A New Subadditive Approach to Integer Programming: Theory and Algorithms

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Abstract

Linear programming duality is well understood and the reduced cost of a column is frequently used in various algorithms. On the other hand, for integer programs it is not clear how to define a dual function even though the subadditive dual theory has been developed a long time ago. In this work we propose a family of computationally tractable subadditive dual functions for integer programs. We develop a solution methodology that computes an optimal primal solution and an optimal subadditive dual function. We present computational experiments, which show that the new algorithm is tractable and the subadditive functions give better sensitivity results than its linear programming counterpart.

Keywords: integer programming, duality, algorithms

1 Introduction

Integer programming (IP) has many practical applications and its importance is well documented. There are several algorithms that compute an optimal IP solution, with branch-and-cut algorithms outperforming the field. Most of the algorithms compute the optimal or near optimal IP solution. On the other hand, the field of IP duality is not well studied and to the best of our knowledge there are no practical algorithms that compute the optimal dual function. In this paper we address how to compute dual functions and reduced cost, and use them to perform sensitivity analysis for IPs. Frequently in IP we would like to estimate the change of the objective value if we perturb the right hand side. Sensitivity analysis for IPs is typically done either by considering the dual vector of the LP relaxation or by resolving the problem after changing the right hand side. Is it possible to compute a vector or a function that would measure the change in the objective function of an IP? A slightly different scenario is when we are given a new variable and we wonder how the optimal objective value changes if this variable is added to the formulation. In many real world problems that are modeled as IPs we would like to obtain alternative optimal solutions. For example, a decision maker wants to select the most robust solution among several optimal solutions. All optimal solutions can be found among the variables with zero reduced cost, which requires an optimal dual function.

All of the aforementioned questions are well understood in linear programming (LP). In LP with each feasible primal problem there is an associated dual problem with the same objective value. Many algorithms for LP compute both a primal and a dual solution, e.g. simplex and primal-dual algorithms. The reduced cost of a variable estimates how much would the addition of the variable to the formulation change the objective value. By using duality we can carry out the sensitivity analysis. Column generation is a technique for solving large-scale LPs efficiently, see e.g. Barnhart et al. (1998). In a column generation algorithm, we start by solving an initial formulation that contains only a small subset of the variables of the problem. This formulation is called the restricted master problem. The algorithm progresses as other variables are introduced to the restricted master problem, which is reoptimized in every iteration. Variables with small reduced cost are more likely to improve the incumbent solution and therefore they are appended to the restricted master problem. The variable selection process is called the subproblem. A column generation type algorithm for solving large-scale IPs is of great interest.

Alcaly and Klevorick (1966) and Baumol (1960) give several interpretations of the LP dual vector in many business related problems. In this case the dual prices measure the change of the objective value, i.e. cost or revenue, if one of the resources is changed. In many applications the underlying model is an IP and therefore it would be useful to have such an interpretation. A recent such application is auctioning, Schrage (2001), Bikhcandani et al. (2001). In auctioning an optimal allocation of bids is sought that maximizes the seller's profit. This problem can be modeled as an IP. The dual values correspond to a bidder's marginal value and they can be used to explain to the losers how much higher should they have bid to win.

For integer programs subadditive duality developed first by Johnson (1973) gives us a partial answer to these questions.

Definition 1. *A function $F : \mathbb{R}^m \to \mathbb{R}$ is subadditive on $Z \subseteq \mathbb{R}^m$ if $F(x+y) \le F(x) + F(y)$ for all $x \in Z, y \in Z$ such that $x + y \in Z$.*

If Z is not specified, we assume $Z = \mathbb{R}^m$. Johnson showed that for a feasible IP

$$\begin{array}{rcll} \min\ cx & & \max\ F(b) & \\ Ax = b & = & F(a_i) \le c_i & i = 1, \ldots, n \\ x \in \mathbb{Z}^n_+ & & F \ \text{subadditive}\,, & \end{array} \tag{1}$$

where $A = (a_1, \ldots, a_n) \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$. We refer to the second problem as the *subadditive dual problem*. At least theoretically the answer to all of the raised questions are in the *optimal subadditive function* (OSF). In other words, the analog to the optimal dual vector in LP is the OSF. Given a subadditive function F, the reduced cost of a column i can be defined as $c_i - F(a_i)$ and most of the other properties from LP carry over to IP, e.g. complementary slackness, $F(b)$ provides a lower bound on the optimal IP value, if $F(a_i) \le c_i$ for all $i = 1, \ldots, n$, and all optimal solutions can be found only among the columns i with $c_i = F(a_i)$, if F is an OSF. However there are still two fundamental issues that need to be addressed; how to encode F and how to compute F. Theory tells us that an OSF can always be obtained as a composition of C-G inequalities, see e.g. Nemhauser and Wolsey (1988), pages 304-308, but such a function would be hard to encode and hard to evaluate. Very little is known about how to compute an OSF. Llewellyn and Ryan (1993) show how an OSF can be constructed from Gomory cuts. Our work originates from the work done by Burdet and Johnson (1977), where an algorithm for solving an IP based on subadditivity is presented. Both of these two works do not present any computational experiments.

Subadditive duality was pioneered by Gomory (1969). He shows strong duality for the group problem and a characterization of all facet-inducing subadditive functions. This work was later extended to the mixed integer case, Gomory and Johnson (1972a,b). The treatment in terms of integer programs is given in Johnson (1980b, 1981, 1987). Araoz (1973) studies the master problem, i.e. the problem were all possible columns are present, and he extends the results from the group problem to the semi-group problem. An excellent summary of results on subadditive duality is given by Johnson (1980a). Wolsey (1981) discusses separable subadditive functions and the relation of Gomory cuts and subadditive functions. He also shows how to construct a subadditive function from a branch-and-bound tree but his functions are hard to encode and compute.

We first give a new family of subadditive functions that is easy to encode and often easy to evaluate. We present an algorithm that computes an OSF. As part of the algorithm we give several new theorems that further shed light on OSFs. The contribution of this research goes beyond a novel methodology for computing an OSF. In addition to sensitivity analysis, new approaches for large-scale IPs can be developed (generalized column generation, Benders' decomposition for IPs). We elaborate on these potential applications in Section 5. Further details on the implementation and the computational results are presented in the subsequent paper Klabjan (2003).

In Section 2 we present a new family of subadditive functions that is easy to encode. We give several interesting properties of these functions. In addition we generalize the concept of reduced cost fixing. In Section 3 we give a finite family of these functions that suffices to obtain strong duality (1). Section 4 first outlines the algorithm that computes an optimal primal solution to an IP and an OSF. The rest of this section presents the entire algorithm. The last section presents possible applications of subadditive duality.

Notation

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. In the rest of the paper we assume that $(A, b) \in \mathbb{Z}_+^{m \times (n+1)}$ and that $c \in \mathbb{Z}^n$. A column i of A is denoted by $a_i \in \mathbb{Z}_+^m$ and a row j of A is denoted by $a^j \in \mathbb{Z}_+^n$. For any $E \subseteq N = \{1, 2, \dots, n\}$, we denote by $A^E \in \mathbb{Z}_+^{m \times |E|}$ the submatrix of A consisting of the columns with indices in E and similarly we define c^E. Let $\mathbf{1}$ be the vector with $\mathbf{1}_i = 1$ for every i and let e_i be the ith unit vector. By $\text{supp}(x)$ we denote the support set of x, i.e. $\text{supp}(x) = \{i \in N : x_i > 0\}$.

In this paper we address the IPs of the form $\min\{cx : Ax = b, x \in \mathbb{Z}_+^n\}$. An $x \in \mathbb{Z}_+^n$ is *feasible to the IP* if $Ax = b$. We say that a subadditive function F is *dual feasible* or simply feasible if $F(a_i) \le c_i$ for all $i \in N$. It is easy to see that for every feasible subadditive function we have $F(b) \le z^{\text{IP}}$, where z^{IP} is the optimal value of the IP. We call $F(b)$ the *objective value* of F. Based on these definitions, a subadditive function F is an OSF if and only if F is feasible and $F(b) = z^{\text{IP}}$.

Example. We demonstrate some of the results on the following example.

$$b = (1, 2, 2, 1)$$

$$\begin{array}{rccccccccccccccc} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 12} & {\scriptstyle 13} & {\scriptstyle 14} & {\scriptstyle 15} \\ c = (& 10 & 1 & 3 & 5 & 0 & 2 & -0.5 & -2 & 3 & 4 & -5 & -3 & -1 & 3 & -1\,) \end{array}$$

$$A = \begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 12} & {\scriptstyle 13} & {\scriptstyle 14} & {\scriptstyle 15} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The two rows above c and A show column indices and are given for better readability.

The optimal primal solution to the LP relaxation is $x_8 = 1.5, x_{12} = 0.5, x_{13} = 0.5$, the optimal dual vector of the LP relaxation is $y_1 = -3, y_2 = -2, y_4 = 2$ and the objective value of the LP relaxation is -5. The optimal IP solution is $x_7 = 1, x_8 = 2$ and $z^{\text{IP}} = -4.5$. $\square$

2 The Generator Subadditive Functions

We start by defining a new family of subadditive functions.

Definition 2. *Given a vector $\alpha \in \mathbb{R}^m$, we define a generator subadditive function $F_\alpha : \mathbb{R}_+^m \to \mathbb{R}$ as*

$$\begin{aligned} F_\alpha(d) = \alpha d - \max \sum_{i \in E} (\alpha a_i - c_i) x_i & \\ A^E x \le d & \\ x \in \mathbb{Z}_+^{|E|} &, \end{aligned}$$

where

$$E = \{i \in N : \alpha a_i > c_i\} \tag{2}$$

is the generator set. Similarly, given a vector $\beta \in \mathbb{R}^m$, we define a ray generator subadditive function $\bar{F}_\beta : \mathbb{R}^m_+ \to \mathbb{R}$ as

$$\begin{aligned} \bar{F}_\beta(d) = \beta d - \max \sum_{i\in E} &(\beta a_i)x_i \\ A^E x &\le d \\ x &\in \mathbb{Z}_+^{|E|}\,, \end{aligned}$$

where $E = \{i \in N : \beta a_i > 0\}$.

The generator set E depends on α but for simplicity of notation we do not show this dependence in our notation. Whenever an ambiguity can occur, we write $E(\alpha)$. In addition, for simplicity of notation we write $H = N \setminus E$.

Lemma 1. *For any α we have*

1. *F_α is subadditive,*
2. *$F_\alpha(a_i) \le \alpha a_i \le c_i$ for all $i \in H$,*
3. *$F_\alpha(a_i) \le c_i$ for all $i \in E$.*

Proof. 1. Let $d_1, d_2 \in \mathbb{R}^m_+$. The statement $F_\alpha(d_1 + d_2) \le F_\alpha(d_1) + F_\alpha(d_2)$ is equivalent to

$$\begin{aligned} &\max\{(\alpha A^E - c^E)x : A^E x \le d_1, x \in \mathbb{Z}_+^{|E|}\} + \max\{(\alpha A^E - c^E)x : A^E x \le d_2, x \in \mathbb{Z}_+^{|E|}\} \\ &\le \max\{(\alpha A^E - c^E)x : A^E x \le d_1 + d_2, x \in \mathbb{Z}_+^{|E|}\}\,. \end{aligned} \tag{3}$$

If x_1^*, x_2^* are optimal solution to the two maximums on the right hand side of (3), then $x_1^* + x_2^*$ is a feasible solution to the optimization problem on the left hand side, which shows the claim.

2. $x = 0$ is feasible to $\max\{(\alpha A^E - c^E)x : A^E x \le a_i, x \in \mathbb{Z}_+^{|E|}\}$ and it yields 0 objective value. By definition of F_α, $F_\alpha(a_i) \le \alpha a_i$ and since $i \in H$ the statement follows.

3. For $i \in E$ we consider $x = e_i$ in $\max\{(\alpha A^E - c^E)x : A^E x \le a_i, x \in \mathbb{Z}_+^{|E|}\}$. This yields that in this case $F_\alpha(a_i) \le c_i$.

□

Lemma 1 shows that F_α is a feasible subadditive function and therefore $F_\alpha(b)$ provides a lower bound on z^{IP}. The vector α is a generalization of dual vectors of the LP relaxation. Every dual feasible vector α to the LP relaxation has to satisfy $\alpha a_i \le c_i$ for all $i \in N$, however α in the definition of F_α can violate some of these constraints. Indeed, if y^* is an optimal solution to the dual of the LP relaxation of the IP, then $E = \emptyset$ and F_{y^*} gives the value of the LP relaxation.

Remark 1. Generator subadditive functions can also be derived via Lagrangian relaxation. To this end, let us rewrite the IP as $\min\{cx : Ax \le b, Ax \ge b, x \in \mathbb{Z}^n_+\}$ and for any multipliers $\alpha \in \mathbb{R}^m_+$ with respect to $Ax \ge b$ consider the Lagrangian relaxation

$$LR(\alpha) = \min\{cx - \alpha(Ax - b) : Ax \le b, x \in \mathbb{Z}^n_+\}\,.$$

From Lagrangian theory, see e.g. Nemhauser and Wolsey (1988), page 324, it follows that $LR(\alpha) \le z^{\text{IP}}$. Since $(A, b) \in \mathbb{Z}_+^{m\times(n+1)}$, it follows that $LR(\alpha) = \alpha b + \max\{\sum_{i\in E}(\alpha a_i - c_i)x_i : A^E x \le b, x \in \mathbb{Z}_+^{|E|}\}$, where E is defined as in (2). Therefore $LR(\alpha) = F_\alpha(b)$. □

The ray generator subadditive function is clearly dual feasible to the IP $\min\{0x : Ax = b, x \in \mathbb{Z}^n_+\}$. We show later that these functions assist us in detecting IP infeasibility.

From the computational point of view, note that to describe F_α we only need to specify α and therefore encoding F_α is easy. To evaluate F_α we need to solve an IP with $|E|$ integer variables, which in general is NP-hard. It is desirable that $|E|$ is small since it makes the evaluation of F_α easier. The computational results for set partitioning instances, Klabjan (2003), show that in practice this is indeed the case; even problems with 100,000 columns have only up to 300 columns in E.

Example (continued). Consider $\alpha = (-3.2, -1.75, 0.25, 2.75)$. For this α we have $\alpha a_7 - c_7 = 0.05, \alpha a_8 - c_8 = 0.5, \alpha a_{12} - c_{12} = 0.8, \alpha a_{13} - c_{13} = 0.8$ and all other columns are in H. F_α is defined as

$$
\begin{aligned}
F_\alpha((d_1, d_2, d_3, d_4)) = -3.2d_1 - 1.75d_2 + 0.25d_3 + 2.75d_4 - \max 0.05x_1 + 0.5x_2 + 0.8x_3 + 0.8x_4 & \\
x_1 + x_3 + x_4 \le d_1 & \\
x_2 + x_3 \le d_2 & \\
x_2 + x_4 \le d_3 & \\
x_1 + x_3 + x_4 \le d_4 & \\
x \in \mathbb{Z}^4_+ \, . &
\end{aligned}
$$

It is easy to check that $F_\alpha(b) = F_\alpha((1, 2, 2, 1)) = -4.75$. Therefore F_α is not an OSF but it does provide a better lower bound than the optimal dual vector of the LP relaxation. □

Strong duality (1) states that among all the subadditive dual functions there is one that attains the equality, however, it does not guarantee equality for specially structured subadditive functions like the generator subadditive functions.

Theorem 1. *If the IP is feasible, then there exists an α such that F_α is a generator OSF, i.e. $F_\alpha(b) = z^{IP}$. If the IP is infeasible, then there exists a ray generator subadditive function $\bar{F}_\beta$ such that $\bar{F}_\beta(b) > 0$.*

The first statement in the theorem follows from Lagrangian duality discussed in Remark 1 and the standard convexification argument, Nemhauser and Wolsey (1988), page 327. Here we show a different argument, which is used later on several occasions.

Proposition 1. *Let the IP be feasible and let $\pi^j x \le \pi^j_0, j \in J$ be valid inequalities for $\{Ax \le b, x \in \mathbb{Z}^n_+\}$. Let*

$$
\begin{aligned}
z^* = \min \ & cx \\
Ax &= b \\
\pi^j x &\le \pi^j_0 \qquad\qquad j \in J \\
x &\ge 0
\end{aligned}
\tag{4}
$$

and let (α, γ) be an optimal dual vector, where α corresponds to constraints (4). Then $F_\alpha(b) \ge z^*$.

Proof. The dual of the LP stated in the proposition reads

$$
\begin{aligned}
\max \ & b\alpha - \sum_{j \in J} \pi^j_0 \gamma_j \\
a_i\alpha - \sum_{j \in J} \pi^j_i \gamma_j &\le c_i \qquad\qquad i \in N \\
\alpha \text{ unrestricted}, \ & \gamma \ge 0 \, .
\end{aligned}
\tag{5}
$$

The optimal value of this LP is z^* and let (α, γ) be an optimal vector. The statement $F_\alpha(b) \ge z^*$ is equivalent to

$$
\max\{(\alpha A^E - c^E)x : A^E x \le b, x \text{ nonnegative integer}\} \le b\alpha - z^* \, . \tag{6}
$$

Let x be a nonnegative integer vector such that $A^E x \leq b$. We have

$$(\alpha A^E - c^E)x \leq \sum_{i\in E}\sum_{j\in J} x_i \pi_i^j \gamma_j \tag{7}$$
$$= \sum_{j\in J}\gamma_j \sum_{i\in E}\pi_i^j x_i \leq \sum_{j\in J}\gamma_j \pi_0^j = b\alpha - z^* \,, \tag{8}$$

where (7) follows from (5), and the inequality in (8) holds since $\pi^j x \leq \pi_0^j, j \in J$ are valid inequalities for $\{A^E x \leq b, x \text{ nonnegative integer}\}$ and $\gamma \geq 0$. This shows (6) and it proves the claim. □

Proof of Theorem 1. If the IP is feasible, then the first statement follows either by using Lagrangian duality or by using (1).

Assume now that the IP is infeasible. Consider $z = \min\{\mathbf{1}u : Ax + u = b, x \in \mathbb{Z}_+^n, u \in \mathbb{Z}_+^m\}$. Since the original IP is infeasible, $z > 0$. By the above case, there exists β such that $z = b\beta - \max\{\beta A^E \bar{x} + \sum_{i\in\tilde{E}}(\beta_i - 1)\tilde{x}_i : (A^E\bar{x})_i \leq b_i \text{ for all } i \neq \tilde{E}, (A^E\bar{x})_i + \tilde{x}_i \leq b_i \text{ for all } i \in \tilde{E}, \bar{x} \in \mathbb{Z}_+^{|E|}, \tilde{x} \in \mathbb{Z}_+^{|\tilde{E}|}\}$, where $E = \{i \in N : \beta a_i > 0\}$ and $\tilde{E} = \{i : \beta_i > 1\}$. But then we have $0 < z \leq b\beta - \max\{\beta A^E \bar{x} : A^E\bar{x} \leq b, \bar{x} \in \mathbb{Z}_+^n\}$, which shows that $\bar{F}_\beta$ has the desired property. □

Note that a ray generator subadditive function shows that the subadditive dual problem is unbounded from above. Theorem 1 states that if the IP is infeasible, then there exists a ray generator subadditive dual function and therefore the subadditive dual problem is unbounded. Clearly the opposite holds as well, i.e. if the subadditive dual problem is unbounded, then the IP is infeasible.

Example (continued). Consider $\alpha = (-4, -1.5, 0.5, 3.5)$. We have $E = \{8, 12, 13\}$ and $\alpha a_8 - c_8 = \alpha a_{12} - c_{12} = \alpha a_{13} - c_{13} = 1$. It is easy to check that $\alpha b = -2.5$ and that $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}_+^3\} = 2$. Therefore $F_\alpha(b) = -4.5$ and this is a generator OSF.

If we add valid inequalities $x_8 + x_{11} + x_{12} \leq 2, x_8 + x_{12} + x_{13} \leq 2$ to the LP relaxation, we obtain an objective value of -4.5 and the corresponding optimal dual vector is indeed $(-4, -1.5, 0.5, 3.5)$. Therefore we can establish $F_\alpha(b) = -4.5$ also directly from Proposition 1.

Consider now the set $Q = \{Ax = (1, 2, 2, 1), x_i = 0 \text{ for all } i \notin R, x \in \mathbb{Z}_+^{15}\}$, where $R = \{8, 9, 10, 12, 13, 14\}$. In other words, we consider only columns in R. The LP relaxation is nonempty since $x_8 = 1.5, x_{12} = 0.5, x_{13} = 0.5$ is primal feasible. Let $\beta = (3, 1, 1, -2)$. We have $\beta a_8 = \beta a_{12} = \beta a_{13} = 2$ and for all other columns $i \in R$ we have $\beta a_i \leq 0$. $\bar{F}_\beta$ is defined as

$$\begin{aligned}
\bar{F}_\beta((d_1, d_2, d_3, d_4)) = 3d_1 + 1d_2 + 1d_3 - 2d_4 - \max\ & 2x_1 + 2x_2 + 2x_3 \\
& x_2 + x_3 \leq d_1 \\
& x_1 + x_2 \leq d_2 \\
& x_1 + x_3 \leq d_3 \\
& x_2 + x_3 \leq d_4 \\
& x \in \mathbb{Z}_+^4 \,.
\end{aligned}$$

It is easy to see that $\bar{F}_\beta(b) = 1 > 0$. Therefore this is a ray generator subadditive function with $\bar{F}_\beta(b) = 1 > 0$. By Theorem 1 we have that $Q = \emptyset$, which we can also easily check by hand. □

Next we give two theorems that have a counterpart in LP and are used in our algorithm.

Theorem 2 (Complementary slackness). *Let x^* be an optimal IP solution. If $x_i^* > 0$, then $\alpha a_i \geq c_i$ in any generator OSF.*

Proof. Let i be such that $x_i^* > 0$. The complementary slackness condition says that for any OSF we have $x_i^*(c_i - F(a_i)) = 0$, see e.g. Nemhauser and Wolsey (1988), page 305. This means that $F(a_i) = c_i$ in any OSF. If F is a generator OSF F_α, then if $i \in H$ it follows that $c_i = F_\alpha(a_i) \leq \alpha a_i$ and if $i \in E$ it follows by definition $\alpha a_i > c_i$. □

In IP reduced cost fixing based on solutions to LP relaxations is a commonly used technique for fixing variables to 0, see e.g. Wolsey (1998), page 109, and similarly variable fixing based on Lagrangian multipliers is known. The next theorem establishes an equivalent property based on subadditive dual functions (not necessarily those derived from a Lagrangian relaxation).

Theorem 3 (Reduced cost fixing). *Let F be a feasible subadditive dual function and let $\hat{z}^{IP}$ be an upper bound on z^{IP}. If $c_k - F(a_k) > 0$ and*

$$v = \left\lceil \frac{\hat{z}^{IP} - F(b)}{c_k - F(a_k)} \right\rceil > 0 \tag{9}$$

for a column $k \in N$, then there is an optimal IP solution x^* with $x^*_k \leq v - 1$.

Proof. Let k be an index such that $c_k - F(a_k) > 0$ and $v > 0$. Then by definition of v it follows that $F(b) + (c_k - F(a_k))l \geq \hat{z}^{\mathrm{IP}}$ for every $l \geq v, l$ integer. Consider the IP $\min\{cx : Ax = b, x_k = l, x \in \mathbb{Z}^n_+\}$ for an $l \geq v$. We show that the optimal value of this IP is greater or equal to $\hat{z}^{\mathrm{IP}}$.

The subadditive dual problem of this IP reads

$$\begin{aligned} \max \quad & G(b,l) \\ G(a_i,0) &\leq c_i & & i \in N - \{k\} \\ G(a_k,1) &\leq c_k \\ & G \text{ subadditive}, \end{aligned} \tag{10}$$

where the extra coordinate in columns corresponds to the constraint $x_k = l$. Consider the feasible subadditive function $\bar{G}(d,s) = F(d) + (c_k - F(a_k))s$ to (10). The objective value of this function is $\bar{G}(b,l) = F(b) + (c_k - F(a_k))l \geq \hat{z}^{\mathrm{IP}}$ and therefore the objective value of the subadditive dual problem (10) is at least $\hat{z}^{\mathrm{IP}}$. This in turn implies that the objective value of the IP with $x_k = l$ is at least $\hat{z}^{\mathrm{IP}}$, which concludes the proof. □

If F is a feasible subadditive function, then

$$\sum_{i \in N} F(a_i)x_i \geq F(b) \tag{11}$$

is a valid inequality for $\{Ax = b, x \in \mathbb{Z}^n_+\}$, see e.g. Nemhauser and Wolsey (1988), page 229. Therefore for any α by considering $F = F_\alpha$ we get that

$$\sum_{i \in E} c_i x_i + \sum_{i \in H} (\alpha a_i)x_i \geq F_\alpha(b) \tag{12}$$

is a valid inequality. These inequalities are used in the computational experiments, Klabjan (2003).

Example (continued). For $\alpha = (-3.2, -1.75, 0.25, 2.75)$ and $\alpha = (-4, -1.5, 0.5, 3.5)$ valid inequality (12) reads

$$\begin{aligned} &-3.2x_1 - 1.75x_2 + 0.25x_3 + 2.75x_4 - 4.95x_5 - 2.95x_6 - 0.5x_7 - 2x_8 + \\ &\qquad x_9 + 3x_{10} - 4.7x_{11} - 3x_{12} - x_{13} + 1.25x_{14} - 1.95x_{15} \geq -4.75 \\ &-4x_1 - 1.5x_2 + 0.5x_3 + 3.5x_4 - 5.5x_5 - 3.5x_6 - 0.5x_7 - 2x_8 \\ &\qquad +2x_9 + 4x_{10} - 5x_{11} - 3x_{12} - x_{13} + 2.5x_{14} - 1.5x_{15} \geq -4.5, \end{aligned}$$

respectively. □

3 Basic Generator Subadditive Functions

Here we show that the set of all generator subadditive functions is convex and we give a finite subset of generator subadditive functions that yield strong duality. In addition, we discuss minimal generator subadditive functions.

Proposition 2. *If F_α and F_β are generator subadditive functions and $0 \le \lambda \le 1$, then $\lambda F_\alpha + (1-\lambda) F_\beta \le F_\gamma$, where $\gamma = \lambda\alpha + (1-\lambda)\beta$ and $E(\gamma) \subseteq E(\alpha) \cup E(\beta)$.*

Proof. First note that $E(\gamma) \subseteq E(\alpha) \cup E(\beta)$. Let $d \in \mathbb{R}^m_+$ and let

$$\begin{aligned}
\bar{z} &= \max\{(\gamma A^{E(\gamma)} - c^{E(\gamma)})x : A^{E(\gamma)} x \le d, x \in \mathbb{Z}_+^{|E(\gamma)|}\} \\
\tilde{z} &= \max\{(\alpha A^{E(\alpha)} - c^{E(\alpha)})x : A^{E(\alpha)} x \le d, x \in \mathbb{Z}_+^{|E(\alpha)|}\} \\
\hat{z} &= \max\{(\beta A^{E(\beta)} - c^{E(\beta)})x : A^{E(\beta)} x \le d, x \in \mathbb{Z}_+^{|E(\beta)|}\} .
\end{aligned}$$

We show that $\bar{z} \le \lambda\tilde{z} + (1-\lambda)\hat{z}$. Let $\bar{x}$ be the optimal solution to $\bar{z}$. Let us define the vector $\tilde{x}$ for each $i \in E(\alpha)$ as $\tilde{x}_i = \bar{x}_i$ for all $i \in E(\alpha) \cap E(\gamma)$ and 0 otherwise. Similarly let the vector $\hat{x}$ for each $i \in E(\beta)$ be defined as $\hat{x}_i = \bar{x}_i$ for all $i \in E(\beta) \cap E(\gamma)$ and 0 otherwise. Since $A^{E(\alpha)}\tilde{x} \le d$ and $A^{E(\beta)}\hat{x} \le d$, we have

$$\begin{aligned}
\bar{z} &= (\gamma A^{E(\gamma)} - c^{E(\gamma)})\bar{x} = \lambda(\alpha A^{E(\gamma)} - c^{E(\gamma)})\bar{x} + (1-\lambda)(\beta A^{E(\gamma)} - c^{E(\gamma)})\bar{x} \\
&\le \lambda(\alpha A^{E(\alpha)} - c^{E(\alpha)})\tilde{x} + (1-\lambda)(\beta A^{E(\beta)} - c^{E(\beta)})\hat{x} \le \lambda\tilde{z} + (1-\lambda)\hat{z} .
\end{aligned}$$

The claim now easily follows by definition. □

We denote

$$\mathcal{S} = \{F : \mathbb{R}^m_+ \to \mathbb{R} | F \text{ feasible subadditive function and there exists an } \alpha \text{ such that } F \le F_\alpha\} .$$

Clearly the set of all generator subadditive functions is a subset of $\mathcal{S}$. Next we show that $\mathcal{S}$ is convex and we give some of the extreme directions.

Corollary 1. *$\mathcal{S}$ is convex.*

Proof. Let $F_1 \in \mathcal{S}$ and $F_2 \in \mathcal{S}$. By Proposition 2 we have $\lambda F_1 + (1-\lambda)F_2 \le \lambda F_\alpha + (1-\lambda)F_\beta \le F_{\lambda\alpha+(1-\lambda)\beta}$, where $F_1 \le F_\alpha$ and $F_2 \le F_\beta$. □

The *asymptotic cone* of $\mathcal{S}$ is the set of all functions $\tilde{F}$ such that $F + \lambda\tilde{F} \in \mathcal{S}$ for all $\lambda > 0$ and for an $F \in \mathcal{S}$, see e.g. Hiriart-Urruty and Lemaréchal (1993), page 109.

Corollary 2. *Every ray generator subadditive function is in the asymptotic cone of $\mathcal{S}$.*

Proof. Let $\bar{F}_\beta$ be a ray generator subadditive function, let $F \in \mathcal{S}$ and let $\lambda > 0$. Then $F + \lambda\bar{F}_\beta \le F_\alpha + \lambda\bar{F}_\beta \le F_{\alpha+\lambda\beta}$, where the first inequality follows since $F \in \mathcal{S}$ and the second inequality can easily be proven by using the technique from the proof of Proposition 2. □

So far we have studied the generator subadditive functions as functions of α and given α we defined E. However we can also reverse this view. Suppose we are given a subset E of N. We would like to find a generator subadditive function F_{α^*} with the best objective value and such that $E(\alpha^*) \subseteq E$. It is easy to see that the objective value η^* and α^* have to be an optimal solution to the LP

$$\max\{\eta : (\eta, \alpha) \in Q_b(E)\} , \tag{13}$$

where

$$\begin{aligned}
Q_b(E) = \{\eta + \alpha(A^E x - b) &\le c^E x & & x \in \mathbb{Z}_+^{|E|}, A^E x \le b & (14)\\
\alpha a_i &\le c_i & & i \in H & (15)\\
(\eta, \alpha) &\in \mathbb{R} \times \mathbb{R}^m\} .
\end{aligned}$$

Constraints (14) express that η is a lower bound on $F_\alpha(b)$ and (15) guarantee that $F_\alpha(a_i) \le c_i$ for all $i \in H$. If (η^*, α^*) is an optimal solution to (13), then $F_{\alpha^*}(b) = \eta^*$ and clearly $E(\alpha^*) \subseteq E$. The LP (13) forms the basis of our algorithm. Note that $Q_b(E)$ might have a large number of constraints and therefore row generation is needed to solve (13). The details about solving (13) are described in Klabjan (2003).

Definition 3. *A generator subadditive function F_α is called a basic generator subadditive function, or a BG function, if $(F_\alpha(b), \alpha)$ is an extreme point of the polyhedron $Q_b(E(\alpha))$.*

A ray generator subadditive function $\bar{F}_\beta$ is called a basic ray generator subadditive function, or a DG function, if $(\bar{F}_\beta(b), \beta)$ is an extreme ray of the polyhedron $Q_b(E(\beta))$.

Note that since there is only a finitely many choices for E and for each E the polyhedron $Q_b(E)$ has only a finite number of extreme points and extreme rays, there is only a finite number of BG and DG functions. Let $F_{\alpha_k}, k \in K(b)$ be all the BG functions and let $\bar{F}_{\beta_j}, j \in J(b)$ be all the DG functions. The sets K and J here depend on b but in LP duality this is not the case. Next we show that these finite subsets of generator subadditive functions suffice for solving the IP or showing infeasibility.

Theorem 4. *If the IP is feasible, then in (1) it suffices to consider only BG functions $F_{\alpha_k}, k \in K(b)$. If the IP is infeasible, then there is a DG function $\bar{F}_{\beta_j}$ for a $j \in J(b)$ such that $\bar{F}_{\beta_j}(b) > 0$.*

Proof. If the IP is feasible, then the first statement follows either from Lagrangian duality, see e.g. Nemhauser and Wolsey (1988), pages 327-328, or by using Theorem 1 and the Minkowski's theorem.

Let now the IP be infeasible. Then there is a ray generator subadditive function $\bar{F}_{\hat{\beta}}$ with $\bar{F}_{\hat{\beta}}(b) > 0$. This function shows that $\max\{\eta : (\eta, \alpha) \in Q_b(E(\hat{\beta}))\}$ is unbounded and therefore there exists an extreme ray $(\tilde{\eta}, \tilde{\beta})$ of $Q_b(E(\hat{\beta}))$ with $\bar{F}_{\tilde{\beta}}(b) \geq \tilde{\eta} > 0$. By definition, $\bar{F}_{\tilde{\beta}}$ is a DG function. □

Next we show that only BG functions yield facet-defining (12) and that BG functions suffice to solve the IP as an LP.

Proposition 3. *If (12) is facet-defining, then F_α is a BG function.*

Proof. Suppose that F_α is not a BG function. Then $(F_\alpha(b), \alpha) = \sum_{i \in I} \lambda_i (F_{\alpha_i}(b), \alpha_i)$, where $(F_{\alpha_i}(b), \alpha_i) \in Q(E(\alpha))$ and $\sum_{i \in I} \lambda_i = 1, \lambda_i > 0$ for every $i \in I$. By Proposition 2 it is easy to see that $\sum_{j \in N} F_\alpha(a_j) x_j \geq F_\alpha(b)$ is dominated by the convex combination of valid inequalities $\sum_{j \in N} F_{\alpha_i}(a_j) x_j \geq F_{\alpha_i}(b), i \in I$ and therefore clearly cannot be a facet. □

Proposition 4.

$$
\begin{aligned}
z^{IP} = \min \ & cx \\
& Ax = b \\
\sum_{i \in N} F_{\alpha_k}(a_i) x_i & \geq F_{\alpha_k}(b) \qquad\qquad k \in K(b) \\
& x \geq 0 \,.
\end{aligned}
$$

Proof. Let z^* be the optimal value and x^* the optimal solution of the LP given in the proposition. Since $\sum_{i \in N} F_{\alpha_k}(a_i) x_i \geq F_{\alpha_k}(b), k \in K(b)$ are valid inequalities for $\{Ax = b, x \in \mathbb{Z}^n_+\}$, it follows that $z^* \leq z^{\mathrm{IP}}$. On the other hand for an optimal BG function $F_{\alpha_{\bar{k}}}, \bar{k} \in K(b)$ we have

$$
z^* = cx^* \geq \sum_{i \in N} F_{\alpha_{\bar{k}}}(a_i) x^*_i \geq F_{\alpha_{\bar{k}}}(b) = z^{\mathrm{IP}} \,,
$$

where the first inequality follows by dual feasibility of $F_{\alpha_{\bar{k}}}$ and the second one by primal feasibility of x^*. It follows that $z^* = z^{\mathrm{IP}}$. □

3.1 Minimal Generator Subadditive Functions

Generator subadditive functions yield valid inequalities (12). Clearly the inequalities that are dominated by other inequalities are redundant.

Definition 4. *A subadditive function F is minimal if there does not exist a subadditive function G such that $F(a_i) \geq G(a_i)$ for all $i \in N$, $F(b) \leq G(b)$, and at least one inequality is strict.*

From the definition it follows that F is minimal if and only if there exists a nonnegative integral vector x such that $Ax = b$ and $\sum_{i\in N} F(a_i)x_i = F(b)$. Minimal subadditive functions for master problems, i.e. the problems where A consists of all the columns a_i with $a_i \le b$, have been studied extensively by Araoz (1973). Note also that every minimal subadditive function defines a face. Next we characterize minimal generator subadditive functions.

Theorem 5. *Assume that A does not have dominated columns, i.e. $\{Ax \le a_i, x_i = 0, x \ne 0, x \in \mathbb{Z}^n_+\} = \emptyset$ for every $i \in N$. Then F_α is minimal if and only if there exists an optimal solution x^* to $\max\{(\alpha A^E - c^E)x : A^E x \le b, x$ nonnegative integer$\}$ such that $\{A^H x = b - A^E x^*, x$ nonnegative integer$\} \ne \emptyset$.*

Proof. Since by assumption A does not have dominated columns, it follows that $F_\alpha(a_i)$ equals to αa_i for all $i \in H$ and it equals to c_i for all $i \in E$. Let us denote $z = \max\{(\alpha A^E - c^E)x : A^E x \le b, x \text{ nonnegative integer}\}$.

Assume first that F_α is minimal. Then there is a nonnegative integral vector x such that $Ax = b$ and $\sum_{i\in N} F_\alpha(a_i)x_i = F_\alpha(b)$. Denote $x = (x^*, \tilde{x})$, where x^* corresponds to the coordinates in E. Then we have

$$F_\alpha(b) = \alpha b - z = \alpha(A^E x^* + A^H \tilde{x}) - z = \alpha A^E x^* + \alpha A^H \tilde{x} - z\,, \tag{16}$$

$$\sum_{i\in N} F_\alpha(a_i)x_i = \sum_{i\in E} c_i x_i + \sum_{i\in H} \alpha a_i x_i = c^E x^* + \alpha A^H \tilde{x}\,. \tag{17}$$

Since $\sum_{i\in N} F_\alpha(a_i)x_i = F_\alpha(b)$ and because of (16) and (17), we get $z = (\alpha A^E - c^E)x^*$, which shows the claim.

Suppose now that we have an x^* that attains the maximum in $\max\{(\alpha A^E - c^E)x : A^E x \le b, x \in \mathbb{Z}^{|E|}_+\}$ and $A^H \tilde{x} + A^E x^* = b$ for a nonnegative integral vector $\tilde{x}$. If we denote $x = (x^*, \tilde{x})$ it follows

$$\sum_{i\in N} F_\alpha(a_i)x_i = \alpha A^H \tilde{x} + c^E x^* = \alpha b - (\alpha A^E - c^E)x^* = F_\alpha(b)\,,$$

which completes the proof. □

Theorem 5 essentially shows that if F_α is minimal, then there is an optimal solution to $\max\{(\alpha A^E - c^E)x : A^E x \le b, x \text{ nonnegative integer}\}$ that can be 'extended' to a feasible IP solution. Next we give another sufficient condition for minimal generator subadditive functions that reveals further structure on the optimal solutions to $\max\{(\alpha A^E - c^E)x : A^E x \le b, x \text{ nonnegative integer}\}$.

Lemma 2. *Let F be a minimal subadditive function. Then for every $k \in N$ such that $\{Ax = b, x \in \mathbb{Z}^n_+, x_k \ge 1\} \ne \emptyset$, there exists an integer $l = l(k) > 1$ such that $lF(a_k) + F(b - la_k) = F(b)$.*

Proof. Let l be a nonnegative integer. Then $lF(a_k) + F(b - la_k) \ge F(la_k) + F(b - la_k) \ge F(b)$ since F is subadditive. We show the claim by contradiction. Suppose that for every integer $l, l \ge 1$ we have $lF(a_k) + F(b - la_k) > F(b)$. Let

$$t = \max_{s\ge 1, s \text{ integer}} \{\frac{F(b) - F(b - sa_k)}{s} : \text{ there exists } \bar{x} \in \mathbb{Z}^n_+ \text{ such that } \bar{x}_k = 0, A\bar{x} + sa_k = b\}\,.$$

By assumption t is well defined. We define a new inequality $\pi x \ge \pi_0$ as $\pi_i = F(a_i)$ for all $i \in N \setminus \{k\}$, $\pi_k = t$, and $\pi_0 = F(b)$. Next we show that $\pi x \ge \pi_0$ is a valid inequality for $\{Ax = b, x \in \mathbb{Z}^n_+\}$ that dominates (11).

Let $x \in \{Ax = b, x \in \mathbb{Z}^n_+\}$. If $x_k = 0$, then $\pi x \ge \pi_0$ since (11) is valid. Let us assume now that $x_k \ge 1$. Then

$$\begin{aligned}\sum_{i\in N} \pi_i x_i &= \sum_{i\in N\setminus\{k\}} \pi_i x_i + \pi_k x_k = \sum_{i\in N\setminus\{k\}} F(a_i)x_i + tx_k \\ &\ge F(\sum_{i\in N\setminus\{k\}} a_i x_i) + tx_k = F(b - a_k x_k) + tx_k \qquad (18)\\ &\ge F(b) = \pi_0\,, \qquad (19)\end{aligned}$$

where (18) follows from subadditivity of F and (19) from the definition of t. This shows validity.

From $lF(a_k) + F(b - la_k) > F(b)$ for all integer $l, l \geq 1$ it follows that $(F(b) - F(b - la_k))/l < F(a_k)$ and therefore $\pi_k = t < F(a_k)$. Since every valid inequality can be written in the form (11), it follows that there exists a subadditive function G that dominates F. This is a contradiction to minimality of F. $\square$

Theorem 6. *If F_α is minimal and A^E does not have dominated columns, then for every $k \in E$ with $\{Ax = b, x \in \mathbb{Z}^n_+, x_k \geq 1\} \neq \emptyset$ there exists an optimal solution $\tilde{x}$ to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x$ nonnegative integer$\}$ with $\tilde{x}_k > 0$.*

Proof. Let F_α be minimal and $k \in E$. By Lemma 2 there is an integer $l, l \geq 1$ such that $lF_\alpha(a_k) + F_\alpha(b - la_k) = F_\alpha(b)$. Since A^E does not have dominated columns, this condition is equivalent to

$$f(b) = f(b - la_k) + l(\alpha a_k - c_k),$$

where we denote $f(d) = \max\{(\alpha A^E - c^E)x : A^E x \leq d, x \text{ nonnegative integer}\}$.

Let $\bar{x}$ be an optimal solution to $f(b - la_k)$ and let us denote $\tilde{x} = \bar{x} + le_k$. Then

$$f(b - la_k) = (\alpha A^E - c^E)\bar{x} = (\alpha A^E - c^E)\tilde{x} - l(\alpha a_k - c_k) \leq f(b) - l(\alpha a_k - c_k) = f(b - la_k)\,,$$

where the inequality follows from $A^E \tilde{x} \leq b$. This shows that $\tilde{x}$ is an optimal solution to $f(b)$ and clearly $\tilde{x}_k > 0$. $\square$

Note that the extra condition $\{Ax = b, x \in \mathbb{Z}^n_+, x_k \geq 1\}$ only states that x_k is not 0 in any feasible solution. Theorem 6 reveals a peculiar structure of $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \text{ nonnegative integer}\}$. Namely, for every column $k \in E$ there is an optimal solution with the positive kth coordinate. The condition that A^E does not have dominated columns can be replaced with a weaker statement that $F_\alpha(a_i) = c_i$ for all $i \in E$.

Example (continued). We cannot use Theorem 5 since A has dominated columns.

For $\alpha = (-3.2, -1.75, 0.25, 2.75)$ we have $E = \{7, 8, 12, 13\}$, which has dominated columns, however, it is easy to check that $F_\alpha(a_i) = c_i$ for all $i \in E$ and we can use Theorem 6. The optimal solutions to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}^4_+\}$ are $x_2 = x_4 = 1$ and $x_2 = x_3 = 1$, and therefore by Theorem 6 F_α is not minimal.

For $\alpha = (-4, -1.5, 0.5, 3.5)$, we have $E = \{8, 12, 13\}$ and F_α is minimal since $x = e_7 + 2e_8$ satisfied the inequality at equality. We can verify Theorem 6 since the optimal solutions to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}^4_+\}$ are $x_1 = 2$ and $x_1 = x_2 = 1$, and $x_1 = x_3 = 1$. $\square$

4 Solution Methodology

We first briefly describe the main ideas of our algorithm that finds an optimal IP solution and it simultaneously computes a generator OSF. As is the case with almost all linear programming algorithms, in order to prove optimality we need a primal feasible solution and a dual feasible solution with the same value. Our algorithm as well maintains a primal feasible solution and a generator subadditive function. We try simultaneously to find a better primal solution and to improve the objective value of the generator function.

4.1 Algorithmic Framework

The main idea of the algorithm is as follows. Given E, we choose based on the current α a column i in H. Next we set $E = E \cup \{i\}$ and we update α by solving (13). The procedure is repeated until we find an α such that $F_\alpha(b) = cx$ for a nonnegative integer vector x with $\alpha_i x \geq c_i$ for all i with $x_i > 0$ and $Ax = b$.

Computational experiments have shown that solving (13) is difficult and therefore this framework needs to be enhanced. (13) has to be solved by row generation since we can have a large number of constraints (14). Instead we keep a subset $\hat{E} \subseteq \mathbb{Z}^n_+$ such that $\hat{E} \subseteq \{x \in \mathbb{Z}^n_+ : A^E x \leq b\}$, where

$$E = \cup_{x \in \hat{E}} \operatorname{supp}(x)\ . \tag{20}$$

By definition for every $x \in \hat{E}$ we have $A^E x \leq b$ but if $A^E x \leq b$, then x is not necessarily in $\hat{E}$. If $|\hat{E}|$ is much smaller than $|\{x \in \mathbb{Z}^n_+ : A^E x \leq b\}|$, then it should be easier to compute the new α. However now subadditivity is no longer automatic. In order to maintain subadditivity, we use ideas from Burdet and Johnson (1977).

We now work with functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $S(x) = \{y \in \mathbb{Z}^n_+ : y \leq x\}$. Given $\hat{E} \subseteq \mathbb{Z}^n_+$ and a vector α we define

$$\pi(x) = \alpha A x - \max_{\substack{y \in \hat{E} \\ y \in S(x)}} \{(\alpha A - c)y\} . \tag{21}$$

We say that π is dual feasible if $\pi(e_i) \leq c_i$ for every $i \in N$. If π is dual feasible and subadditive, and x is feasible to the IP, then

$$\pi(x) \leq \sum_{i \in N} \pi(e_i) x_i \leq cx . \tag{22}$$

Therefore π provides weak duality. By Theorem 1 there is a generator OSF F_α and consider $\hat{E} = \{x \in \mathbb{Z}^{|E|}_+, A^E x \leq b\}$, where E is defined based on α. It follows that there is a π that attains equality in (22) and therefore it gives strong duality. If $|\hat{E}|$ is much smaller than $|\{x \in \mathbb{Z}^n_+ : A^E x \leq b\}|$, then π has the advantage over the generator functions since it is easier to evaluate. On the other hand it is harder to encode π since we need to store α and $\hat{E}$. π does solve the IP but however it does not serve the purpose of the generator subadditive functions since it is defined on $\mathbb{R}^n$. Computational experiments have shown that in many instances π can be converted to a generator OSF without much effort by using relation (20).

We have converted our problem to the problem of solving

$$\begin{aligned} \max_{\pi} \max_{\substack{x \in \mathbb{Z}^n_+ \\ Ax = b}} & \pi(x) \\ & \pi(e_i) \leq c_i \qquad\qquad i \in N \\ & \pi \text{ subadditive} . \end{aligned}$$

We solve this problem by using the concept from the first paragraph of this section. We start with $\hat{E} = \emptyset$ and we gradually enlarge it. After every expansion we recompute α so as to maximize the objective value of π. Given an $\hat{E}$, we define $\hat{H} = \{x \in \mathbb{Z}^n_+ : x \notin \hat{E}, S(x) \setminus \{x\} \subseteq \hat{E}\}$. It follows from Burdet and Johnson (1977) that π is subadditive if and only if $\alpha A x \leq cx$ for every $x \in \hat{H}$. Given $\hat{E}$ and $\hat{H}$, α that gives the largest dual objective value is the optimal solution to

$$D(\hat{E}, \hat{H}) \qquad \begin{aligned} & \max \ \pi_0 \\ & \pi_0 + \alpha(Ay - b) \leq cy && \text{for all } y \in \hat{E} && (23) \\ & \alpha A x \leq cx && x \in \hat{H} && (24) \\ & \alpha \text{ unrestricted}, \pi_0 \ \text{ unrestricted} . \end{aligned}$$

(23) capture the objective value and (24) assure that π stays subadditive. In other words, we maximize the dual objective value while maintaining subadditivity.

The algorithm is given in Algorithm 1. Steps 3 and 4 expand $\hat{E}$ and update $\hat{H}$. It is easy to check that $\hat{H}$ satisfies the definition. In step 5 we update α and steps 6 and 7 update the dual and the primal value, respectively.

```
1: Ê = {0}, Ĥ = {e_i : i ∈ N}, α = optimal dual vector of the LP relaxation, w^D = −∞.
2: loop
3:   Choose a vector x̄ ∈ Ĥ.
4:   Ê = Ê ∪ {x̄}, Ĥ = Ĥ ∪ {x̄ + e_i : i ∈ N, y ∈ Ê for every y ≤ x̄ + e_i, y ∈ Z^n_+}
5:   Update α by solving D(Ê, Ĥ). Let π*_0 be the optimal value.
6:   w^D = max{w^D, π*_0}
7:   If w^D = min{cx : x ∈ Ĥ, Ax = b}, then we have solved the IP and exit.
8: end loop
```

Algorithm 1: The basic algorithm

The overall algorithm for computing a generator OSF has 3 stages. In the first stage we find π based on Algorithm 1 and the corresponding optimal IP solution x^*. The basic enhancement to Algorithm 1 is to observe that it suffices that π be subadditive on $\{x \in \mathbb{Z}^n_+, Ax = b\}$. Furthermore, if we know that $x_i = 0$ in an optimal solution for all $i \in G \subseteq N$ (e.g. by using Theorem 3), then it suffices that π is subadditive on $\{x \in \mathbb{Z}^{|N \setminus G|}_+, A^{N \setminus G} x = b\}$. In addition, if we have an IP solution with value $\hat{z}^{\mathrm{IP}}$, then it is enough that π is subadditive on $\{x \in \mathbb{Z}^{|N \setminus G|}_+, A^{N \setminus G} x = b, c^{N \setminus G} x < \hat{z}^{\mathrm{IP}}\}$. When $\hat{z}^{\mathrm{IP}} = z^{\mathrm{IP}}$, this set is empty and therefore in stage 1 we stop when $D(\hat{E}, \hat{H})$ becomes larger than $\hat{z}^{\mathrm{IP}}$. Stage 2 then finds a π, which is subadditive in $\{x \in \mathbb{Z}^{|S|}_+ : A^S x = b, c^S x \leq z^{\mathrm{IP}}\}$ for an $S \subseteq N$ that includes $\mathrm{supp}(x^*)$. Note that this π has to satisfy the complementary slackness conditions. In the last stage we obtain a generator OSF. We start with E as defined in (20), where $\hat{E}$ is obtained in stage 2. Now we gradually keep expanding E and this time we cannot avoid solving (13). In the last stage we use the fact that $\alpha a_i \geq c_i$ for every i with $x^*_i > 0$.

In view of the above discussion, at every iteration of the algorithm every column is either *active* or *non active*. A column i is non active if we know that $x_i = 0$ in an optimal solution. We say that a generator OSF F or π is *optimal over* $S \subseteq N$ if it is an optimal subadditive dual function for the IP $\min\{c^S x : A^S x = b, x \text{ nonnegative integer}\}$.

It is important that in Algorithm 1 we keep the size of $\hat{E}$ and $\hat{H}$ low. During the computation, the algorithm dynamically strengthens the upper bounds on the variables by using reduced cost fixing. At the end π is not necessarily dual feasible since columns i with the imposed upper bound $x_i \leq 0$ (i.e. non active columns) may have $\pi(e_i) > c_i$. This is the main reason why stages 2 and 3 are needed.

Next we give a detailed description of each stage.

4.2 Stage 1: Obtaining an Optimal Primal IP Solution with the Use of Subadditive Functions π

We first elaborate on the observation that it suffices to have subadditivity on $\{Ax = b, cx < \hat{z}^{\mathrm{IP}}, x \in \mathbb{Z}^n_+\}$. This means that we can remove from $\hat{H}$ and $\hat{E}$ all the elements that yield an objective value larger than $\hat{z}^{\mathrm{IP}}$. We call this operation pruning. If $\bar{x} \in \hat{H}$ and $t(\bar{x}) + c\bar{x}$ is greater or equal to $\hat{z}^{\mathrm{IP}}$, then we can remove $\bar{x}$ from $\hat{H}$, where

$$P(\bar{x}) \qquad \begin{aligned} t(\bar{x}) = \min\ & cx \\ Ax &= b - A\bar{x} \\ x &\geq 0\,. \end{aligned}$$

Ideally we would like to solve $P(\bar{x})$ over all nonnegative integer vectors x however this is computationally intractable and we consider the LP relaxation $P(\bar{x})$. Note that the cardinality of $\hat{H}$ can increase by n in each iteration (see step 4 of Algorithm 1) and therefore solving this LP relaxation at every iteration for each element of $\hat{H}$ is too time consuming. Instead, in step 3 of Algorithm 1, after we select an element $\bar{x}$ from $\hat{H}$ that is moved to $\hat{E}$, we compute $P(\bar{x})$. If the element is pruned, then it is removed from $\hat{H}$ and the selection process is repeated.

The second major enhancement we employ is the selection of an element from $\hat{H}$ that is added to $\hat{E}$. In step 3 of Algorithm 1 we have to select an element from $\hat{H}$ that is appended to $\hat{E}$. We choose an element from $\hat{H}$ judiciously based on the ideas of pseudocosts, see e.g. Linderoth and Savelsbergh (1999).

Given a generator set $\hat{E}$, vector α that gives the largest dual objective value is the optimal solution to $D(\hat{E}, \hat{H})$. It is clear that if we select an arbitrary element from $\hat{H}$, append it to $\hat{E}$, update $\hat{H}$ and $\hat{E}$, and we compute α from $D(\hat{E}, \hat{H})$, we obtain a feasible subadditive dual function with the objective value that equals to the objective value of this LP. Given an $\tilde{\alpha}$ from the previous iteration, next we describe our approach to selecting a vector from $\hat{H}$.

After moving an element to $\hat{E}$, the candidate set $\hat{H}$ is expanded. Note that for $y = 0$, which is always in $\hat{E}$, (23) reads $\pi_0 - \alpha b \leq 0$. The new $D(\hat{E}, \hat{H})$ differs from the previous one by relaxing $\alpha A\bar{x} \leq c\bar{x}$ to $\pi_0 + \alpha(A\bar{x} - b) \leq c\bar{x}$ for $\bar{x}$ that is moved from $\hat{H}$ to $\hat{E}$, and by introducing additional constraints (24) corresponding to the new elements in $\hat{H}$. For the former, we would like to move the most binding constraint (24) and therefore the candidate elements are all $\bar{x} \in H$ with $c\bar{x} - \tilde{\alpha} A\bar{x}$ below a given small number κ. For simplicity of notation, let $\tilde{H} = \{\bar{x} \in \hat{H} : c\bar{x} - \tilde{\alpha} A\bar{x} < \kappa\}$.

The optimality is achieved if the objective value of a feasible subadditive function equals to the value of an IP feasible solution and therefore it is also important to obtain good IP solutions. Given $\bar{x} \in \tilde{H}$, there exists $\hat{x} \in \mathbb{Z}^n_+$ with $\hat{x} \geq \bar{x}$ and $A\hat{x} = b$ if and only if there exists a nonnegative integer vector x satisfying $Ax = b - A\bar{x}$. We would like to obtain a vector x that yields the smallest overall cost. Since this requires solving IPs, we relax the integrality of x to $x \geq 0$. However in this case the reduced cost $c_i - \tilde{\alpha} a_i$ is a better measure of improvement than the cost (see the discussions in Section 5). To each $\bar{x} \in \tilde{H}$ we assign a score

$$
\begin{aligned}
s(\bar{x}) = \min (c - \tilde{\alpha} A)x & \\
Ax &= b - A\bar{x} \\
x &\geq 0\,.
\end{aligned}
$$

If we select the element as $\min\{s(\bar{x}) : \bar{x} \in \tilde{H}\}$, then we move an element that minimizes the slack of the newly introduced constraints (24) since the new constraints read $\alpha A(\bar{x} + e_i) = \alpha A\bar{x} + \alpha a_i \leq c(\bar{x} + e_i) = c\bar{x} + c_i$. On the other hand, if we select the element as $\max\{s(\bar{x}) : \bar{x} \in \tilde{H}\}$, then we increase the chance that the selected element is pruned and therefore permanently removed from consideration in stage 1. Computational experiments have shown that the latter strategy performs substantially better.

Computing $s(\bar{x})$ for every $\bar{x} \in \tilde{H}$ is computationally too expensive and therefore we use the idea of pseudocosts. First we rewrite the objective function as $(c - \tilde{\alpha} A)x = cx - \tilde{\alpha} Ax = cx - \tilde{\alpha}(b - A\bar{x}) = cx + \tilde{\alpha} A\bar{x} - \tilde{\alpha} b$ and therefore $s(\bar{x}) = t(\bar{x}) + \tilde{\alpha} A\bar{x} - \tilde{\alpha} b$. Let $D_i, i \in N$ approximate the per unit change of $t(\bar{x})$ if x_i is fixed at $\bar{x}_i + 1$. In Linderoth and Savelsbergh (1999) D_i is called the up pseudocost. Note that in our case based on the definition of the candidate set in the iterations that follow we are only interested in $t(\hat{x})$ for $\hat{x} \geq \bar{x}$ and therefore we only need to consider up pseudocosts. If $\bar{x}_i \geq 1$, then we estimate $t(\bar{x}) \approx t(\bar{x} - e_i) + (1 - f_i)D_i$, where f_i is the fractional part of variable i in the optimal LP solution to $t(\bar{x} - e_i)$.

The up pseudocosts are averaged over all the observed values. Whenever we solve $t(\bar{x})$, we compute

$$
u_i = \frac{t(\bar{x}) - t(\bar{x} - e_i)}{1 - f_i}\,,
$$

if variable i is fractional in $t(\bar{x} - e_i)$. D_i is then updated by taking the average of the observed values so far and u_i. The up pseudocosts are initialized as suggested in Linderoth and Savelsbergh (1999). They suggest a lazy evaluation by initializing them only when needed for the first time. Each time a pseudocost D_i is needed, a given number of simplex iterations is carried out on $P(e_i)$. Observe that $P(\bar{x})$ can be rewritten as $\min\{cx : Ax = b, x \geq \bar{x}, x \geq 0\}$ and therefore only lower bound changes on variables are needed. It means that from one iteration to the next iteration we can efficiently compute $t(\bar{x})$ by using the dual simplex algorithm.

At every iteration in stage 1 we apply reduced cost fixing and after a given number of iterations we add to $P(\bar{x})$ the valid inequality (12) resulting from the current subadditive function.

Note that since we prune all elements $\bar{x}$ with $c\bar{x} < \hat{z}^{\mathrm{IP}}$, after obtaining an optimal IP solution the set $\{Ax = b, cx < \hat{z}^{\mathrm{IP}}, x \in \mathbb{Z}^n_+\}$ is empty and therefore the dual problem is unbounded from above. Therefore the stopping criteria is when the dual objective π_0 of $D(\hat{E}, \hat{H})$ becomes greater or equal to $\hat{z}^{\mathrm{IP}}$.

The algorithm is presented in Algorithm 2. Besides $\hat{H}$ and $\hat{E}$ to keep track of pseudocosts we have two additional sets $C^{\hat{H}}$ and $C^{\hat{E}}$. There is a one to one correspondence among the elements in $\hat{E}$ and $C^{\hat{E}}$, and the elements in $\hat{H}$ and $C^{\hat{H}}$. Every element in $C^{\hat{E}}$ and $C^{\hat{H}}$ is a triple (l, f, i). For $x \in \hat{H}$ or $x \in \hat{E}$, we denote by $p(x)$ the corresponding element in $C^{\hat{H}}$ or $C^{\hat{E}}$ and for (l, f, i) we denote by $\tilde{x}(l, f, i)$ the corresponding vector in either $\hat{H}$ or $\hat{E}$. Given $\bar{x} \in \hat{H}$, by definition of the candidate set, there is a unique i such that $\bar{x} = \tilde{x} + e_i$ and $\tilde{x} \in E$. In a triplet (l, f, i) from $C^{\hat{H}}$, i is this unique variable index, $l = t(\tilde{x})$ and $f = x_i^*$, where x^* is the optimal solution to $P(\tilde{x})$. $C^{\hat{E}}$ is needed when we move vectors from $\hat{E}$ to $\hat{H}$. In the algorithm π_0 denotes the best lower bound obtained, i.e. the objective value of the best feasible subadditive function obtained so far.

Note that in $D(\hat{E}, \hat{H})$ we do not require that $\alpha a_i > c_i$ for all $e_i \in \hat{E}$ and therefore $\hat{E}$ may consist of some vectors that can be moved to $\hat{H}$ without affecting feasibility and the dual objective value. Moving such vectors from $\hat{E}$ to $\hat{H}$ is carried out only occasionally since it requires a major rebuilt of $D(\hat{E}, \hat{H})$. We do this cleaning in steps 7-14 only if $|\hat{E}|$ is too large or we have obtained a better lower bound. Here and only here we add valid inequalities (12). The function `Solve_And_Update`, given in Function 1, solves $D(\hat{E}, \hat{H})$ and updates π_0. Next in steps 16-26 the algorithm performs reduced cost fixing. In step 27 we call the function `Select`, which selects a vector from $\hat{H}$, performs pruning and updates pseudocosts and $\hat{z}^{\mathrm{IP}}$. This function is given in Function 2. In steps 28-30 we expand the candidate set $\hat{H}$ and we update the data for pseudocosts. At the end, in step 31, we solve $D(\hat{E}, \hat{H})$ to obtain α for the next iteration.

1: Form and solve $P(0)$. Let x^*, y^* be an optimal primal and dual solution, respectively.
2: $\alpha = y^*, \hat{z}^{\text{IP}} = \infty, \pi_0 = b\alpha, \hat{H} = \{e_i : i \in N\}, C^{\hat{H}} = \{(w^D, x_i^*, i) : i \in N\}, \hat{E} = C^{\hat{E}} = \emptyset$
3: Form the linear program $D(\hat{E}, \hat{H})$.
4: **loop**
5: $\quad S = \{i \in N : \alpha a_i > c_i\}$
6: $\quad$ /* Remove unnecessary elements from $\hat{E}$. */
7: $\quad$ **if** $|\hat{E}|$ is too large or π_0 has increased in the previous iteration **then**
8: $\quad\quad$ **for all** $\bar{x} \in \hat{E}$ such that $\text{supp}(\bar{x}) \not\subseteq S$ **do**
9: $\quad\quad\quad \hat{E} = \hat{E} \setminus \{\bar{x}\}, \hat{H} = \hat{H} \cup \{\bar{x}\}, C^{\hat{E}} = C^{\hat{E}} \setminus \{p(\bar{x})\}, C^{\hat{H}} = C^{\hat{H}} \cup \{p(\bar{x})\}$
10: $\quad\quad$ **end for**
11: $\quad\quad$ Solve_and_Update( )
12: $\quad\quad$ Add (12) to $P(0)$.
13: $\quad\quad$ Goto step 5.
14: $\quad$ **end if**
15: $\quad$ /* reduced cost fixing */
16: $\quad$ **for all** $k \notin S$ **do**
17: $\quad\quad$ **if** $c_k - \alpha a_k > 0$ **then**
18: $\quad\quad\quad$ Add $x_k \leq v - 1$ to $P(0)$, where v is defined by (9).
19: $\quad\quad\quad$ Remove from $C^{\hat{H}}$ all elements (l, f, i) with $\tilde{x}(l, f, i)_k \geq v$.
20: $\quad\quad\quad$ Remove from $\hat{H}$ all elements $\bar{x}$ with $\bar{x}_k \geq v$.
21: $\quad\quad$ **end if**
22: $\quad$ **end for**
23: $\quad$ **if** $\hat{H}$ has been reduced **then**
24: $\quad\quad$ Solve_and_Update( )
25: $\quad\quad$ Goto step 5.
26: $\quad$ **end if**
27: $\quad (\bar{x}, x^*, j)$=Select( ) /* Select and element from $\hat{H}$ that is moved to $\hat{E}$. $\bar{x}$ is the selected vector, x^* is the optimal solution to $P(\bar{x})$ and $\bar{x} = x + e_j$ for a unique $x \in \hat{E}$. */
28: $\quad \hat{E} = \hat{E} \cup \{\bar{x}\}, C^{\hat{E}} = C^{\hat{E}} \cup \{(cx^*, x_j^*, j)\}, \hat{H} = \hat{H} \setminus \{\bar{x}\}, C^{\hat{H}} = C^{\hat{H}} \setminus \{p(\bar{x})\}$
29: $\quad Q = \{i \in N : y \in \hat{E}$ for every $y \leq \bar{x} + e_i, y \in \mathbb{Z}_+^n\}$ /* Update $\hat{H}$ to be the candidate set. */
30: $\quad \hat{H} = \hat{H} \cup \{\bar{x} + e_i : i \in Q\}, C^{\hat{H}} = C^{\hat{H}} \cup \{(cx^*, x_i^*, i) : i \in Q\}$
31: $\quad$ Solve_and_Update( )
32: **end loop**

Algorithm 2: Stage 1 algorithm

Update $D(\hat{E}, \hat{H})$ to be consistent with the current $\hat{E}$ and $\hat{H}$.
Solve $D(\hat{E}, \hat{H})$.
if $D(\hat{D}, \hat{H})$ is unbounded and $z^{\text{IP}} = \infty$ **then**
$\quad$ IP is infeasible and exit.
else
$\quad$ Let (α^*, π_0^*) be an optimal solution to $D(\hat{E}, \hat{H})$.
$\quad \alpha = \alpha^*, \pi_0 = \max\{\pi_0, \pi_0^*\}$
end if
if $\pi_0 \geq \hat{z}^{\text{IP}}$ **then**
$\quad \hat{z}^{\text{IP}}$ is the optimal IP value and exit.
end if

Function 1: Solve_and_Update( )

1: **for all** $(l, f, i) \in C^{\hat{H}}$ with $c\tilde{x}(l, f, i) - \alpha A\tilde{x}(l, f, i) \leq \kappa$ **do**
2: Select the element (l, f, i) with the largest value $l + (1 - f)D_i + \alpha A\tilde{x}(l, f, i) - \alpha b$.
3: **end for**
4: Let $(\bar{l}, \bar{f}, \bar{i})$ be the element where the maximum is attained. Denote $\tilde{x} = \tilde{x}(\bar{l}, \bar{f}, \bar{i})$.
5: Solve $P(\tilde{x})$ and let x^* be an optimal solution.
6: Let $u_{\bar{i}} = (t(\tilde{x}) - \bar{l})/(1 - \bar{f})$ and update $D_{\bar{i}}$ to be the average among all $u_{\bar{i}}$ computed up to this point.
7: /* pruning */
8: **if** $t(\tilde{x}) \geq \hat{z}^{\text{IP}}$ **then**
9: Remove $\tilde{x}$ from $\hat{H}$ and $(\bar{l}, \bar{f}, \bar{i})$ from $C^{\hat{H}}$.
10: **if** $\tilde{x} = e_{\bar{i}}$ **then**
11: Fix $x_{\bar{i}} = 0$ in $P(0)$.
12: **end if**
13: Goto step 1.
14: **end if**
15: **if** x^* is integral and $c(\bar{x} + x^*) < \hat{z}^{\text{IP}}$ **then**
16: $\hat{z}^{\text{IP}} = c(\bar{x} + x^*)$
17: **end if**
18: Return $(\tilde{x}, x^*, \bar{i})$.

Function 2: Select( )

4.3 Stage 2: Improving the Feasibility of the Subadditive Dual Function π

Let x^* be the optimal IP solution and let α^* be the vector associated with the final subadditive function π^* from stage 1. Recall that π^* does not necessarily satisfy complementary slackness and it may have non active columns i with $x_i^* \geq 1$. The objective of this stage is to obtain an optimal subadditive function π that is subadditive on $Z = \{x \in \mathbb{Z}_+^{|S|} : A^S x = b, c^S x \leq z^{\text{IP}}\}$, where S includes all columns in $\text{supp}(x^*)$. Therefore if $\tilde{x}$ is an optimal IP solution, then $\tilde{x} \in Z$ and π has to satisfy the complementary slackness condition.

Stage 2 uses α^* to warm start π and it applies Algorithm 2 with some adjustments. In stage 1 every time we update the upper bound $\hat{z}^{\text{IP}}$ in step 16 of Function 2, we also record all the pseudocost values and the active global upper bounds on variables. At the beginning of stage 2 we set the pseudocost values to the last recorded values, we change the upper bounds on variables to the recorded values and we remove from $P(0)$ all the inequalities that were added since the last update to $\hat{z}^{\text{IP}}$ in stage 1. These operations guarantee that all the active columns include all the optimal IP solutions, i.e. if $\tilde{x}$ is an optimal IP solution, then the columns in $\text{supp}(\tilde{x})$ are active. Next we set $E = \{i \in N : \alpha^* a_i > c_i, i \text{ active}\}, H = \{i \in N : \alpha^* a_i \leq c_i, i \text{ active}\}$ and we solve (13) by including only those constraints (14) for which there exists $\tilde{x} \in \mathbb{Z}_+^n$ such that $Ax + A\tilde{x} = b$. These constraints suffice to have subadditivity on Z. Let α be the computed vector, which is used to obtain an initial subadditive function π as follows. Initially let $\hat{E} = \{i \in N : \alpha a_i > c_i, i \text{ active}\}$ and then we make it subinclusive by adding subsets. Based on the computed subinclusive $\hat{E}$ we construct the candidate set $\hat{H}$. We next solve $D(\hat{E}, \hat{H})$ and we apply reduced cost fixing. Computational experiments have shown that the optimal value of $D(\hat{E}, \hat{H})$ is a much better starting point than starting from the value of the linear programming relaxation.

Next we again run Algorithm 2 but with slight modifications. We skip the initialization steps 1-3 since they were described above. Step 18 in Algorithm 2 is modified by replacing $v-1$ with v and in steps 19 and 20 we change v to $v+1$. The condition in step 8 of Function 2 is replaced by $t(\tilde{x}) > z^{\text{IP}}$. Step 1 in Function 2 is enhanced to reflect that we must satisfy complementary slackness for x^*. Namely, in addition to considering all the elements in $C^{\hat{H}}$ as described in step 1, we also consider the elements with $\tilde{x}(l, f, i) = e_i, x_i^* \geq 1$. For such elements $\alpha a_i \geq c_i$ and therefore they are likely candidates for $\hat{E}$. It is easy to see that if we apply these adjustments, the resulting π is subadditive on $\{x \in \mathbb{Z}_+^{|S|} : A^S x = b, c^S x \leq z^{\text{IP}}\}$ for $\text{supp}(x^*) \subseteq S$ and it has the objective value z^{IP}.

4.4 Stage 3: Computing a Generator Optimal Subadditive Function

In this stage we obtain a generator OSF.

Let us denote by S the set of all the active columns as computed in stage 2. We first obtain a generator OSF F_α for the IP $\min\{c^S x : A^S x = b, x \text{ nonnegative integer}\}$, i.e. F_α is optimal over S. Since typically the cardinality of S in stage 2 is low, this can easily be achieved by solving (13) with $E = S, H = \emptyset$ and $A = A^S$. In all performed computational experiments a generator OSF for this IP is F_α, where α is obtained in stage 2. We denote by F_α the computed OSF.

The remaining part of stage 3 gradually introduces non active columns. First observe that F_α is a generator OSF over $\bar{S} = S \cup \{i \in N \setminus S : \alpha a_i \le c_i\}$ and therefore we set $S = \bar{S}$. Typically at this point the size of $N \setminus S$ is several thousand columns. Note that $\alpha a_i > c_i$ for every $i \in N \setminus S$. We can view F_α as either a feasible subadditive function on N (but not necessarily optimal) or as a subadditive function with value z^{IP} that violates some of the dual feasibility constraints. In the former case we have to add the columns in $N \setminus S$ to E, which can decrease the objective value of the function. We first use a simple heuristic based on the following theorem to expand S.

Proposition 5. *Let F_α be a generator OSF over S and let $C \subseteq N \setminus S$ be such that the objective value of the LP*

$$\begin{aligned} \min \quad & \sum_{i\in E}(c_i - F_\alpha(a_i))x_i + \sum_{i\in H}(c_i - \alpha a_i)x_i + \sum_{i\in C}(c_i - \alpha a_i)y_i \\ & A^S x + A^C y = b \\ & 0 \le x, 0 \le y \end{aligned} \tag{ELP}$$

is greater or equal to 0. Let γ be the optimal dual vector to this LP. Then $F_{\alpha+\gamma}$ is a generator OSF over $S \cup C$.

Proof. First note that the objective value of ELP is 0 since the optimal IP solution gives the objective value 0. Therefore $\gamma b = 0$. For all $i \in H \cup C$ clearly $(\alpha + \gamma)a_i \le c_i$ since γ is dual feasible to ELP. This implies that $E(\alpha + \gamma) \subseteq E$.

It suffices to show that

$$\max\{((\alpha+\gamma)A^E - c^E)x : A^E x \le b, x \in \mathbb{Z}_+^{|E|}\} \le (\alpha+\gamma)b - F_\alpha(b) = \alpha b - F_\alpha(b)\,. \tag{25}$$

Let x be a nonnegative integer vector with $A^E x \le b$. For every $i \in E$ let $F_\alpha(a_i) = \alpha a_i - (\alpha A^E - c^E)z^i$, where $A^E z^i \le a_i, z^i \in \mathbb{Z}_+^{|E|}$. Then we have

$$\begin{aligned} \sum_{i\in E}(\gamma a_i + \alpha a_i - c_i)x_i &\le \sum_{i\in E}(\alpha A^E - c^E)z^i x_i = (\alpha A^E - c^E)\sum_{i\in E} z^i x_i \\ &\le \max\{(\alpha A^E - c^E)x : A^E x \le b, x \in \mathbb{Z}_+^{|E|}\} = \alpha b - F_\alpha(b)\,, \end{aligned}$$

where the first inequality follows from dual feasibility to ELP of γ and the second one from

$$A^E(\sum_{i\in E} z^i x_i) = \sum_{i\in E}(A^E z^i)x_i \le \sum_{i\in E} a_i x_i \le b\,.$$

The last equality holds by optimality of F_α over S. This shows (25) and therefore the claim. $\square$

Note that in ELP all the objective coefficients of the variables in S are nonnegative and the optimal IP solution yields an objective value 0. Therefore if $y_i = 0$ for all $i \in C$, then the objective value of ELP is 0. The objective coefficients of variables in C are negative and therefore these variables can push the objective value below 0.

The algorithm employs the following heuristic, called the *LP based expansion heuristic*, to expand S. We start with $C = N \setminus S$ and we repeat the following steps. We solve ELP and let (x^*, y^*) be an optimal solution. If the objective value of ELP is negative, we set $C = C \setminus \bar{C}$, where $\bar{C} = \{i \in C : y_i^* > 0\}$, and we repeat the procedure. Otherwise we abort the loop. At the end the objective value of ELP is 0 and therefore by Proposition 5 we have a readily available generator OSF over $S \cup C$. The heuristic essentially 'kills' all the y variables that are positive in the optimal solution and therefore such variables contribute toward the negative objective value. The proposed heuristic does not necessarily find the maximum cardinality set C but it performs well in practice. After applying the LP based expansion heuristic, we have a generator OSF over $S = S \cup C$. In our computational experiments this heuristic decreases the size of $N \setminus S$ from several thousand to just a dozen of columns.

In the following we show how to expand S to N, i.e. how to include the remaining few columns in $N \setminus S$. The algorithm gradually expands S and is given in Algorithm 3. We first append a column from $N \setminus S$ to S and we update E to reflect this change. At this point F_α is not necessarily a generator OSF over S since the new column that is added to E may decrease the objective value. Recall that given a set E, (13) computes the generator subadditive dual function with the largest objective value and therefore in step 3 we compute this LP. If the new F_α is not a generator OSF over S, we expand E by moving some elements from $S \setminus E$ to E and we repeat the procedure. We perform this in step 13. Note that if $E_1 \subseteq E_2$, then $Q_b(E_1) \subseteq Q_b(E_2)$ and therefore the value of the new generator subadditive function is only greater or equal to the value of the current function. If some of the columns in $N \setminus S$ can be included in S for 'free' without decreasing the objective value of the current F_α, we expand S, step 14. If $S = N$ and the objective value of F_α is z^{IP}, then clearly F_α is a generator OSF over N. If $S \neq N$ and $F_\alpha(b) = z^{\mathrm{IP}}$, then F_α is a generator OSF over S and we first try to further expand S by applying the LP based column expansion heuristic and then we expand S as shown in steps 9 and 10.

1: Let j be the column where $\max\{\alpha a_i - c_i : i \in N \setminus S\}$ is attained and set $S = S \cup \{j\}, E = E \cup \{j\}$.
2: **loop**
3:   Solve (13) with $A = A^S, S = E \cup H$, and let α be the optimal solution and η^* the objective value.
4:   **if** $\eta^* = z^{\mathrm{IP}}$ **then**
5:     **if** $S = N$ **then**
6:       F_α is a generator OSF and quit.
7:     **else**
8:       Apply the LP based column expansion heuristic, which possibly yields a new S and a new OSF F_α over S.
9:       Let j be the column where $\max\{\alpha a_i - c_i : i \in N \setminus S\}$ is attained.
10:       $S = S \cup \{j\}, E = E \cup \{j\}$.
11:     **end if**
12:   **else**
13:     Select a subset $\tilde{E}$ of columns from $S \setminus E$ and set $E = E \cup \tilde{E}$.
14:     $S = S \cup \{i \in N \setminus S : \alpha a_i \leq c_i\}, H = S \setminus E$
15:   **end if**
16: **end loop**

Algorithm 3: Stage 3 algorithm

Next we describe how to perform step 13, i.e. how to select columns that are appended to E. This selection procedure, called *generator expansion*, is crucial since efficient generator expansion decreases the number of steps 3 in the algorithm, which turned out to be the bottleneck due to the large number of rows in (13). In order to understand generator expansion, we need the following proposition.

Proposition 6. *Let x^* be an optimal IP solution and F_α a generator OSF. Then $\bar{x}$ defined for the columns $i \in E$ by $\bar{x}_i = x_i^*$ is an optimal solution to* $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}_+^{|E|}\}$.

Proof. By definition of E and H we have $A^E \bar{x} + A^H \tilde{x} = b$, where $\tilde{x}$ is defined for the columns in H as

$\tilde{x}_i = x_i^*$. Then we have

$$z^{\mathrm{IP}} = F_\alpha(b) = b\alpha - \max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}_+^{|E|}\} \leq b\alpha - (\alpha A^E - c^E)\bar{x} = \tag{26}$$

$$= \alpha(b - A^E\bar{x}) + c^E\bar{x} = \alpha A^H \tilde{x} + c^E \bar{x}$$

$$= \sum_{i\in H}(\alpha a_i)\tilde{x}_i + c^E\bar{x} \leq \sum_{i \in H} c_i \tilde{x}_i + c^E \bar{x} = cx^* = z^{\mathrm{IP}}\ , \tag{27}$$

where the inequality (26) holds since $\bar{x}$ is feasible to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}_+^{|E|}\}$ and the inequality (27) is true since for all $i \in H$ we have $\alpha a_i \leq c_i$. Since the lower bound equals to the upper bound, it follows that (26) is an equality, meaning that $\bar{x}$ is optimal to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \in \mathbb{Z}_+^{|E|}\}$. □

We apply 2 strategies for generator expansion. Both of them are based on the dual prices of (13). After this LP is solved in step 3, let y^* be the optimal dual vector. In order to decrease the dual value of (13), it is beneficial to move columns $i \in H$ with large y_i^*. Let $d = \max\{y_i^* : i \in H\}$. The problem we face is that many columns i may have $y_i^* = d$ and therefore all such columns would be indifferent based solely on this criteria. Another important observation is based on Proposition 6. Let $R = \{i \in E, x_i^* > 0\}$ and let $r = \sum_{i\in R} a_i$. If there is a column $j \in E$ with $x_j^* = 0$ and $\mathrm{supp}(a_j) \cap \mathrm{supp}(r) = \emptyset$, then clearly F_α is not optimal by Proposition 6. Therefore promising candidates are columns $j \in H$ with the property that $\mathrm{supp}(a_j) \cap \mathrm{supp}(r) \neq \emptyset$. In many instances the initial generator F_α, i.e. the one before Algorithm 3 is applied, has $E = \emptyset$ and therefore this observation applies only after some number of iterations of the loop in steps 2-16 in Algorithm 3. We design two different generator expansion strategies called *initial* and *disjoint generator expansion*. The initial generator expansion is applied at early iterations of the algorithm. Once R becomes large enough, we switch to disjoint generator expansion.

The two generator expansion procedures are as follows. We first describe initial generator expansion. Let $D = \{i \in H : y_i^* = d\}$. If $|D| \leq 5$, then $\tilde{E} = D$. Otherwise to select the entire D would make E unnecessarily large and therefore (13) would be difficult to solve. We select a subset of D based on the following strategy. After solving (13) by a row generation procedure based on constraints (14) in step 3, we obtain the final LP, which consists only of a subset of constraints (14). From this LP we remove all the constraints corresponding to the columns in D and we solve this new LP to obtain the optimal vector $(\tilde{\eta}, \tilde{\alpha})$. The selected columns $\tilde{E}$ consist of all columns $i \in D$ with the largest $\tilde{\alpha} a_i - c_i$. Note that the LP without columns corresponding to D gives an upper bound on the updated (13), i.e. (13) after step 14 of Algorithm 3, since constraints (14) for vectors x with $\mathrm{supp}(x) \subseteq D$ are not included. Columns with large $\tilde{\alpha} a_i - c_i$ are better candidates for E than columns with low $\tilde{\alpha} a_i - c_i$. In disjoint generator expansion we first compute R and r. Next we find among all columns $j \in H$ with $\mathrm{supp}(a_j) \cap \mathrm{supp}(r) \neq \emptyset$, a column with the largest y_j^*. We select $\tilde{E} = \{i \in H : y_i^* = y_j^*, \mathrm{supp}(a_i) \cap \mathrm{supp}(r) \neq \emptyset\}$.

5 Applications of the Generator OSF

In this section we present two potential applications of the generator OSF.

5.1 Integral SPRINT or Sifting

Suppose we want to solve a large-scale IP with a large number of variables, which are given explicitly. A computationally efficient methodology for solving the LP relaxations resulting from such large-scale problems is by SPRINT or sifting. Sifting is essentially column generation except that the columns are given explicitly (an underlying combinatorial structure is not assumed). When solving IPs, we can solve the restricted master problem by computing an OSF F. In the subproblem we find columns with low reduced cost $c_i - F_\alpha(a_i)$, which are then appended to restricted master problem. As long as the columns are given explicitly, finding low reduced cost columns should be tractable.

5.2 An all Integer Benders Decomposition

Suppose we want to solve $z^{\mathrm{IP}} = \min\{cx + dy : Ax + By = b, x \in \mathbb{Z}_+^p, y \in \mathbb{Z}_+^q\}$. We show how we can reformulate this problem as a nonlinear problem with p integer variables and a single continuous variable. We also give an algorithm to solve this nonlinear mixed integer program. The main ideas are very similar to the Benders decomposition approach for MIPs, see e.g. Nemhauser and Wolsey (1988), pages 337-341.

Proposition 7. *If $\{Ax + By = b, x \in \mathbb{Z}_+^p, y \in \mathbb{Z}_+^q\} \neq \emptyset$, then the optimal value of the nonlinear mixed integer program*

$$
\begin{aligned}
\min \ & \eta && && \text{(28a)}\\
-\eta + cx + F_{\alpha_k}(b - Ax) &\leq 0 && k \in K && \text{(28b)}\\
\bar{F}_{\beta_j}(b - Ax) &\leq 0 && j \in J && \text{(28c)}\\
Ax &\leq b && &&\\
x \in \mathbb{Z}_+^p, &\eta \textit{ unrestricted}\,, && &&
\end{aligned}
$$

where

$$K = \cup_{x \in \mathbb{Z}_+^p, Ax \leq b} K(b - Ax),\ J = \cup_{x \in \mathbb{Z}_+^p, Ax \leq b} J(b - Ax),$$

equals to z^{IP}.

Proof. Let (x^*, η^*) be an optimal solution to (28).

We first show that there is a $y \in \mathbb{Z}_+^q$ such that $Ax^* + By = b$. If not, then the IP $\min\{dy : By = b - Ax^*, y \in \mathbb{Z}_+^q\}$ is infeasible. Therefore by Theorem 4 there exists a $\beta_j, j \in J(b - Ax^*)$ such that $\bar{F}_{\beta_j}(b - Ax^*) > 0$, which contradicts that x^* satisfies (28c).

Let y^* be an optimal solution to $\min\{dy : By = b - Ax^*, y \in \mathbb{Z}_+^q\}$ and let $F_{\alpha_{\tilde{k}}}, \tilde{k} \in K(b - Ax^*)$ be a generator OSF for this problem. We claim that (x^*, y^*) is an optimal solution to $\min\{cx + dy : Ax + By = b, x \in \mathbb{Z}_+^p, y \in \mathbb{Z}_+^q\}$.

First we show that $\eta^* = cx^* + dy^*$. Suppose that $\eta^* < cx^* + dy^*$. Then from (28b) we obtain that $F_{\alpha_k}(b - Ax^*) < dy^*$ for every $k \in K$. For $k = \tilde{k}$ this is a contradiction since by choice of $\tilde{k}$ we have $dy^* = F_{\alpha_{\tilde{k}}}(b - Ax^*)$.

Suppose that there is an optimal solution $(\bar{x}, \bar{y})$ to the IP such that $c\bar{x} + d\bar{y} < cx^* + dy^*$. Since $\min\{dy : By = b - A\bar{x}, y \in \mathbb{Z}_+^q\}$ is feasible, it follows that $\bar{F}_{\beta_j}(b - A\bar{x}) \leq 0$ for all $j \in J$. For all $k \in K$ we also have that $F_{\alpha_k}(b - A\bar{x}) \leq \min\{dy : By = b - A\bar{x}, y \in \mathbb{Z}_+^q\} = dy^*$, which implies that $(\bar{x}, c\bar{x} + d\bar{y})$ is a feasible solution to (28). The value of this solution is $c\bar{x} + d\bar{y} < cx^* + dy^* = \eta^*$, which is a contradiction to the optimality of (x^*, η^*). □

Based on Proposition 7 we can design an all integer Benders decomposition algorithm that is given in Algorithm 4. The algorithm is suitable for problems, where the subproblem decomposes. For example, in two stage linear recourse stochastic integer programs, see e.g. Birge and Louveaux (1997), the subproblem decomposes into l smaller subproblems, where l is the number of scenarios. In addition, each subproblem in the decomposition has the same constraint matrix.

$K = J = \emptyset$
loop
  Solve the master problem $\min\{\eta : -\eta + cx + F_{\alpha_k}(b - Ax) \le 0$ for all $k \in K, F_{\beta_j}(b - Ax) \le 0$ for all $j \in J, x \in \mathbb{Z}_+^p, \eta$ unrestricted$\}$. If the problem is infeasible, then the IP is infeasible and exit. Otherwise let (x^*, η^*) be an optimal solution.
  Solve the subproblem $\min\{dy : By = b - Ax^*, y \in \mathbb{Z}_+^q\}$ and its subadditive dual.
  if the subproblem is infeasible **then**
    Set $J = J \cup \{j\}$, where $\bar{F}_{\beta_j}$ is a DG function with $\bar{F}_{\beta_j}(b - Ax^*) > 0$.
  else
    Let F_{α_k} be a BG OSF.
    if $\eta^* + cx^* + F_{\alpha_k}(b - Ax^*) > 0$ **then**
      $K = K \cup \{k\}$
    else
      (x^*, y^*) is an optimal IP solution, where y^* is an optimal solution to the subproblem. Exit.
    end if
  end if
end loop

Algorithm 4: The all integer Benders decomposition algorithm

Unfortunately in Algorithm 4 the master problem is a nonlinear mixed integer program. We suggest to solve it iteratively as follows. Suppose that for each $k \in K$ we have a nonnegative integer vector w^k such that $B^{E(\alpha_k)} w^k \le b$ and for each $j \in J$ we have a nonnegative integer vector u^k such that $B^{E(\beta_j)} u^k \le b$. At each iteration we solve

$$
\begin{aligned}
&\min\ \eta \\
-\eta + (c - \alpha_k A)x &\le -\alpha_k b + (\alpha_k B^{E(\alpha_k)} - c^{E(\alpha_k)}) w^k && k \in K \\
-\beta_j A x &\le -\beta_j b + \beta_j B^{E(\beta_J)} u^j && j \in J \\
x \in \mathbb{Z}_+^p&, \eta \text{ unrestricted}
\end{aligned}
$$

and let $(\bar{x}, \bar{\eta})$ be an optimal solution. For all $k \in K$ let $\bar{w}^k$ be an optimal solution to $\max\{(\alpha_k B^{E(\alpha_k)} - c^{E(\alpha_k)})w : B^{E(\alpha_k)} w \le b - A\bar{x}, w$ nonnegative integer$\}$ and for all $j \in J$ let $\bar{u}^j$ be an optimal solution to $\max\{\beta_j B^{E(\beta_j)} u : B^{E(\beta_j)} u \le b - A\bar{x}, u$ nonnegative integer$\}$ for all $j \in J$. If $(\alpha_k B^{E(\alpha_k)} - c^{E(\alpha_k)})\bar{w}^k = (\alpha_k B^{E(\alpha_k)} - c^{E(\alpha_k)}) w^k$ for all $k \in K$ and $(\beta_k B^{E(\beta_j)} - c^{E(\beta_j)})\bar{u}^j = (\beta_k B^{E(\beta_j)} - c^{E(\beta_j)}) u^j$ for all $j \in J$, then $(\bar{\eta}, \bar{x})$ is an optimal solution to the master problem. Otherwise for all $k \in K$ that do not satisfy the first equality we set $w^k = \bar{w}^k$, for all $j \in J$ that violate the second equality we set $u^j = \bar{u}^j$, and we repeat the procedure.

6 Conclusions

We presented a new family of subadditive functions that is easy to encode and in most practical instances also easy to evaluate. We gave several properties of these functions. We show several applications and we also presented an algorithm to compute an optimal subadditive function. Further enhancements, e.g. an extended Proposition 5, how to efficiently solve (13), and implementation details for the set partitioning problem are given in Klabjan (2003). This work, which is a sequel article to the presented one, also provides extensive computational results.

In linear programming the dual vector shows that linear programming is in co-$\mathcal{NP}$. If we attempt to show that integer programming is in co-$\mathcal{NP}$, we might guess an α and an optimal solution x^* to $\max\{(\alpha A^E - c^E)x : A^E x \le b, x$ nonnegative integer$\}$. Now we can easily check that $F_\alpha(b) \ge K$, which would show that the optimal IP value is greater or equal to K. However to complete the proof, we must also verify in polynomial time that x^* is an optimal solution to $\max\{(\alpha A^E - c^E)x : A^E x \leq b, x \text{ nonnegative integer}\}$. Clearly we do not know how to do this verification in polynomial time and if $\mathcal{NP} \neq \text{co}-\mathcal{NP}$, we cannot verify this statement in polynomial time. From this discussion we also conclude that if $\mathcal{NP} \neq \text{co}-\mathcal{NP}$, then $|E|$ grows with n.

We hope that this research spawns other approaches to computing the subadditive dual function. We have given just a few applications of having an optimal subadditive function. We wonder if subadditive dual functions other than those arising from rounding procedures can be efficiently used in LP based branch-and-bound algorithms. They can be used to provide lower bounds and cuts. The bottleneck of our algorithm is stage 3 and therefore it is important to improve the generation expansion strategy or design a completely different algorithm to compute a generator OSF.

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