Duality and Existence of Optimal Policies in Generalized Joint Replenishment

D. Klabjan • D. Adelman

Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 West Green Street, Urbana, Illinois, 61801, USA

Graduate School of Business, University of Chicago, 1101 E. 58th Street, Chicago, Illinois, 60637, USA

klabjan@uiuc.edu • dan.adelman@gsb.uchicago.edu

---

1. Introduction

We study the following problem. A controller continuously monitors inventories for a finite set of items $\mathcal{I}$. An item may represent a product, a location, or a product-location pair. The inventory of each item $i \in \mathcal{I}$ is infinitely divisible, is consumed at a constant deterministic rate of $0 < \lambda_i < \infty$, and costs the firm $0 \leq h_i < \infty$ per unit per time to hold. It also cannot exceed a maximum allowable inventory level of $0 < \overline{X}_i \leq \infty$. For each item i, to avoid degenerate cases, we assume that either $h_i > 0$ or $\overline{X}_i < \infty$ (or both). As inventories continuously deplete, the controller may at any time replenish a subset $I \subseteq \mathcal{I}$ of items, which incurs an ordering cost of $0 < C_I < \infty$ and is completed instantaneously. Without loss of generality, we assume $C_{I_1} \leq C_{I_2}$ if $I_1 \subseteq I_2$, since otherwise the controller can replenish I_1 by executing I_2 without replenishing items $I_2 \setminus I_1$. Although we can accommodate different item sizes, we assume for simplicity that all demands and inventories are measured in the same units, e.g. liters, and that no more than $0 < \overline{A} \leq \infty$ total units can be replenished across all items in a single replenishment. The controller's problem is to minimize the long-run time average cost, subject to allowing no stockouts. We call this problem the generalized joint replenishment problem.

We provide a new formulation of the generalized joint replenishment problem as a semi-Markov decision process on continuous spaces, which extends the model of Adelman [2003] to include holding costs. We prove the existence of an optimal stationary, deterministic policy. This existence question is stated in Federgruen and Zheng [1992] as an open problem. We also give the following two new results with respect to cyclic schedules: (1) an example showing that cyclic schedules need not be optimal, and (2) cyclic schedules are ϵ-optimal for every $\epsilon > 0$. The full version of this document appears in Adelman and Klabjan [2005]. We omit most proofs since they can be found in this manuscript.

We resolve the existence question using the powerful and elegant machinery of infinite linear programming duality, Anderson and Nash [1987]. Using this approach, we can accomodate constraints on replenishment quantities, which are essential in real-world applications such as inventory routing. We formulate the underlying problem as an infinite dimensional linear program. This program seeks a measurable bounded function subject to uncountably many constraints and an objective function. The dual of this linear program is a linear program having variables correspond to finite measures. Among other results, we show solvability and strong duality of these two linear programs. Our duality results are important not only because they lead to a resolution of the existence question. They also provide, at least theoretically, a way to verify whether a given policy, or cyclic schedule, is optimal. Such a *certificate of optimality* has been missing in the inventory literature, and is essential if optimal control policies are ever to be identified. Whereas previous models in the literature yield bounds on optimal cost, our models are the first to provide the exact optimal cost. Having a complete duality theory will enable future researchers to not only better understand problems in this arena, but also to create brand new classes of math programming solution algorithms to solve them.

In Section 2 we formulate the generalized joint replenishment problem as a semi-Markov decision process. The existence result is stated in Section 3. Section 4 presents the results about cyclic schedules.

2. Semi-Markov Decision Formulation

We start by formally stating the generalized joint replenishment problem. Suppose quantity a_i is replenished of item i when its inventory level is x_i. We assign all future holding cost that results to the current replenishment. Consequently, the inventory holding cost associated with x_i is sunk, because it is assigned to the previous replenishments. The delivery of a_i moves the inventory level to $x_i + a_i$ and incurs additional holding cost. The newly accumulated holding cost is $\frac{(a_i+x_i)^2}{2\lambda_i} - \frac{x_i^2}{2\lambda_i}$, or $(1/2\lambda_i)(a_i^2 + 2a_i x_i)$. Therefore, for every $(x, a) \in \mathbb{R}^{|\mathcal{I}|} \times \mathbb{R}^{|\mathcal{I}|}$

the cost of replenishment vector a is the sum of fixed ordering costs and holding costs, i.e.

$$c(x,a) = C_{\mathrm{supp}(a)} + \sum_{i\in\mathcal{I}} \frac{h_i}{2\lambda_i}(2a_ix_i + a_i^2)\,, \tag{1}$$

where we denote by $\mathrm{supp}(a)$ the support set of a.

The problem is to find an infinite sequence of replenishments $\{(x_n, a_n, t_n)\}_{n=0,1,\ldots}$, where x_n and a_n denote the vectors of item inventory levels and replenishment quantities respectively, at decision epoch n, and t_n represents the elapsed time between replenishments n and $n+1$. The notation $x_{i,n}$ and $a_{i,n}$ denotes the inventory level and replenishment quantity, respectively, of item i on replenishment n. Given a fixed initial inventory state $x_0 = x$, the control problem can be formulated as

$$J^*(x) = \inf \limsup_{N\to\infty} \frac{\sum_{n=0}^{N} c(x_n,a_n)}{\sum_{n=0}^{N} t_n} \tag{2a}$$

$$x_{n+1} = x_n + a_n - \lambda t_n \quad n\in\mathbb{Z}_+ \tag{2b}$$

$$x_n + a_n \le \overline{X} \quad n\in\mathbb{Z}_+ \tag{2c}$$

$$\sum_{i\in\mathcal{I}} a_{i,n} \le \overline{A} \quad n\in\mathbb{Z}_+ \tag{2d}$$

$$x_0 = x \tag{2e}$$

$$x,a,t \ge 0, \tag{2f}$$

where $\mathbb{Z}_+ = \{0,1,\ldots\}$ and $\lambda = (\lambda_1,\ldots,\lambda_{|\mathcal{I}|})$. Constraints (2b) maintain inventory flow balance, constraints (2c) ensure that the storage limits $\overline{X}_i$ are not violated, and constraints (2d) ensure that no replenishment delivers more than $\overline{A}$ in total across all items. The objective function minimizes the lim sup of the long-run time average cost.

3. Existence of Optimal Policies

The central existence result of this paper is the following theorem.

Theorem 1. *There exists a function $f(\cdot)$ and a constant J^* such that for all initial feasible inventory states $x_0 = x$, the infimum in (2a) equals $J^*(x) = J^*$ and an optimal sequence $\{(x_n^*, a_n^*, t_n^*)\}_{n=0,1,\ldots}$ that attains J^* is given by*

$$\begin{aligned} a_n^* &= f(x_n^*), \\ t_n^* &= \min_{i\in\mathcal{I}}\left\{\frac{x_{i,n}^*+a_{i,n}^*}{\lambda_i}\right\}, \text{ and} \\ x_{n+1}^* &= x_n^* + a_n^* - \lambda t_n^* \end{aligned} \tag{3}$$

for all $n \in \mathbb{Z}_+$.

The proof of this result uses linear programming duality on infinite-dimensional Borel spaces. A primal/dual pair of linear programs are formulated. Next it is shown that both of them are solvable and that there is no duality gap. By using complementary slackness an optimal deterministic stationary policy, i.e. of the form (3), is constructed.

4. Results on Cyclic Schedules

For ease of notation let $\tau(x,a) = \min_{i \in \mathcal{I}} \left\{ \frac{x_i + a_i}{\lambda_i} \right\}$ and $s(x,a) = x + a - \lambda \tau(x,a)$. If currently the observed inventory is x and a replenishment a is used, then the next replenishment will occur $\tau(x,a)$ time units from now and the inventory level is going to be $s(x,a)$.

Definition 1. *A sequence* $\{(x_n, a_n)\}_{n=0,\dots,N-1}$ *of* $N < \infty$ *of steps is called a* **cyclic schedule** *if*

$$x_n = \begin{cases} s(x_{N-1}, a_{N-1}) & \text{for } n = 0 \\ s(x_{n-1}, a_{n-1}) & \text{for } n = 1, \dots, N-1. \end{cases}$$

Cyclic schedules might not be optimal as shown by the following example.

Proposition 1. *All cyclic schedules are suboptimal for the following instances:* $\mathcal{I} = \{1,2\}$, $\lambda_1 = \lambda_2 = 1$, $C_{\{1\}} = C_{\{2\}} = 1$, $C_{\{12\}} = 2$, $h_1 = h_2 = 0$, *one of* $\overline{X}_1$ *and* $\overline{X}_2$ *is rational and the other is irrational,* $\overline{A} = \infty$.

Proof. An optimal policy manages the two items independently because $C_{\{1\}} + C_{\{2\}} = 2 \leq C_{\{12\}}$, i.e. there is no economic incentive to replenish items together. Hence, the optimal policy replenishes quantity $\overline{X}_i$ of item i whenever it stocks out. Now suppose there exists an optimal cyclic schedule, and $(x, 0)$ (or $(0, x)$) is some state on it. Then there exists a cycle length $T < \infty$ such that state $(x, 0)$ (or $(0, x)$) is revisited. Hence, by flow balance (2b), there must exist $n_1, n_2 \in \mathbb{N}$ such that $x + n_1\overline{X}_1 - T = x$, which implies $T = n_1\overline{X}_1$ and similarly $T = n_2\overline{X}_2$. However, if one of $\overline{X}_1$ and $\overline{X}_2$ is rational and the other irrational, then $n_1\overline{X}_1 = n_2\overline{X}_2$ equates an irrational number with a rational number, which is a contradiction. $\square$

The same scenerio can occur even when all of the input data are rational. For instance, consider $h_1 = 1$, $h_2 = 2$, and $\overline{X}_1 = \overline{X}_2 = \infty$. Then, as before, it is optimal to manage the items indepedently, but in this case each item follows the classical economic order quantity, which equals quantity $\sqrt{2}$ for item 1 and quantity 1 for item 2. Because the former is irrational and the later is rational, the same argument holds, so that there does not exist an optimal cyclic schedule.

On a positive note, cyclic schedules can approximate optimal policies arbitrarily close. Cyclic schedules are said to be *ϵ-optimal* if for every $\epsilon > 0$ there exists a cyclic schedule $\{(x_n, a_n)\}_{n=0,\dots,N-1}$ such that

$$\frac{\sum_{n=0}^{N-1} c(x_n, a_n)}{\sum_{n=0}^{N-1} \tau(x_n, a_n)} - J^* \le \epsilon,$$

i.e. they can get ϵ close to any optimal policy. Here J^* is the value of the optimal policy.

Theorem 2. *Cyclic schedules are ϵ-optimal.*

References

D. Adelman. Price-directed replenishment of subsets: Methodology and its application to inventory routing. *Manufacturing & Service Operations Management*, 5:348–371, 2003.

D. Adelman and D. Klabjan. Duality and existence of optimal policies in generalized joint replenishment. *Mathematics of Operations Reserach*, 30:28–50, 2005.

E.J. Anderson and P. Nash. *Linear Programming in Infinite-dimensional Spaces*. John Wiley & Sons, 1987.

A. Federgruen and Y. S. Zheng. The joint replenishment problem with general joint cost structures. *Operations Research*, 40:384–403, 1992.