad\qquad i \in C \tag{23b}$$

has dimension $|C| - 1$. Hence there exist $k = |C|$ affinely independent vectors $\tilde{x}^i$ which are feasible to (23a)-(23b). Let

$$\varepsilon = \min_{i\in N\setminus C} \{u'_i, d' - b'_1 - \sum_{i\in C^+} u'_i + v'_1 + \sum_{i\in N_1^-\setminus C^-} u'_i\}.$$

Then $\varepsilon > 0$ since $\mu > b'_1$. Consider vectors

$$\begin{array}{rcccccccccccccccc}
z_1 = & (& \varepsilon & \dots & 0 & u'_{p+1} & \dots & u'_q & u'_{q+1}-v'_{q+1} & \dots & u'_r & 0 & \dots & 0 & 1 &) \\
 & & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
 & (& 0 & \dots & \varepsilon & u'_{p+1} & \dots & u'_q & u'_{q+1}-v'_{q+1} & \dots & u'_r & 0 & \dots & 0 & 1 &) \\
 & (& 0 & \dots & 0 & \varepsilon & \dots & 0 & u'_{q+1}-v'_{q+1} & \dots & u'_r & 0 & \dots & 0 & 1 &) \\
 & & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
 & (& 0 & \dots & 0 & 0 & \dots & \varepsilon & u'_{q+1}-v'_{q+1} & \dots & u'_r & 0 & \dots & 0 & 1 &) \\
 & (& 0 & \dots & 0 & u'_{p+1} & \dots & u'_q & \tilde{x}^1_1 & \dots & \dots & \dots & \dots & \tilde{x}^1_k & 0 &) \\
 & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \\
 & & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \\
 & (& 0 & \dots & 0 & u'_{p+1} & \dots & u'_q & \tilde{x}^k_1 & \dots & \dots & \dots & \dots & \tilde{x}^k_k & 0 &) \\
z_{n+1} = & (& 0 & \dots & 0 & 0 & \dots & 0 & u'_{q+1}-v'_{q+1} & \dots & u'_r & 0 & \dots & 0 & 1 &). \\
 & & & {\scriptstyle N_1^+\setminus C^+} & & & {\scriptstyle N_1^-\setminus C^-} & & & {\scriptstyle C^+} & & & {\scriptstyle C^-} & & {\scriptstyle y_1} &
\end{array}$$

Similar arguments to those used in Theorem 6 show that zs are affinely independent. Therefore (20) is facet-defining. ∎

4.2 Full description of the convex hull

In this section we show that inequalities derived in Theorem 6 and Theorem 7, combined with the trivial inequalities, provide the full description of P^1 when P^1 is full-dimensional. The proof is based on the following concept. Given a set of valid inequalities, if all optimal solutions corresponding to an arbitrary objective function over P^1, satisfy one of the inequalities in the family at equality, then these inequalities describe P^1.

Theorem 8. Inequalities (14a)-(14c), (15) and (20) completely describe P^1.

Proof. Let us consider the maximization problem with arbitrary objective function (a, c), where a corresponds to x and c to y_1. Let $M(a, c)$ denote the corresponding set of optimal solutions. We consider the following cases.

1. If $a_k < 0$ for some $k \in N_1^+$, then $M(a, c) \subseteq \{(x, y_1) : x_k = 0\}$.
2. If $a_k < 0$ for some $k \in N_1^-$ and $k \neq 1$, then $M(a, c) \subseteq \{(x, y_1) : x_k = u'_k\}$.
3. If $a_1 < 0$ and $1 \in N_1^-$, then $M(a, c) \subseteq \{(x, y_1) : x_1 = u'_1 \pm v'_1 y_1\}$.
4. Let us consider the case when $a_i > 0$ for $i \in C = C^+ \cup C^-$ and $a_i = 0$ for $i \in N \setminus C$. Here $C^+ \subseteq N_1^+$ and $C^- \subseteq N_1^-$. Note that if $(\hat{x}, \hat{y}_1)$ is an optimal solution, then $\hat{x}_i = 0$ for any $i \in N_1^+ \setminus C^+$ and $\hat{x}_i = u'_i$ for any $i \in N_1^- \setminus C^-$ ($\hat{x}_1 = u'_1 \pm v'_1 y_1$ if $1 \in N_1^- \setminus C^-$). We now consider several cases. The general strategy is to consider several intervals for the capacity constraint (14a) with respect to the upper bound d'. For large values of d', continuous variables are limited only by their upper bounds for any choice of y, so optimal solutions satisfy constraints $x_i = u'_i$ for $i \in C^+$, $i \neq 1$ at equality. For small values of d', the capacity constraint becomes active and is therefore satisfied at equality by an optimal solution. In the intermediate case, the capacity constraint plays a role for either $y_1 = 0$ or $y_1 = 1$, but not both. In this case the inequalities described by Theorem 6 and Theorem 7 are used.
   (a) $1 \in \left(N_1^+ \setminus C^+\right) \cup C^-$
      i. If $\sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i + b'_1 \leq d'$, then we claim $M(a, c) \subseteq \{(x, y_1) : x_i = u'_i, i \in C^+\}$. To show this, assume $(\hat{x}, \hat{y}_1)$ is an optimal solution and $\hat{x}_j < u'_j$ for some $j \in C^+$. Consider $(\tilde{x}, \tilde{y}_1)$ such that $\tilde{x}_i = \hat{x}_i$, $i \neq j$, $\tilde{x}_j = u'_j$ and $\tilde{y}_1 = \hat{y}$. Then $(\tilde{x}, \tilde{y}_1)$ is feasible and provides a larger objective value, since $a_j > 0$. Therefore, $(\hat{x}, \hat{y}_1)$ is not optimal, and this contradiction proves our claim.
      ii. If $\sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i < d' < \sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i + b'_1$, we have two possibilities.
         A. $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$ and there is nothing to prove.
         B. $M(a, c) \subsetneq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$, i.e. there exists an optimal solution $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$, where $\hat{x}^+$ corresponds to $i \in N^+$ and $\hat{x}^-$ corresponds to $i \in N^-$, satisfying (14a) as a strict inequality. Consider first $\hat{y}_1 = 1$. Since $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$ satisfies (14a) at inequality, all $\hat{x}_j^+$ are equal to u'_j and all $\hat{x}_j^-$ are equal to 0. Hence

$$\sum_{i \in N^+} \hat{x}_i^+ - \sum_{i \in N^-} \hat{x}_i^- + b'_1 \hat{y}_1 = \sum_{i \in C^+} \hat{x}_i^+ - \sum_{i \in C^-} \hat{x}_i^- - \sum_{i \in N_1^- \setminus C^-} u'_i + b'_1 \hat{y}_1 = \sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i + b'_1,$$

         which is larger than d' by assumption. Therefore $\hat{y}_1 = 0$, and as above $\hat{x}^+ = u'$ and $\hat{x}^- = 0$. Thus if $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$ is an optimal solution, then it either satisfies (14a) at equality, or $\hat{y}_1 = 0, \hat{x}^+ = u', \hat{x}^- = 0$. Combining these two possibilities together we obtain $M(a, c) \subseteq \{(x, y) : (x, y)$ satisfies (15) at equality$\}$.
      iii. Finally, let us consider the remaining case $d' \leq \sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i$. Let $(\hat{x}, \hat{y}_1)$ be an optimal solution, which satisfies (14a) at inequality. Since $d' \leq \sum_{i \in C^+} u'_i - \sum_{i \in N_1^- \setminus C^-} u'_i$, there exists $j \in C^+$, such that $\hat{x}_j < u'_j$, or there exists $j \in C^-$ such that $\hat{x}_j > 0$. Consider

$(\tilde{x}, \tilde{y}_1)$, such that $\tilde{x}_i = \hat{x}_i$, $i \neq j$, $\tilde{x}_j = \hat{x}_j + \varepsilon$ if $j \in C^+$, $\tilde{x}_j = \hat{x}_j - \varepsilon$ if $j \in C^-$ and $\tilde{y}_1 = \hat{y}_1$. There exists ε small enough such that $(\tilde{x}, \tilde{y}_1)$ is feasible. Vector $(\tilde{x}, \tilde{y}_1)$ provides a larger objective value and therefore $(\hat{x}, \hat{y}_1)$ is not optimal. This contradiction proves that $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$.

(b) $1 \in C^+ \cap N_2^+$

i. If $\sum_{i\in C^+} u'_i + v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1 \leq d'$, then for both $y_1 = 0$ and $y_1 = 1$ all continuous variables x_i, $i \in C^+$ are bounded from above only by their upper bounds, and all continuous variables x_i, $i \in C^-$ are bounded from below only by 0. Therefore $M(a, c) \subseteq \{(x, y_1) : x_1 = u'_1 + v'_1 y_1, x_i = u'_i, i \in C^+\}$.

ii. If $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i < d' < \sum_{i\in C^+} u'_i + v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1$, we have two subcases.

A. $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$ and there is nothing to prove.

B. $M(a, c) \subsetneq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$, i.e. there exists an optimal solution $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$, where $\hat{x}^+ \in C^+$ and $\hat{x}^- \in C^-$, satisfying (14a) as a strict inequality. Using the same argument as in case a.ii.B we can prove that $\bar{y}_1 = 0$. Thus if $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$ is an optimal solution, then it either satisfies (14a) at equality, or $\hat{y}_1 = 0$. Combining these two possibilities together, we obtain $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (15) at equality$\}$.

iii. For the remaining case $d' \leq \sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i$, similarly to the case (a).iii, we have $M(a, c) \subseteq \{(x, y) : (x, y)$ satisfies (14a) at equality$\}$.

(c) $1 \in C^+ \cap N_2^-$ and $v'_1 \leq b'_1$

In this case we consider three subcases with the arguments identical to those used in the previous case. The only difference is the intervals for the value of d'. Here they correspond to:

i. $\sum_{i\in C^+} u'_i - v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1 \leq d'$,

ii. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i < d' < \sum_{i\in C^+} u'_i - v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1$,

iii. $d' \leq \sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i$.

(d) $1 \in C^+ \cap N_2^-$ and $v'_1 > b'_1$

i. If $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i \leq d'$, then for both $y_1 = 0$ and $y_1 = 1$ all continuous variables x_i, $i \in C^+$ are bounded from above only by their upper bounds, and all continuous variables x_i, $i \in C^-$ are bounded from below only by 0. Therefore $M(a, c) \subseteq \{(x, y_1) : x_1 = u'_1 + v'_1 y_1, x_i = u'_i, i \in C^+\}$.

ii. If $\sum_{i\in C^+} u'_i - v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1 < d' < \sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i$, we have two subcases.

A. $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$ and there is nothing to prove.

B. $M(a, c) \subsetneq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$. That is, some optimal solution $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$, where $\hat{x}^+$ corresponds to $i \in N^+$ and $\hat{x}^-$ corresponds to $i \in N^-$, satisfies (14a) as a strict inequality. Consider first $\hat{y}_1 = 0$. Since $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$ satisfies (14a) at inequality, $\hat{x}_i^+ = u'_i$ for any $i \in C^+$ and $\hat{x}_i^- = 0$ for any $i \in C^-$. Hence

$$\sum_{i\in N^+} \hat{x}_i^+ - \sum_{i\in N^-} \hat{x}_i^- + b'_1 \hat{y}_1 = \sum_{i\in C^+} \hat{x}_i^+ - \sum_{i\in C^-} \hat{x}_i^- - \sum_{i\in N_1^- \setminus C^-} u'_i = \sum_{i\in C^+} u'_i - \sum_{i\in N_1^- \setminus C^-} u'_i,$$

which is greater than d' by assumption. Therefore $\hat{y}_1 = 1$, and as before $\hat{x}_1^+ = u'_1 - v'_1$, $\hat{x}_i^+ = u'_i$ for $i \in C^+$ and $i \neq 1$, $\hat{x}_i^- = 0$ for $i \in C^-$. Thus if $(\hat{x}^+, \hat{x}^-, \hat{y}_1)$ is an optimal solution, then either it satisfies (14a) at equality, or $\hat{y}_1 = 1$. Combining these two possibilities together we obtain $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (20) at equality$\}$.

iii. For the remaining case $d' \leq \sum_{i\in C^+} u'_i - v'_1 - \sum_{i\in N_1^- \setminus C^-} u'_i + b'_1$, similar to the case (a).iii, we have $M(a, c) \subseteq \{(x, y_1) : (x, y_1)$ satisfies (14a) at equality$\}$.

(e) $1 \in \left(N_1^- \setminus C^-\right) \cup N_2^-$

In this case we consider three subcases with the arguments identical to the previous case. The only difference is the intervals for the value of d', which in this case correspond to the following.

i. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i + v'_1 + b'_1 \le d'$: Vector $(u'_1 - v'_1, u', 0, 1)$ and $(u'_1, u', 0, 0)$ are both feasible, hence upper bounds (14c) are satisfied at equality.

ii. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i + v'_1 + b'_1 < d' < \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Setting $y_1 = 0$ is feasible, $y_1 = 1$ is not. Hence (15) is satisfied at equality.

iii. $d' \le \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Inequality (14a) is satisfied at equality.

(f) $1 \in \left(N_1^- \setminus C^-\right) \cup N_2^+$ and $v'_1 \le b'_1$
In this case we consider three subcases with the arguments identical to the previous case. The only difference is the intervals for the value of d'. They correspond to the following.

i. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i - v'_1 + b'_1 \le d'$: $(u'_1 - v'_1, u', 0, 1)$ and $(u'_1, u', 0, 0)$ are both feasible, hence upper bounds (14c) are satisfied at equality.

ii. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i - v'_1 + b'_1 < d' < \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Setting $y_1 = 0$ is feasible, $y_1 = 1$ is not. Hence (15) is satisfied at equality.

iii. $d' \le \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Inequality (14a) is satisfied at equality.

(g) $1 \in \left(N_1^- \setminus C^-\right) \cup N_2^+$ and $v'_1 > b'_1$
In this case we consider three subcases with the arguments identical to the previous case. The only difference is the intervals for the value of d'. They correspond to the following.

i. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i - v'_1 + b'_1 \le d'$: $(u'_1 - v'_1, u', 0, 1)$ and $(u'_1, u', 0, 0)$ are both feasible, hence upper bounds (14c) are satisfied at equality.

ii. $\sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i - v'_1 + b'_1 < d' < \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Setting $y_1 = 1$ is feasible, $y_1 = 0$ is not. Hence (20) is satisfied at equality.

iii. $d' \le \sum_{i\in C^+} u'_i - \sum_{i\in N_1^-\setminus C^-} u'_i$: Inequality (14a) is satisfied at equality.

This covers all the cases and proves the statement of the theorem. □

Example (continued). This theorem shows that the trivial inequalities and all those given in Table 1 completely describe the convex hull of P^1. □

4.3 Lifting coefficients for binary variables

Next we show how to compute lifting coefficients for binary variables based on (15). We define a lifting order $i_2, \dots, i_{|N|}$ for y such that $2 \le i_j \le |N|$ and $i_j \ne i_k$ for $j \ne k$. Let also $i_1 = 1$. Let us assume that variables $y_{i_2}, \dots, y_{i_{k-1}}$ have already been lifted, which means that $I_y = \{i_2, \dots, i_{k-1}\}$. Then for $i_k \in L_0$ the lifting coefficient β_{i_k} is defined by

$$
\begin{aligned}
\beta_{i_k} = \quad & \min \quad \Big(\sum_{j\in C^+} u_j - \sum_{j\in C^+} x_j + \sum_{j\in C^-} x_j - (\lambda \mp \delta v_1) y_1 - \sum_{j\in I_y\cap L_0} \beta_j y_j - \sum_{j\in I_y\cap L_1} \beta_j (y_j - 1) \Big) \\
& \text{s.t.} \quad \sum_{j\in N_1^+} x_j - \sum_{j\in N_1^-} x_j + \sum_{j=1}^{k-1} b_{i_j} y_{i_j} \le d - \sum_{i\in L_1\setminus I_y} b_i \\
& \qquad 0 \le x_j \le u_j \pm v_j \qquad j = i_k \\
& \qquad 0 \le x_j \le u_j \pm v_j y_j \qquad j \in \{i_1, \dots, i_{k-1}\} \\
& \qquad 0 \le x_j \le u_j \pm v_j \bar{y}_j \qquad j \in \{i_{k+1}, \dots, i_{|N|}\} \\
& \qquad y \text{ binary.}
\end{aligned}
$$

For $i_k \in L_1$ we have

$$
\begin{array}{rll}
\beta_{i_k} = & \max & \Big(\sum_{j\in C^+} u_j - \sum_{j\in C^+} x_j + \sum_{j\in C^-} x_j - (\lambda \mp \delta v_1)y_1 - \sum_{j\in I_y\cap L_0} \beta_j y_j - \sum_{j\in I_y\cap L_1} \beta_j (y_j - 1)\Big) \\
& \text{s.t.} & \sum_{j\in N_1^+} x_j - \sum_{j\in N_1^-} x_j + \sum_{j=1}^{k-1} b_{i_j} y_{i_j} \le d - \sum_{i\in L_1\setminus(I_y\cup\{i_k\})} b_i \\
& & 0 \le x_j \le u_j \qquad j = i_k \\
& & 0 \le x_j \le u_j \pm v_j y_j \qquad j \in \{i_1, \dots, i_{k-1}\} \\
& & 0 \le x_j \le u_j \pm v_i \bar{y}_i \qquad j \in \{i_{k+1}, \dots, i_{|N|}\} \\
& & y \text{ binary.}
\end{array}
$$

These two optimization problems can be solved by the dynamic programming algorithm developed in Theorem 1. Similar procedure is applied to lifting of (20).

Example (continued). Consider lifting of the first inequality given in Table 1; $x_1 + x_3 - x_4 + 2y_1 \le 7$. We lift binary variables in the order y_2, y_3, y_4, and y_5. To find β_2 we need to solve the following problem

$$
\begin{array}{rlll}
\beta_2 = & \min & 7 - x_1 - x_3 + x_4 - 2y_1 & \\
& \text{s.t.} & x_1 + x_2 + x_3 - x_4 - x_5 + 5y_1 \le 6 & \\
& & 0 \le x_1 \le 3 - y_1 & 0 \le x_4 \le 3 \\
& & 0 \le x_2 \le 4 & 0 \le x_5 \le 2 \\
& & 0 \le x_3 \le 4 & \\
& & y_1 \text{ binary,} &
\end{array}
$$

which gives $\beta_2 = 0$. Similarly we obtain $\beta_3 = 3$, $\beta_4 = -1$ and $\beta_5 = -2$. Thus the resulting inequality is

$$
x_1 + x_3 - x_4 + 2y_1 + 3y_3 - y_4 - 2y_5 \le 4. \qquad \square
$$

5 Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant No. 0084826.

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A Proof of Theorem 1

Proof. Let (x_n, y_n) be the nth components of an optimal solution (x, y) to $f_n(\tilde{d})$. To prove the statement of the theorem consider the following possible cases. The difference between $n \leq k$ and $n \geq k+1$ is considered later in the proof.

1. If $y_n = 0$ and $x_n = 0$, then clearly $f_n(\tilde{d}) = f_{n-1}(\tilde{d})$ (first case in (6) and (7)).

2. If $y_n = 1$ and $x_n = 0$, then $f_n(\tilde{d}) = f_{n-1}(\tilde{d} - \hat{c}_n) + \hat{b}_n$ (second case in (6) and (7)).

3. If $y_n = 0$ and $x_n = \hat{u}_n$, then $f_n(\tilde{d}) = f_{n-1}(\tilde{d} - \delta_n \hat{u}_n) + \hat{a}_n \hat{u}_n$ (third case in (6) and (7)).

4. If $y_n = 1$ and $x_n = \hat{u}_n + \hat{v}_n$, then similarly $f_n(\tilde{d}) = f_{n-1}(\tilde{d} - \delta_n(\hat{u}_n + \hat{v}_n) - \hat{c}_n) + \hat{a}_n(\hat{u}_n + \hat{v}_n) + \hat{b}_n$ (fourth case in (6) and (7)).

5. The case $0 < x_n < \hat{u}_n + \hat{v}_n y_n$ is more involved.

   We first show that (5b) is satisfied at equality at an optimal solution or we end up in one of the already considered cases. Assume that (x, y) does not have this property, i.e. it is an optimal solution that satisfies (5b) as strict inequality. We consider a solution $(\tilde{x}, \tilde{y})$, defined by $\tilde{y} = y$ and $\tilde{x} = x + \delta_n \varepsilon e_n$, where ε is positive but small enough to make $(\tilde{x}, \tilde{y})$ feasible. Clearly, $(\tilde{x}, \tilde{y})$ provides equal or larger objective value and therefore it is optimal. If $(\tilde{x}, \tilde{y})$ satisfies (5b) at equality, we obtain the claim. Otherwise either $\tilde{x}_n = \hat{u}_n + \hat{v}_n \tilde{y}_n$ and we apply cases 3 and 4 or $\tilde{x}_n = 0$ and we use cases 1 and 2.

   Next we consider two cases.

   (a) Let us first assume $n \geq k + 1$. Then

      i. for every $i \geq k + 1$ and $\hat{a}_i < \hat{a}_n$ we have $x_i = 0$ in every optimal solution,

      ii. if $\hat{a}_{q-1} < \hat{a}_q = \hat{a}_{q+1} = \cdots = \hat{a}_n$, then there exists an optimal solution $(\tilde{x}, \tilde{y})$ with either $\tilde{x}_n = \hat{u}_n + \hat{v}_n \tilde{y}_n$ or $\tilde{x}_q = \tilde{x}_{q+1} = \cdots = \tilde{x}_{n-1} = 0$ and $0 < \tilde{x} < \hat{u}_n + \hat{v}_n \tilde{y}_n$,

      iii. for every $i \leq k$ and $|\hat{a}_i| > \hat{a}_n$ we have $x_i = 0$ in every optimal solution,

      iv. if $i \leq k$ and $|\hat{a}_i| < \hat{a}_n$, then $x_i = \hat{u}_i + \hat{v}_i y_i$ in every optimal solution,

      v. if $i \leq k$ and $|\hat{a}_{q+1}| < |\hat{a}_q| = |\hat{a}_{q-1}| = \cdots = |\hat{a}_p| = \hat{a}_n < |\hat{a}_{p-1}|$ for a $p \leq k$, then there exists an optimal solution $(\tilde{x}, \tilde{y})$ with either $\tilde{x}_n = \hat{u}_n + \hat{v}_n \tilde{y}_n$ or $\tilde{x}_q = \tilde{x}_{q+1} = \cdots = \tilde{x}_p = 0$ and $0 < \tilde{x} < \hat{u}_n + \hat{v}_n \tilde{y}_n$.

      We show only case (i) since all other cases can be proved similarly. Let us assume that $x_i > 0$. Consider vector $(\bar{x}, \bar{y})$, defined by $\bar{y} = y$, $\bar{x} = x - \varepsilon e_i + \varepsilon e_n$, where $0 < \varepsilon = \min\{\hat{u}_n + \hat{v}_n y_n - x_n, x_i\}$. Vector $(\bar{x}, \bar{y})$ is feasible, since $\sum_{q=1}^{M} \delta_q x_q = \sum_{q=1}^{M} \delta_q \bar{x}_q$, however due to $\hat{a}_i < \hat{a}_n$ the objective value is strictly greater than the objective value of (x, y), which contradicts optimality of (x, y).

      As a conclusion of all these claims we either end up in one of the cases 1-4 or we have that

$$x_n = \tilde{d} - \sum_{i=1}^{n} \hat{c}_i y_i + \sum_{i=t(n)}^{k} (\hat{u}_i + \hat{v}_i y_i). \tag{24}$$

      In turn we obtain that

$$\begin{aligned} f_n(\tilde{d}) = \max \Bigg\{ &\sum_{i=t(n)}^{k} \hat{a}_i(\hat{u}_i + \hat{v}_i y_i) + \sum_{i=1}^{n} \hat{b}_i y_i + \hat{a}_n\big(\tilde{d} - \sum_{i=1}^{n} \hat{c}_i y_i + \sum_{i=t(n)}^{k} (\hat{u}_i + \hat{v}_i y_i)\big) : \\ &- \sum_{i=t(n)}^{k} (\hat{u}_i + \hat{v}_i y_i) + \sum_{i=1}^{n} \hat{c}_i y_i \leq \tilde{d}, &(25) \\ &- \sum_{i=t(n)}^{k} (\hat{u}_i + \hat{v}_i y_i) + \sum_{i=1}^{n} \hat{c}_i y_i \geq \tilde{d} - (\hat{u}_n + \hat{v}_n y_n), &(26) \\ &y \text{ binary} \Bigg\}. \end{aligned}$$

      We have replaced x_n by (24) and we use properties (i)-(v). The condition $x_n \geq 0$ is equivalent to (25). On the other hand, $x_n \leq \hat{u}_n + \hat{v}_n y_n$ is imposed by (26).

      By using the standard argument of differentiating between $y_n = 0$ and $y_n = 1$ we obtain the fifth and the sixth terms in (7).

(b) Let us now assume $1 \le n \le k$. In this case either there exists an optimal solution under cases 1-4 or $x_i = 0$ for every $1 \le i < n$. This statement can be shown similarly as the equivalent statement in the case $n \ge k+1$. Thus in this case we have

$$x_n = \sum_{i=1}^{n} \hat{c}_i y_i - \tilde{d}$$

and thus

$$\begin{aligned} f_n(\tilde{d}) = \max \Bigg\{ & \sum_{i=1}^{n} \hat{b}_i y_i + \hat{a}_n (\sum_{i=1}^{n} \hat{c}_i y_i - \tilde{d}) \\ & \sum_{i=1}^{n} \hat{c}_i y_i \ge \tilde{d} \\ & \sum_{i=1}^{n} \hat{c}_i y_i \le \tilde{d} + \hat{u}_n + \hat{v}_n y_n \\ & y \text{ binary} \Bigg\} . \end{aligned}$$

The fifth and the sixth terms in (6) can easily be justified by setting $y_n = 0$ and $y_n = 1$, respectively.

This completes the proof. $\square$

B Computation of $g, \tilde{g}, h, \tilde{h}$ by dynamic programming

The recursion for h and $\tilde{h}$ are simple. For any integers $1 \le l \le k$ and $2 \le p \le k$ we have

$$h_p^l(\tilde{d}) = \max\{h_{p-1}^l(\tilde{d}), h_{p-1}^l(\tilde{d} - \hat{c}_p) + \hat{b}_p + \hat{a}_l \hat{c}_p\} .$$

It is easy to explicitly write h_1^l. A similar recursive relation holds for $\tilde{h}$.

For g we exhibit three recursions that hold for every $\tilde{d} \in \Omega$. The first one reads

$$g_{p,1}^{ls}(\tilde{d}) = \max\{g_{p-1,1}^{ls}(\tilde{d}), g_{p-1,1}^{ls}(\tilde{d} - \hat{c}_p) + \hat{b}_p - \hat{a}_l \hat{c}_p\} \tag{27}$$

and it is valid for $k+1 \le p \le M$. The second one is

$$g_{k,j}^{ls}(\tilde{d}) = \max\{g_{k,j+1}^{ls}(\tilde{d}), g_{k,j+1}^{ls}(\tilde{d} - \hat{c}_j) + \hat{b}_j - \hat{a}_l \hat{c}_j\} \tag{28}$$

and it holds for $1 \le j \le s-1$. The last one reads

$$g_{k,s}^{ls}(\tilde{d}) = \max\{g_{k,s+1}^{l,s+1}(\tilde{d} + \hat{s}), g_{k,s+1}^{l,s+1}(\tilde{d} + \hat{u}_s - \hat{c}_s + \hat{v}_s) + (\hat{a}_s + \hat{a}_l)\hat{v}_s + \hat{b}_s - \hat{a}_l \hat{c}_s\} \tag{29}$$

and it is applied for $1 \le s \le k-1$.

The boundary condition reads

$$g_{kk}^{lk}(\tilde{d}) = \begin{cases} ((\hat{a}_k + \hat{a}_l)\hat{v}_k + \hat{b}_k - \hat{a}_l \hat{c}_k)^+ & \hat{c}_k - \hat{u}_k \le \tilde{d} \le \hat{c}_k + \hat{u}_l - \hat{u}_k \\ 0 & \text{otherwise.} \end{cases}$$

We can compute all g_{pj}^{ls} by following the next steps for every l. Each iteration in what follows is assumed to be carried out for every $\tilde{d} \in \Omega$.

1. Fixed $\bar{s}, 1 \le \bar{s} \le k$.

(a) For every $s = k, k-1, \dots, \bar{s}$ we use (29) to obtain g^{ls}_{ks}. This gives us $g^{l,\bar{s}}_{k\bar{s}}$.

(b) Next for $j = \bar{s}-1, \bar{s}-2, \dots, 1$ we compute $g^{l\bar{s}}_{k,1}$ by using (28). This step gives us $g^{l,\bar{s}}_{k,1}$.

2. At this point we have $g^{ls}_{k,1}$ for every $s, 1 \leq s \leq k$. Now for $p = k+1, k+2, \dots, M$ we apply (27) to obtain $g^{ls}_{p,1}$

Similar recursive formulas can be obtained for $\tilde{g}$, which leads to a similar procedure to calculate $\tilde{g}$.

After obtaining $h, \tilde{h}$ and $g, \tilde{g}$, the complete procedure to calculate f_M is now simple. We first use (6) to compute $f_1, f_2, \dots, f_k$. Next by using (7) we obtain $f_{k+1}, f_{k+2}, \dots, f_M$.