

# Sequence Independent Lifting for Mixed Integer Programs with Variable Upper Bounds

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## Abstract

We investigate the convex hull of the set defined by a single inequality with continuous and binary variables with variable upper bound constraints. We extend the traditional flow cover inequality, and show that it is valid for a restriction of the set in which some variables are fixed. We also give conditions under which this inequality is facet-defining and, when it is not, we show how it can be lifted to obtain valid inequalities for the entire set using sequence independent lifting. In general, computing the lifting function is NP-hard, but under an additional restriction on the cover we obtain a closed form. Finally, we show how these results imply and extend known results about the single node fixed charge flow polyhedron.

## 1 Introduction

Many optimization problems arising from practical applications are formulated as mixed integer programs. In many of these applications *variable upper bound constraints* are present. This is the case for the facility location problem (see e.g. [Aardal \(1998\)](#)), the lot-sizing problem (see e.g. [Salomon \(1991\)](#)), and network design problems (see e.g. [Bienstock and Günlük \(1996\)](#)). Even if these constraints are not present in the initial formulation, they can be generated by preprocessing, [Savelsbergh \(1994\)](#). A successful approach for solving such mixed integer programs is by branch-and-cut, [Marchand et al. \(2002\)](#), [Johnson et al. \(2000\)](#), which requires generating valid inequalities for the underlying polyhedron or a relaxation of it. In this paper we study the polyhedron defined as the convex hull of feasible solutions to a single inequality involving continuous and binary variables that are additionally related by variable upper bounds.

We study the set  $\bar{S}$  described by

$$\begin{aligned} \sum_{i \in N} \bar{a}_i \bar{x}_i + \sum_{i \in N} \bar{b}_i \bar{y}_i &\leq \bar{d} \\ 0 \leq \bar{x}_i &\leq \bar{u}_i + \bar{v}_i \bar{y}_i \quad i \in N \\ \bar{x} \geq 0, \bar{y} &\text{ binary,} \end{aligned}$$

where  $\bar{a}_i, \bar{b}_i, \bar{u}_i, \bar{v}_i \in \mathbb{Q}$  for every  $i \in N$  and  $\bar{d} \in \mathbb{Q}$ . Note that all the variables in  $\bar{S}$  are bounded. By defining the new variables  $x_i = |\bar{a}_i|\bar{x}_i$ ,  $y_i = \bar{y}_i$  if  $\bar{b}_i \geq 0$ , and  $y_i = 1 - \bar{y}_i$  if  $\bar{b}_i < 0$ , we obtain the set  $S$ , which is equivalent to  $\bar{S}$  and given by

$$\begin{aligned} \sum_{i \in N_1^+} x_i - \sum_{i \in N_1^-} x_i + \sum_{i \in N_2^+} b_i y_i + \sum_{i \in N_2^-} b_i y_i &\leq d' \\ 0 \leq x_i \leq u_i + v_i y_i & \quad i \in N_2^+ \\ 0 \leq x_i \leq u_i - v_i y_i & \quad i \in N_2^- \\ x \geq 0, y \text{ binary}, & \end{aligned} \tag{1}$$

where  $b_i \geq 0$ ,  $v_i \geq 0$  for  $i \in N = N_1^+ \cup N_1^- = N_2^+ \cup N_2^-$ . Note that with respect to  $\bar{S}$  we have

- $b_i = |\bar{b}_i|$ ,
- $d' = \bar{d} - \sum_{\bar{b}_i < 0} \bar{b}_i$ ,
- $u_i = |\bar{a}_i|\bar{u}_i$  if  $\bar{b}_i \geq 0$ ,
- $u_i = |\bar{a}_i|(\bar{u}_i + \bar{v}_i)$  if  $\bar{b}_i < 0$ ,
- $v_i = |\bar{a}_i|\bar{v}_i$  if  $\bar{b}_i \geq 0$  and  $|\bar{a}_i|\bar{v}_i \geq 0$ , or  $\bar{b}_i < 0$  and  $|\bar{a}_i|\bar{v}_i < 0$ ,
- $v_i = -|\bar{a}_i|\bar{v}_i$  if  $\bar{b}_i \geq 0$  and  $|\bar{a}_i|\bar{v}_i < 0$ , or  $\bar{b}_i < 0$  and  $|\bar{a}_i|\bar{v}_i \geq 0$ ,

and sets  $N_1^+, N_1^-, N_2^+, N_2^-$  are defined as

- $N_1^+ = \{i \in N : \bar{a}_i \geq 0\}$ ,
- $N_2^+ = \{i \in N : \bar{v}_i \geq 0\}$ ,
- $N_1^- = \{i \in N : \bar{a}_i < 0\}$ ,
- $N_2^- = \{i \in N : \bar{v}_i < 0\}$ .

We let  $P$  be the convex hull of  $S$ . We say that variable  $i$  has a *zero constant bound* if  $u_i = 0$ . If  $u_i > 0$ , then we say that it has a *positive constant bound*.

The basic special case not involving binary variables in the constraint, i.e.  $b_i = 0$  for every  $i \in N$ , and the seminal study on the topic is the work by [Padberg et al. \(1985\)](#), which is extended and enhanced in [Van Roy and Wolsey \(1986\)](#), [Gu et al. \(1999\)](#), and [Atamtürk et al. \(2001\)](#). They all build on the notion of a cover, which is also a building block in our work. Recently, [Richard et al. \(2003a,b\)](#) studied a similar polyhedron. That polyhedron is a particular case of  $S$  obtained by choosing  $v_i = 0$  for all  $i \in N$ . Set  $S$  has more structure, which is embedded in with the variable upper bound constraints. The structure of these additional constraints is heavily used in our work. Another special case of our polyhedron is studied by [Miller et al. \(2003\)](#) in the context of multi-item lot-sizing. Their case corresponds to  $N_1^- = N_2^- = \emptyset$ ,  $u_i = 0, v_i = K - b_i$  for every  $i \in N$ , where  $K$  is a constant. [Goemans \(1989\)](#) considers the case  $u_i$  for all  $i \in N$ , and  $N_2^- = \emptyset, N_1^- = \emptyset$ . [Atamtürk et al. \(2001\)](#) study the problem with no binary variables in the constraint ( $b_i = 0$  for all  $i \in N$ ) but however uses more general variable upper bounds. The polyhedron considered by [Marchand and Wolsey \(1999\)](#) can be obtained from our polyhedron if  $v_i = 0$  for all  $i \in N$  and  $u_i = 0$  for all  $i \in N$  but one. This paper also shows that their model is a relaxation of the standard single node fixed charge flow model. By using the same technique, it can be seen that it is also a relaxation of  $S$ . A framework for obtaining many of previously studied inequalities is given by [Louveaux and Wolsey \(2003\)](#).

In [Section 2](#) we first give basic polyhedral results about  $S$ . Next we introduce the generalized flow cover inequality and give sufficient conditions under which it is facet-defining. When it is not, we use sequence independent lifting to strengthen it. [Section 3](#) first outlines sequence independent lifting. Then we consider sequence independent lifting of the flow cover inequality. A difficulty with lifting is not only that the lifting

function is not superadditive, but we show that it is NP-hard to compute the lifting function. To circumvent this difficulty we lower bound the lifting function by its LP relaxation. This section also presents the resulting valid inequality. [Section 4](#) discusses a special case of flow cover inequalities for which the lifting function is given in a closed form. Nevertheless it is not superadditive and therefore we provide a superadditive lower bound of the lifting function. Finally we show that the resulting inequalities generalize and strengthen results by [Gu et al. \(1999\)](#).

## 2 Valid inequalities

In this section we first give basic polyhedral results for  $P$ . We then focus on covers and the resulting flow cover inequalities.

### 2.1 Basic polyhedral results

To guarantee that  $P$  is full-dimensional we impose some restrictions on its parameters. In the rest of the paper, we make the following assumptions:

1.  $u_i - v_i \geq 0$  for  $i \in N_2^-$ ,
2.  $u_i \geq 0$  for  $i \in N_2^+$ ,
3.  $u_i + v_i > 0$  for  $i \in N_2^+$ ,
4.  $u_i > 0$  for  $i \in N_2^-$ ,

which are justified as follows. Note that  $u_i + v_i \geq 0$  for  $i \in N_2^+$  and  $u_i - v_i \geq 0$  for  $i \in N_2^-$ , otherwise  $y_i = 0$  for any feasible solution. Second, we need  $u_i \geq 0$  for any  $i$ , which is clear for  $i \in N_2^-$ . For  $i \in N_2^+$  observe that if  $u_i < 0$ , then  $y_i = 1$  for any feasible solution. In addition,  $u_i + v_i > 0$  for  $i \in N_2^+$ , since if not, then  $u_i = v_i = 0$ , which implies  $x_i = 0$ . Finally  $u_i > 0$  for  $i \in N_2^-$ , since if not, then we have  $x_i = 0$ .

If assumptions 1-4 hold, then the following conditions are necessary and sufficient for  $P$  to be full-dimensional.

**Proposition 1.** Let

$$p_k = \begin{cases} b_k & k \in N_1^+ \\ b_k + v_k & k \in N_1^- \cap N_2^- \\ |b_k - v_k| & k \in N_1^- \cap N_2^+, \end{cases}$$

$p = \max p_k$ , and  $M = \{k \in N : p_k = p\}$ . If  $p > 0$  and  $u_k > 0$  for all  $k \in M \cap ((N_1^+ \cap N_2^+) \cup \{i \in N_1^- \cap N_2^+ : b_i \geq v_i\})$ , then  $P$  is full-dimensional if and only if

$$p + \sum_{\substack{N_1^- \cap N_2^+ \\ b_i < v_i}} b_i - \sum_{N_1^- \cap N_2^-} u_i - \sum_{\substack{N_1^- \cap N_2^+ \\ b_i \geq v_i}} u_i - \sum_{\substack{N_1^- \cap N_2^+ \\ b_i < v_i}} (u_i + v_i) \leq d'. \quad (2)$$

If either  $p = 0$  or there exists  $k \in M \cap ((N_1^+ \cap N_2^+) \cup \{i \in N_1^- \cap N_2^+ : b_i \geq v_i\})$  with  $u_k = 0$ , then  $P$  is full-dimensional if and only if (2) is satisfied strictly as inequality.  $\square$

The proof of [Proposition 1](#) is given in [Shebalov \(2004\)](#). Trivial inequalities are those that belong to the description of  $S$ . Using standard techniques the following proposition can be shown, see [Shebalov \(2004\)](#).

**Proposition 2.** The trivial inequalities are facet-defining in the following two cases.

1.  $x_k \leq u_k \pm v_k y_k$  is a facet-defining inequality for  $k \in N_1^-$ .
2.  $x_k \geq 0$  is a facet-defining inequality for  $k \in N_1^+$ .  $\square$

Next we give a general condition that inequality has to satisfy to be facet-defining. Whenever we do not need to distinguish between  $N_2^+$  and  $N_2^-$ , we write  $\pm v_k$ , since they are handled similarly.

**Proposition 3.** If

$$\sum_{i \in N_1^+} \pi_i^+ x_i - \sum_{i \in N_1^-} \pi_i^- x_i + \sum_{i \in N} \rho_i^+ y_i \leq \delta \quad (3)$$

is a nontrivial facet of  $P$ , then  $\pi_i^+ \geq 0$  and  $\pi_i^- \geq 0$  for any  $i \in N$ .

*Proof.* Let  $i \in N$ . Since (3) is nontrivial, there exists  $(x, y) \in P$  such that it satisfies (3) at equality and  $x_i > 0$  for  $i \in N_1^+$ . Therefore  $(\bar{x}, \bar{y}) = (x, y) - (\varepsilon e_i, 0) \in P$  for  $\varepsilon$  small enough, thus  $\pi_i^+ \geq 0$ . Similarly, there exists  $(x, y) \in P$  such that it satisfies (3) at equality and  $x_i < u_i \pm v_i y_i$  for  $i \in N_1^-$ . Therefore  $(\bar{x}, \bar{y}) = (x, y) + (\varepsilon e_i, 0) \in P$  for  $\varepsilon$  small enough, thus  $\pi_i^- \geq 0$ .  $\square$

## 2.2 Flow cover inequality

In this section we introduce the flow cover inequality. We prove that it is valid in the case  $x_i = y_i = 0$  for every  $i \in N_1^-$ , i.e. we assume that  $N_1^- = \emptyset$ . The resulting polyhedron after fixing all variables in  $N_1^-$  to 0 is denoted by  $P^+$ . In Section 3 and Section 4 we remove this assumption by lifting it. For technical reasons we denote the right hand side in (1) by  $d$ . As some variables may be fixed at their upper bound,  $d$  in this section might be different from  $d'$  appearing in the description of  $S$ .

A set  $C \subseteq N_1^+$  is a cover if

$$\lambda = \sum_{i \in C_w} (u_i + v_i + b_i) + \sum_{i \in C_b} (u_i - v_i + b_i) + \sum_{i \in C_v} u_i - d > 0,$$

where  $C_w = C \cap N_2^+$ ,  $C_b = \{i \in C \cap N_2^- : b_i \geq v_i\}$ , and  $C_v = \{i \in C \cap N_2^- : b_i < v_i\}$ , see Figure 1 ( $C_w^+, C_b^+, C_v^+$  are defined and used in Section 3).

We define the *flow cover inequality*

$$\sum_{i \in C} x_i + \sum_{i \in C} b_i y_i \leq d - \sum_{i \in C_w} (v_i + b_i - \lambda)^+ (1 - y_i) - \sum_{i \in C_b} (b_i - v_i - \lambda)^+ (1 - y_i) - \sum_{i \in C_v} (v_i - b_i - \lambda)^+ y_i. \quad (4)$$

If  $u_i = b_i = 0, v_i > 0$  for all  $i \in N$  and  $N_2^- = \emptyset$ , then (4) simplifies to the standard flow cover inequality studied initially by Padberg *et al.* (1985). Note that Richard *et al.* (2003a,b), and Marchand and Wolsey (1999) do not present an equivalent inequality since they do not consider variable upper bounds.

**Theorem 1.** If  $C$  is a cover, then (4) is valid for  $P^+$ .

*Proof.* We use the proof technique given in Louveaux and Wolsey (2003). If  $(x, y) \in P^+$ , then due to nonnegativity we have  $\sum_{i \in C} x_i + \sum_{i \in C} b_i y_i \leq d$ . We first introduce new continuous nonnegative variables  $t$  as follows.

$$t_i = \begin{cases} u_i + v_i y_i - x_i & i \in C_w \\ u_i - v_i y_i - x_i & i \in C_v \cup C_b \end{cases}$$

The resulting inequality reads

$$\sum_{i \in C_w} (v_i + b_i) y_i + \sum_{i \in C_v \cup C_b} (b_i - v_i) y_i \leq d - \sum_{i \in C} u_i + \sum_{i \in C} t_i.$$

Next we make the substitution  $\bar{y}_i = 1 - y_i$  for every  $i \in C_w \cup C_b$  to obtain

$$\sum_{i \in C_v} (b_i - v_i) y_i - \sum_{i \in C_b} (b_i - v_i) \bar{y}_i - \sum_{i \in C_w} (v_i + b_i) \bar{y}_i \leq -\lambda + \sum_{i \in C} t_i. \quad (5)$$

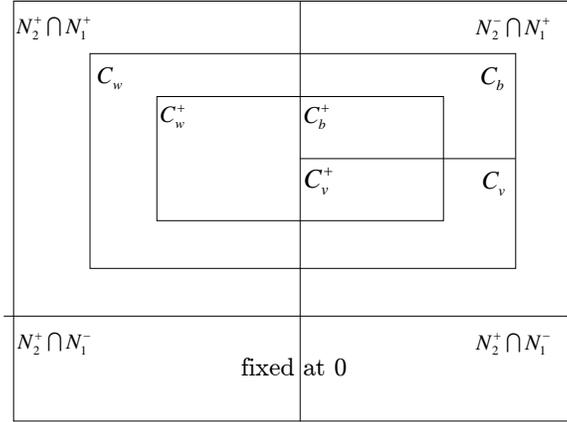


Figure 1: Partition of  $N$  into subsets

Now we use the mixed-integer rounding procedure (see e.g. [Nemhauser and Wolsey \(1988\)](#)). Let  $\bar{\alpha}$  be any number such that  $\bar{\alpha} > \max\{\lambda, \max_{i \in C_w} (v_i + b_i), \max_{i \in C_v \cup C_b} |v_i - b_i|\}$  and let  $\alpha = (\bar{\alpha} - \lambda)/\bar{\alpha}$ . We define two real value functions

$$F_\alpha(d) = \lfloor d \rfloor + \frac{(d - \lfloor d \rfloor - \alpha)^+}{1 - \alpha},$$

$$\bar{F}_\alpha(d) = \min\{0, \frac{d}{1 - \alpha}\} = \lim_{t \rightarrow 0} \frac{F_\alpha(td)}{t}.$$

In the next step we divide (5) by  $\bar{\alpha}$ . We follow by applying  $F_\alpha$  to the coefficients of binary variables  $y, \bar{y}$  and the left-hand side, and  $\bar{F}_\alpha$  to the coefficients of the continuous variables  $t$ . This step is justified by the concept of mixed-integer rounding. We obtain

$$\sum_{i \in C_w} \lambda F_\alpha\left(-\frac{v_i - b_i}{\bar{\alpha}}\right) y_i + \sum_{i \in C_b} \lambda F_\alpha\left(-\frac{b_i - v_i}{\bar{\alpha}}\right) \bar{y}_i + \sum_{i \in C_w} \lambda F_\alpha\left(-\frac{v_i + b_i}{\bar{\alpha}}\right) \bar{y}_i \leq -\lambda + \sum_{i \in C} t_i. \quad (6)$$

It is easy to see that  $\lambda F_\alpha(-\frac{d}{\bar{\alpha}}) = -\min(\lambda, d)$  for every  $d \leq \bar{\alpha}$ . Using this property and substituting back  $y$  and  $x$  in (6) yields the claim.  $\square$

The following theorem gives conditions for flow cover inequalities to be facet-defining.

**Theorem 2.** If  $C$  is a cover and

1. there exists  $i \in C_w$  such that  $u_i + v_i > \lambda$ , and
2. there exists  $i \in C_b$  such that  $u_i - v_i > \lambda$ , and
3. there exists  $i \in C_v$  such that  $u_i > \lambda$ , and
- 4.

$$\lambda + \max_{i \in N \setminus C} b_i < \max\left\{ \max_{i \in C_w, u_i + v_i \geq \lambda} (b_i + v_i), \max_{i \in C_b, u_i - v_i \geq \lambda} (b_i - v_i), \max_{i \in C_v, u_i \geq \lambda} (v_i - b_i) \right\},$$

then (4) is a facet-defining inequality for  $P^+$ .

*Proof.* To prove the statement we show that there exist  $2|N|$  affinely independent vectors in  $P^+$  satisfying (4) at equality. We denote vectors as  $z = (x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4)$ , where  $(x_1, y_1)$  corresponds to the coordinates of  $z$  from  $C_w$ ,  $(x_2, y_2)$  to the coordinates of  $z$  from  $C_b$ ,  $(x_3, y_3)$  to the coordinates of  $z$  from  $C_v$ , and  $(x_4, y_4)$  to the coordinates of  $z$  from  $N \setminus C$ . Let  $|C_w| = p$  and  $|C_b| = q$ , and without loss of generality assume that  $u_1 + v_1 > \lambda$ ,  $u_{p+1} - v_{p+1} > \lambda$ , and  $u_{p+q+1} > \lambda$ . The following vectors, where  $\epsilon$  is a small enough number, belong to  $P$  and satisfy (4) at equality.

a) For  $1 \leq i \leq p$  we define

$$z_1^i = \begin{cases} (u + v - \lambda e_i, u - v, u, 0; 1, 1, 0, 0) & \text{if } u_i + v_i \geq \lambda, \\ (u + v - (\lambda - \epsilon)e_1 - \epsilon e_i, u - v, u, 0; 1, 1, 0, 0) & \text{if } u_i + v_i < \lambda. \end{cases}$$

At any  $z_1$  we have

$$\sum_{j \in C} x_j + \sum_{j \in C} b_j y_j + \sum_{j \in C_w} (v_j + b_j - \lambda)^+ (1 - y_j) + \sum_{j \in C_b} (b_j - v_j - \lambda)^+ (1 - y_j) + \sum_{j \in C_v} (v_j - b_j - \lambda)^+ y_j \quad (7)$$

$$= \sum_{j \in C_w} (u_j + v_j) - \lambda + \sum_{j \in C_b} (u_j - v_j) + \sum_{j \in C_v} u_j + \sum_{j \in C_w} b_j + \sum_{j \in C_b} b_j = d, \quad (8)$$

which shows that  $z_1$  satisfy (4) at equality.

b) For  $p + 1 \leq i \leq p + q$  we have

$$z_2^i = \begin{cases} (u + v, u - v - \lambda e_i, u, 0; 1, 1, 0, 0) & \text{if } u_i - v_i \geq \lambda, \\ (u + v, u - v - (\lambda - \epsilon)e_{p+1} - \epsilon e_i, u, 0; 1, 1, 0, 0) & \text{if } u_i - v_i < \lambda. \end{cases}$$

(8) holds for all  $z_2$  as well, and therefore  $z_2$  satisfy (4) at equality.

c) For  $p + q + 1 \leq i \leq |C|$  we have

$$z_3^i = \begin{cases} (u + v, u - v, u - \lambda e_i, 0; 1, 1, 0, 0) & \text{if } u_i \geq \lambda, \\ (u + v, u - v, u - (\lambda - \epsilon)e_{p+q+1} - \epsilon e_i, 0; 1, 1, 0, 0) & \text{if } u_i < \lambda. \end{cases}$$

(8) holds for  $z_3$  as well, and therefore any  $z_3$  satisfies (4) at equality.

d) Let  $1 \leq i \leq p$  and

$$z_4^i = \begin{cases} (u + v - v_i e_i, u - v, u, 0; 1 - e_i, 1, 0, 0) & \text{if } v_i + b_i \geq \lambda, \\ (u + v - (\lambda - b_i - v_i)e_1 - v_i e_i, u - v, u, 0; 1 - e_i, 1, 0, 0) & \text{if } v_i + b_i < \lambda. \end{cases}$$

At  $z_4$ , (7) is equal to

$$\sum_{j \in C_w} (u_j + v_j) - v_i + \sum_{j \in C_b} (u_j - v_j) + \sum_{j \in C_v} u_j + \sum_{j \in C_w} b_j - b_i + \sum_{j \in C_b} b_j + v_i + b_i - \lambda = d, \quad (9)$$

which shows that all  $z_4$  satisfy (4) at equality.

e) For  $p + 1 \leq i \leq p + q$  we define

$$z_5^i = \begin{cases} (u + v, u - v + v_i e_i, u, 0; 1, 1 - e_i, 0, 0) & \text{if } b_i - v_i \geq \lambda, \\ (u + v, u - v - (\lambda + v_i - b_i)e_{p+1} + v_i e_i, u, 0; 1, 1 - e_i, 0, 0) & \text{if } b_i - v_i < \lambda. \end{cases}$$

At  $z_5$ , (7) gives (9). Hence all  $z_5$  satisfy (4) at equality.

f) Let  $p + q + 1 \leq i \leq |C|$  and we define

$$z_6^i = \begin{cases} (u + v, u - v, u - v_i e_i, 0; 1, 1, e_i, 0) & \text{if } v_i - b_i \geq \lambda, \\ (u + v, u - v, u - (\lambda - v_i + b_i)e_{p+q+1} - v_i e_i, 0; 1, 1, e_i, 0) & \text{if } v_i - b_i < \lambda. \end{cases}$$

At  $z_6$ , (7) gives (9). Hence all  $z_6$  satisfy (4) at equality.

We next present two families of vectors **g**) and **h**) for the particular case where

$$\lambda + \max_{i \in N \setminus C} b_i < \max_{i \in C_w, u_i + v_i \geq \lambda} (b_i + v_i)$$

satisfies assumption 4. Similar points can be derived when

$$\lambda + \max_{i \in N \setminus C} b_i < \max_{i \in C_b, u_i - v_i \geq \lambda} (b_i - v_i)$$

or

$$\lambda + \max_{i \in N \setminus C} b_i < \max_{i \in C_v, u_i \geq \lambda} (v_i - b_i).$$

g) For  $i = |C| + 1, \dots, |N|$  we have

$$z_7^i = (u + v - v_1 e_1, u - v, u, 0; 1 - e_1, 1, 0, e_i).$$

At  $z_7$ , (7) is equal to

$$\sum_{j \in C_w} (u_j + v_j) - v_1 + \sum_{j \in C_b} (u_j - v_j) + \sum_{j \in C_v} u_j + \sum_{j \in C_w} b_j - b_1 + \sum_{j \in C_b} b_j + v_1 + b_1 - \lambda = d, \quad (10)$$

which shows that all  $z_7$  satisfy (4) at equality. It is easy to see that these vectors are feasible using assumption 4 of the theorem.

h) For  $i = |C| + 1, \dots, |N|$  let

$$z_8^i = (u + v - v_1 e_1, u - v, u, \epsilon e_i; 1 - e_1, 1, 0, e_i).$$

At  $z_8$ , (7) gives (10), and hence all  $z_8$  satisfy (4) at equality, and their feasibility follows from the fourth assumption of the theorem.

It remains to prove that they are affinely independent. We subtract  $(u + v - \lambda e_1, u - v, u, 0, 1, 1, 0, 0)$  from all the vectors, and multiply  $z_1$  through  $z_6$  by  $-1$ . Denote the new families of vectors by  $z_j^0$ . Now if we arrange them in the order  $z_1^0, z_2^0, z_3^0, z_4^0, z_5^0, z_6^0, z_8^0 - z_7^0, z_8^0$ , we get the lower triangular matrix.

Since all elements except the very first one on the main diagonal are nonzero, this shows that they are affinely independent. Thus (4) is facet-defining.  $\square$

If only the first three conditions of **Theorem 2** are satisfied, then vectors  $z_7$  and  $z_8$  might not be feasible and (4) is a facet only for

$$\begin{aligned} \sum_{i \in C} x_i + \sum_{i \in C} b_i y_i &\leq d \\ 0 \leq x_i &\leq u_i + v_i y_i & i \in C_w \\ 0 \leq x_i &\leq u_i - v_i y_i & i \in C_b \cup C_v \\ y &\text{ binary,} \end{aligned}$$

where only a subset of the variables and variable upper bound constraints is imposed. We next illustrate the flow cover inequality (4) on an example polytope that will be used throughout the paper.

**Example.** Let  $P$  be given by

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 - x_6 - x_7 - x_8 + 4y_1 + 3y_2 + y_3 + 6y_4 + 3y_5 + 6y_6 + 7y_7 + 2y_8 &\leq 29 \\
0 \leq x_1 \leq 3 + 2y_1 & \quad 0 \leq x_4 \leq 3 - 2y_4 & \quad 0 \leq x_7 \leq 5 - 4y_7 \\
0 \leq x_2 \leq 5 - y_2 & \quad 0 \leq x_5 \leq 5 + 2y_5 & \quad 0 \leq x_8 \leq 3 - y_8. \\
0 \leq x_3 \leq 4 - 3y_3 & \quad 0 \leq x_6 \leq 8y_6
\end{aligned} \tag{11}$$

Thus  $N_1^+ = \{1, 2, 3, 4, 5\}$ ,  $N_1^- = \{6, 7, 8\}$ ,  $N_2^+ = \{1, 5, 6\}$ , and  $N_2^- = \{2, 3, 4, 7, 8\}$ .

Assume that  $y_4 = y_6 = y_7 = 1$  and  $x_4 = 1, x_6 = 8, x_7 = 1$ , which corresponds to fixing  $x_4, y_4, x_6, y_6, x_7, y_7$  to their upper bounds (this is considered in detail in [Section 3](#)). Then (11) becomes

$$x_1 + x_2 + x_3 + x_5 - x_8 + 4y_1 + 3y_2 + y_3 + 3y_5 + 2y_8 \leq 18.$$

Consider  $C = \{1, 2, 3\}$ . Then  $C_w = \{1\}, C_b = \{2\}, C_v = \{3\}$  and  $\lambda = 9 + 7 + 4 - 18 = 2$ , thus  $C$  is a cover. The flow cover inequality for  $C$  is

$$x_1 + x_2 + x_3 + 3y_2 + y_3 \leq 14. \tag{12}$$

Conditions of [Theorem 2](#) are satisfied and therefore (12) is a facet-defining inequality for  $P^+$ , i.e.  $x_8$  and  $y_8$  are fixed to 0.  $\square$

### 3 Valid inequalities from covers via sequence independent lifting

In this section we lift the flow cover inequalities (4) when no additional structure is imposed on the cover. We show that computing exact lifting coefficients is an NP-hard problem. For this reason we carry out approximate lifting. In [Section 4](#) we consider specially structured covers that result in exact lifting.

#### 3.1 Overview of sequence independent lifting

One of the techniques used to construct valid inequalities for a given polyhedron is sequence independent lifting, see [Wolsey \(1977\)](#), and [Gu et al. \(2000\)](#). Let  $L_u$  be a subset of  $N$ , which corresponds to the variables fixed to their upper bounds, i.e.  $x_i = u_i \pm v_i, y_i = 1$  for  $i \in L_u$ . Let  $C \subseteq N \setminus L_u$  and denote  $L_l = N \setminus (L_u \cup C)$ . Variables from  $L_l$  are fixed to their lower bounds, meaning  $x_i = y_i = 0$  for  $i \in L_l$ . Let also  $\delta_i = 1$  for  $i \in N_1^+$ , and  $\delta_i = -1$  for  $i \in N_1^-$ . After fixing the variables in  $N \setminus C$  the resulting set is

$$\begin{aligned}
P^0 = \{ & (x, y) \in \mathbb{R}^{|C|} \times \mathbb{R}^{|C|} : \sum_{i \in C} \delta_i x_i + \sum_{i \in C} b_i y_i \leq d' - \sum_{i \in L_u} (\delta_i (u_i \pm v_i) + b_i), \\
& 0 \leq x_i \leq u_i \pm v_i y_i \quad i \in C, \\
& y \text{ binary} \}.
\end{aligned}$$

Let

$$0 \leq \alpha_0 - \sum_{i \in C} (\alpha_i x_i + \beta_i y_i) \tag{13}$$

be a valid inequality for  $P^0$ . The goal is to construct a valid inequality for  $P$  of the form

$$0 \leq \alpha_0 - \sum_{i \in C} (\alpha_i x_i + \beta_i y_i) - \sum_{i \in L_l \cup L_u} (\alpha_i (x_i - w'_i) + \beta_i (y_i - w''_i)), \tag{14}$$

where  $w'_i = w''_i = 0$  for  $i \in L_l$  and  $w'_i = u_i \pm v_i, w''_i = 1$  for  $i \in L_u$ , by choosing appropriately the values of  $\alpha_i$  and  $\beta_i$ . In order to do so, we introduce the functions

$$h_i(z) = \max \{ \alpha_i (x - w'_i) + \beta_i (y - w''_i) : \delta_i (x - w'_i) + b_i y = z, 0 \leq x \leq u_i \pm v_i y, y \text{ binary} \}, \tag{15}$$

and

$$f(z) = \min \left\{ \alpha_0 - \sum_{i \in C} (\alpha_i x_i + \beta_i y_i) : \sum_{i \in C} (\delta_i x_i + b_i y_i) \leq d - z, 0 \leq x_i \leq u_i \pm v_i y_i, i \in C, y \text{ binary} \right\},$$

where

$$d = d' - \sum_{i \in L_u} (\delta_i (u_i \pm v_i) + b_i). \quad (16)$$

The following theorem from [Gu et al. \(1999\)](#) provides a way to obtain the lifting coefficients  $\alpha_i$  and  $\beta_i$  from the functions  $h_i(z)$  and  $f(z)$  when  $f(z)$  is superadditive. A function  $f$  is *superadditive* on  $Z$  if  $f(z_1) + f(z_2) \leq f(z_1 + z_2)$  for all  $z_1, z_2, z_1 + z_2 \in Z$ .

**Theorem 3.** Assume that (13) is valid for  $P^0$  and that  $(\alpha_i, \beta_i)$  are chosen in such a way that  $h_i(z) \leq f(z)$  for any  $z$  where both functions are defined, and any  $i \in L_l \cup L_u$ . Assume also that  $f(z)$  is superadditive. Then (14) is valid for  $P$ .  $\square$

If  $f$  is not superadditive, [Gu et al. \(1999\)](#) prove that it is sufficient to find a superadditive function  $g$  such that  $g(z) \leq f(z)$  for all  $z$ , and use the inequality  $h_i(z) \leq g(z)$  to find values for  $(\alpha_i, \beta_i)$ . There might be many functions that satisfy these conditions. To obtain the strongest inequalities we choose a *non-dominated*  $g(z)$ , which means that there exist no  $g', g' \neq g$  such that  $g'$  is superadditive and  $g(z) \leq g'(z) \leq f(z)$  for every  $z$ . In addition, the strongest inequalities are obtained by choosing  $(\alpha_i, \beta_i)$  in such a way that  $h_i(z) = g(z)$  for at least two distinct  $z$  values (see [Gu et al. \(1999\)](#) for details).

### 3.2 Lifting functions

In this section we use the flow cover inequality and sequence independent lifting to obtain valid inequalities for  $P$ . To this end, we let  $L_u$  be a subset of  $N$  corresponding to the variables fixed at their upper bounds. Let  $C \subseteq (N_1^+ \setminus L_u)$  be a cover with  $d$  defined by (16). Based on [Theorem 1](#), (4) is a valid inequality of  $P$  whenever  $x_i = y_i = 0$  for all  $i \in N \setminus C$ . In this section we obtain valid inequalities for  $P$  by lifting the flow cover inequality, valid for  $P^+$ , with respect to the variables in  $N \setminus C$ . Let  $C_w^+ = \{i \in C_w : b_i + v_i - \lambda > 0\}$ ,  $C_b^+ = \{i \in C_b : b_i - v_i - \lambda > 0\}$ ,  $C_v^+ = \{i \in C_v : v_i - b_i - \lambda > 0\}$ , and  $C^+ = C_w^+ \cup C_b^+ \cup C_v^+$  (see [Figure 1](#)). In this case, the lifting function  $f(z)$  introduced in [Section 3.1](#) is given by

$$\begin{aligned} f(z) = \min \left( d - \sum_{i \in C_w^+} (b_i + v_i - \lambda)(1 - y_i) - \sum_{i \in C_b^+} (b_i - v_i - \lambda)(1 - y_i) - \sum_{i \in C_v^+} (v_i - b_i - \lambda)y_i - \sum_{i \in C} x_i - \sum_{i \in C} b_i y_i \right) \\ \sum_{i \in C} x_i + \sum_{i \in C} b_i y_i \leq d - z \\ 0 \leq x_i \leq u_i + v_i y_i \quad i \in C_w \\ 0 \leq x_i \leq u_i - v_i y_i \quad i \in C_b \cup C_v \\ y \text{ binary.} \end{aligned}$$

By using a reduction from a variant of the subset-sum problem given in [Shebalov \(2004\)](#), we can prove the following theorem.

**Theorem 4.** Given input of arbitrary nonnegative  $u, v, b$ , and  $d$ , there exists  $z$  such that it is NP-hard to compute  $f(z)$ .  $\square$

This negative result implies that we will obtain a superadditive approximation  $g(z)$  of  $f(z)$  in two steps. First we find a function  $\bar{f}$  such that  $\bar{f}(z) \leq f(z)$  and then we derive a superadditive function  $g$  with  $g(z) \leq \bar{f}(z)$ . We obtain  $\bar{f}(z)$  by combining  $f(z)$  for those  $z$ , where we can derive a closed form formula, and the LP relaxation associated with the IP problem defining  $f(z)$  otherwise.

Without loss of generality we reorder the variables so that  $C_w^+ \cup C_b^+ = \{1, \dots, p\}$ , where  $p = |C_w^+| + |C_b^+|$ , and  $b_i \pm v_i \geq b_{i+1} \pm v_{i+1}$  for  $i = 1, \dots, p$ . Let  $\gamma := d - \sum_{i \in C} u_i - \sum_{i \in C_w \setminus C_w^+} (b_i + v_i) - \sum_{i \in C_b \setminus C_b^+} (b_i - v_i)$ .

Note that  $\gamma = \lambda + \sum_{i \in C_w^+} (b_i + v_i) + \sum_{i \in C_b^+} (b_i - v_i)$ , and therefore  $\gamma > 0$ . It is shown in [Shebalov \(2004\)](#) that  $f(z)$  on  $S = (-\infty, -\lambda + b_1 \pm v_1] \cup [\gamma, d]$  is given by

$$f(z) = \begin{cases} -\lambda & \text{if } -\infty < z \leq -\lambda, \\ z & \text{if } -\lambda < z \leq \min\{-\lambda + \sum_{i=1}^p u_i, 0\}, \\ 0 & \text{if } \min\{-\lambda + \sum_{i=1}^p u_i, 0\} < z \leq -\lambda + b_1 \pm v_1, \text{ and} \\ z - \sum_{i \in C_w^+} (b_i + v_i - \lambda) & \text{if } \gamma \leq z \leq d. \\ -\sum_{i \in C_b^+} (b_i - v_i - \lambda) & \end{cases}$$

Let  $f_{LP}(z)$  be the function obtained by relaxing the binary requirement on  $y$  in the definition of  $f(z)$ . It is clear that  $f_{LP}(z) \leq f(z)$ . A lengthy calculation presented in [Shebalov \(2004\)](#) shows that  $f_{LP}(z)$  is

$$f_{LP}(z) = \begin{cases} d - \sum_{i \in C} u_i - \sum_{i \in C_w} (b_i + v_i) - \sum_{i \in C_b} (b_i - v_i) & \text{if } z \leq d - \sum_{i \in C} u_i - \sum_{i \in C_w} (b_i + v_i) - \sum_{i \in C_b} (b_i - v_i), \\ \tilde{f}_k(z) & \text{if } \gamma - \sum_{i=1}^k (b_i \pm v_i) \leq z < \gamma - \sum_{i=1}^{k-1} (b_i \pm v_i), \text{ and} \\ z - \sum_{i \in C_w^+} (b_i + v_i - \lambda) - \sum_{i \in C_b^+} (b_i - v_i - \lambda) & \text{if } \gamma \leq z \leq d, \end{cases}$$

where for  $1 \leq k \leq |C_w^+| + |C_b^+|$  we define

$$\tilde{f}_k(z) = z + (b_k \pm v_k - \lambda) \frac{\gamma - z - \sum_{i=1}^{k-1} (b_i \pm v_i)}{b_k \pm v_k} + \sum_{i=1}^{k-1} (b_i \pm v_i - \lambda) - \sum_{i \in C_w^+} (b_i + v_i - \lambda) - \sum_{i \in C_b^+} (b_i - v_i - \lambda).$$

Now we define  $\bar{f}(z)$  to be  $f(z)$  for  $z \in S$  and  $\bar{f}(z) = f_{LP}(z)$  otherwise. It can be shown that  $\bar{f}(z)$  is continuous. We claim that the function

$$g(z) = \begin{cases} -Dk & z \in (-kd, -\lambda + b_1 \pm v_1 - kd] \quad k = 0, \dots, \infty \\ f_{LP}(z) - Dk & z \in (-\lambda + b_1 \pm v_1 - kd, (1-k)d] \quad k = 0, \dots, \infty, \end{cases}$$

is a strong superadditive approximation of  $f(z)$  and is such that  $g(z) \leq f(z)$ , where  $D = f_{LP}(d) = d - \sum_{i \in C_w^+} (b_i + v_i - \lambda) - \sum_{i \in C_b^+} (b_i - v_i - \lambda)$ .

Functions  $f(z)$ ,  $f_{LP}(z)$  and  $g(z)$  are represented in [Figure 2](#), where  $a_1 = -\lambda + (b_1 \pm v_1)$ .

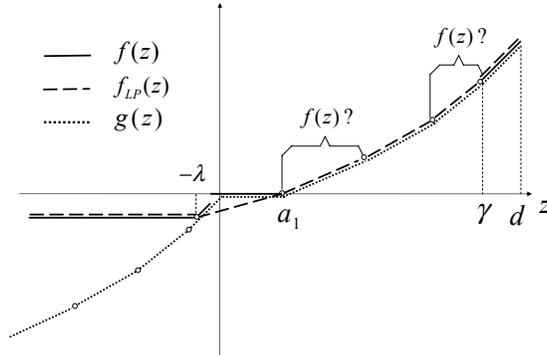


Figure 2:  $f(z)$ ,  $f_{LP}(z)$  and  $g(z)$

**Theorem 5.**  $g(z)$  is continuous,  $g(z) \leq f(z)$ , superadditive and non-dominated by any other superadditive function.

*Proof.* The first two statements are obvious from the definition of  $g(z)$ , and therefore we only prove the last two. To show superadditivity of  $g(z)$  it suffices to show that  $g(z_1) + g(z_2) \leq g(z_1 + z_2)$  for  $z_1 \leq z_2$  such that  $z_1 + z_2 \leq d$ . We consider the following three cases.

1. Let  $0 \leq z_1 < z_2 \leq d$ . Define  $g_1(z) = g(z_1) + g(z - z_1)$  for  $z_1 \leq z \leq d$ , see [Figure 3](#). Note that  $g_1(z_1) = g(z_1)$ , since  $g(0) = 0$ . For all  $z_1 \leq z \leq d$ , where the derivatives of  $g(z)$  and  $g_1(z)$  exist, we

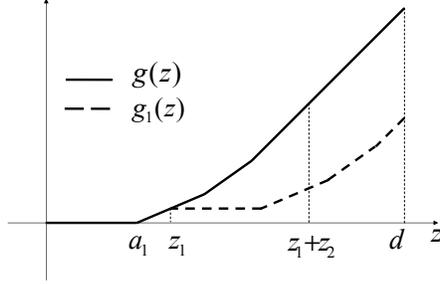


Figure 3:  $g(z)$  and  $g_1(z)$  for case 1

have  $g'(z) \geq g'(z - z_1) = g'_1(z)$  because  $0 \leq z_1 < z_2 \leq d$ . The derivatives of  $g(z)$  and  $g_1(z)$  do not exist at a finite number of points, but since both functions are continuous the inequality  $g(z) \geq g_1(z)$  follows on the entire interval. Since  $z_1 \leq z_1 + z_2 \leq z_1 + d$ , we have  $g(z_1) + g(z_2) = g(z_1) + g((z_1 + z_2) - z_1) = g_1(z_1 + z_2) \leq g(z_1 + z_2)$ .

2. Assume now  $z_1 < 0 \leq z_2$ . Let  $k$  be such that  $-(k+1)d \leq z_1 \leq -kd$  and define  $g_1(z) = g(z_1) + g(z - z_1)$  for  $z_1 \leq z \leq z_1 + d$ , see [Figure 4](#). Note that  $g_1(z_1) = g(z_1)$  and  $g_1(z_1 + d) = g(z_1 + d)$ . We consider the two cases  $z_1 \leq z \leq -kd$  and  $-kd \leq z \leq z_1 + d$ .

For  $z_1 \leq z \leq -kd$  we have  $g'(z) \geq g'_1(z)$  for all but finite number of points, which as in case 1 implies  $g(z) \geq g_1(z)$ . On the other hand, for  $-kd \leq z \leq z_1 + d$  we have  $g'(z) \leq g'_1(z)$  for all but a finite number of points. If there exists  $z_0$  such that  $g(z_0) < g_1(z_0)$ , then  $g(z) < g_1(z)$  for any  $z \geq z_0$ . However  $g_1(z_1 + d) = g(z_1 + d)$ , which implies  $g(z) \geq g_1(z)$ . From above we obtain that for  $z_1 \leq z_1 + z_2 \leq z_1 + d$ , it follows  $g(z_1) + g(z_2) = g(z_1) + g((z_1 + z_2) - z_1) = g_1(z_1 + z_2) \leq g(z_1 + z_2)$ .

3. Assume finally that  $z_1 < z_2 < 0$ . Let  $-(k_1 + 1)d \leq z_1 \leq -k_1d$ ,  $-(k_2 + 1)d \leq z_2 \leq -k_2d$  and define  $g_1(z) = g(z_1) + g(z - z_1)$  for  $z_1 - (k_2 + 1)d \leq z \leq z_1 - k_2d$ , see [Figure 5](#). Note that  $g_1(z_1 - k_2d) = g(z_1 - k_2d)$  and  $g_1(z_1 - (k_2 + 1)d) = g(z_1 - (k_2 + 1)d)$ . Again we consider two cases.

For  $-(k_1 + k_2 + 1)d \leq z \leq z_1 - k_2d$  we have  $g'(z) \leq g'_1(z)$  for all but finite number of points, which as in case 1 implies  $g(z) \geq g_1(z)$ . For  $z_1 - (k_2 + 1)d \leq z \leq -(k_1 + k_2 + 1)d$  we have  $g'(z) \geq g'_1(z)$  for all but finite number of points. If there exists  $z_0$  such that  $g(z_0) < g_1(z_0)$ , then  $g(z) < g_1(z)$  for any  $z \leq z_0$ , however  $g_1(z_1 - (k_2 + 1)d) = g(z_1 - (k_2 + 1)d)$ , which implies  $g(z) \geq g_1(z)$ . For  $z_1 - (k_2 + 1)d \leq z_1 + z_2 \leq z_1 - k_2d$ , we conclude  $g(z_1) + g(z_2) = g(z_1) + g((z_1 + z_2) - z_1) = g_1(z_1 + z_2) \leq g(z_1 + z_2)$ .

Next we show that  $g(z)$  is not dominated by any other superadditive function which is less than or equal to  $f(z)$ . Assume for contradiction that this is not the case and that there exists a superadditive function  $\bar{g}(z)$  that dominates  $g(z)$  and never exceeds  $f(z)$ . Let  $G = \{z \in (\infty, d] | \bar{g}(z) > g(z)\}$ . Note that  $G$  is not empty and that  $z < 0$  for all  $z \in G$  as  $f(z) = g(z)$  for  $z \geq 0$ . In addition,  $g(z + d) = g(z) + g(d)$  for each  $z$ . Consider now  $z \in G$  such that  $z + d \notin G$ . Such a point clearly exists. By definition of  $z$  we have  $\bar{g}(z_0 + d) \leq g(z_0 + d)$ . Then  $\bar{g}(z_0 + d) \leq g(z_0 + d) = g(z_0) + g(d) < \bar{g}(z_0) + \bar{g}(d)$ , which is a contradiction to superadditivity of  $\bar{g}(z)$ .  $\square$

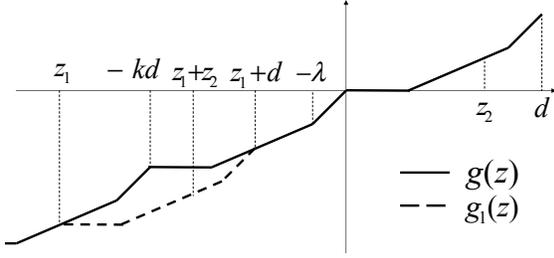


Figure 4:  $g(z)$  and  $g_1(z)$  for case 2

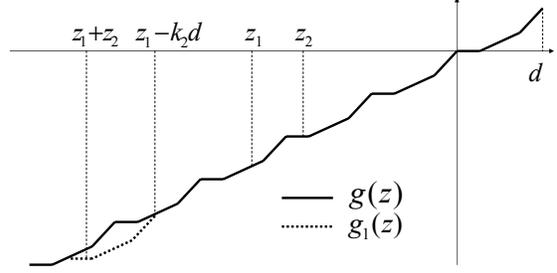


Figure 5:  $g(z)$  and  $g_1(z)$  for case 3

### 3.3 Lifted flow cover inequality

In this section we first give the resulting lifted flow cover inequality. Then we show that the inequality is valid by deriving  $(\alpha_i, \beta_i)$  for  $i \in (N \setminus C) \cap L_l$ . For  $i \in (N \setminus C) \cap L_u$  we only give the final result.

#### 3.3.1 The resulting lifted flow cover inequality

To simplify notation we define  $a_0 = 0$  and  $a_i = -\lambda + \sum_{j=1}^i (b_j \pm v_j)$  for  $i \in \{1, \dots, p\}$ , and  $a_{p+1} = d$ . Note that  $a_p = \gamma$ . We also let  $m_i = \frac{g(a_{i+1}) - g(a_i)}{a_{i+1} - a_i}$  for  $i \in \{0, \dots, p\}$  and observe that  $m_i < m_{i+1}$ , and  $m_0 = 0$ , and  $m_p = 1$ . Finally we let  $l_i = g(a_i) - \frac{g(a_{i+1}) - g(a_i)}{a_{i+1} - a_i} a_i$  for  $i \in \{0, \dots, p\}$ . Note that  $l_i > l_{i+1}$ , and  $l_0 = 0$ . Based on these definitions,  $(m_i, l_i)$  is a slope and  $y$ -intercept of the linear pieces which define  $g(z)$  for  $z \in [0, d]$ .

**Definition 1.**

- For  $i \in N_1^+ \cap N_2^+$  and  $b_i < d$  let  $k_i^1$  be the index with the property  $a_{k_i^1} \leq b_i < a_{k_i^1+1}$ .
- For  $i \in N_1^+ \cap N_2^+$  let  $k_i^2$  be the index with the property  $a_{k_i^2} \leq b_i + v_i < a_{k_i^2+1}$  if  $b_i + v_i < d$  and  $k_i^2 = p$  if  $b_i + v_i \geq d$ .
- For  $i \in N_1^- \cap N_2^+$  let  $k_i^2$  be the index with the property  $a_{k_i^2} \leq b_i < a_{k_i^2+1}$  if  $b_i < d$  and  $k_i^2 = p$  if  $b_i > d$ .
- For  $i \in N_1^- \cap N_2^+$  and  $d > b_i - v_i \geq 0$  let  $k_i^1$  be the index with the property  $a_{k_i^1} \leq b_i - v_i < a_{k_i^1+1}$ .
- For  $i \in N_1^- \cap N_2^+$  and  $b_i - v_i < 0$  let  $k_i^1$  be the index with the property  $a_{k_i^1} - t_i d \leq b_i - v_i < a_{k_i^1+1} - t_i d$  for a uniquely defined integer  $t_i \geq 1$ .

The lifted flow cover inequality is given in the next theorem.

**Theorem 6.** Let  $L_u$  be a subset of  $N$  and  $C$  be a subset of  $N \setminus L_u$  such that

$$\lambda = \sum_{i \in C_w} (u_i + v_i + b_i) + \sum_{i \in C_b} (u_i - v_i + b_i) + \sum_{i \in C_v} u_i - d' > 0,$$

where  $C_w = C \cap N_2^+$ ,  $C_b = \{i \in C \cap N_2^- : b_i \geq v_i\}$ ,  $C_v = \{i \in C \cap N_2^- : b_i < v_i\}$ ,  $d' = d - \sum_{i \in L_u \cap N_1^+} (u_i \pm v_i + b_i) + \sum_{i \in L_u \cap N_1^-} (u_i \pm v_i - b_i)$ , and let  $L_l = N \setminus (L_u \cup C)$ . Then the following *lifted flow cover inequality*,

where  $(\alpha_i, \beta_i) \in J_i$  and the lifting sets  $J_i$  are defined in Appendix A,

$$\begin{aligned}
& \sum_{i \in C} (x_i + b_i y_i) + \sum_{\substack{i \in (N_1^+ \setminus C) \cap L_l \\ u_i > 0}} g(b_i) y_i + \sum_{\substack{i \in N_1^- \cap L_l \\ u_i > 0}} (g(b_i) y_i - x_i) + \sum_{\substack{i \in (N_2^+ \setminus C) \cap L_l \\ u_i = 0}} (\alpha_i x_i + \beta_i y_i) \\
& + \sum_{i \in (N_1^+ \setminus C) \cap L_u} ((x_i - (u_i \pm v_i)) - (\pm v_i + g(\mp v_i - b_i))(y_i - 1)) - \sum_{i \in N_1^- \cap L_u} g(\pm v_i - b_i)(y_i - 1) \\
& \leq d - \sum_{i \in N_1^+ \cap L_u} (u_i \pm v_i + b_i) + \sum_{i \in N_1^- \cap L_u} (u_i \pm v_i - b_i) \\
& - \sum_{i \in C_w} (v_i + b_i - \lambda)^+(1 - y_i) - \sum_{i \in C_b} (b_i - v_i - \lambda)^+(1 - y_i) + \sum_{i \in C_v} (v_i - b_i - \lambda)^+ y_i,
\end{aligned} \tag{17}$$

is valid for  $P$ . □

We derive the second and the third terms in Section 3.3.2, the fourth term together with the sets  $J_i$  in Section 3.3.3, and the fifth and the sixth terms in Section 3.3.4.

**Example (continued).** Consider  $L_u = \emptyset$  and  $C = \{1, 2, 3, 5\}$ . Then  $C_w = \{1, 5\}$ ,  $C_b = \{2\}$ ,  $C_v = \{3\}$  and  $\lambda = 9 + 10 + 7 + 4 - 29 = 1$ , thus  $C$  is a cover. The resulting flow cover inequality is

$$x_1 + x_2 + x_3 + x_5 + y_2 - 2y_5 \leq 17. \tag{18}$$

The lifting function is

$$g(z) = \begin{cases} \dots & \\ z & -14 \leq z \leq 0 \\ 0 & 0 \leq z \leq 7 \\ \frac{1}{6}z - \frac{7}{6} & 7 \leq z \leq 13 \\ \frac{1}{2}z - \frac{11}{2} & 13 \leq z \leq 15 \\ z - 13 & 15 \leq z \leq 29. \end{cases}$$

Now we lift variables  $(x_i, y_i)$  for  $i = 4, 6, 7, 8, 9$ :

- $4 \in (N_1^+ \setminus C) \cap L_l$  and  $u_4 = 3 > 0$ , thus  $(\alpha_4, \beta_4) = (0, g(b_4)) = (0, g(6)) = (0, 0)$ .
- $6 \in (N_2^+ \setminus C) \cap L_l$  with  $u_6 = 0$ , and therefore  $(\alpha_6, \beta_6) = (0, 0)$ .
- $7 \in N_1^- \cap L_l$  and  $u_7 = 5 > 0$ , hence  $(\alpha_7, \beta_7) = (-1, g(b_7)) = (-1, g(7)) = (-1, 0)$ .
- $8 \in N_1^- \cap L_l$  and  $u_8 = 3 > 0$ , hence  $(\alpha_8, \beta_8) = (-1, g(b_8)) = (-1, g(2)) = (-1, 0)$ .
- $9 \in (N_2^+ \setminus C) \cap L_l$  and  $u_9 = 0$ , and therefore  $(\alpha_9, \beta_9) = (0, 0)$  or  $(\frac{1}{6}, -\frac{5}{6})$  or  $(\frac{1}{2}, -\frac{9}{2})$  or  $(1, -11)$ .

After lifting we obtain the following four valid inequalities for  $P$ :

$$\begin{aligned}
& x_1 + x_2 + x_3 + x_5 - x_7 - x_8 + y_2 - 2y_5 \leq 17 \\
& x_1 + x_2 + x_3 + x_5 - x_7 - x_8 + \frac{1}{6}x_9 + y_2 - 2y_5 - \frac{5}{6}y_9 \leq 17 \\
& x_1 + x_2 + x_3 + x_5 - x_7 - x_8 + \frac{1}{2}x_9 + y_2 - 2y_5 - \frac{9}{2}y_9 \leq 17 \\
& x_1 + x_2 + x_3 + x_5 - x_7 - x_8 + x_9 + y_2 - 2y_5 - 11y_9 \leq 17.
\end{aligned} \tag{19}$$

□

### 3.3.2 Lifting coefficients for variables with positive constant bounds

For  $i \in (N_2^+ \setminus C) \cap L_l$  we derive here the function  $h_i(z)$  introduced in (15) for the particular case where  $u_i > 0$ . The case where  $u_i = 0$  is studied in [Section 3.3.3](#).

**Proposition 4.** If  $u_i > 0$ , then  $h_i(z)$  is given by

$$h_i(z) = \begin{cases} \max \left( \varphi_{\alpha_i}^{[0, u_i]}(z), \psi_{\alpha_i}^{[b_i, u_i + v_i + b_i]}(z) \right) & \text{for } i \in N_1^+ \cap N_2^+ \\ \max \left( \varphi_{\alpha_i}^{[0, u_i]}(z), \psi_{\alpha_i}^{[b_i, u_i - v_i + b_i]}(z) \right) & \text{for } i \in N_1^+ \cap N_2^- \\ \max \left( \varphi_{-\alpha_i}^{[-u_i, 0]}(z), \psi_{-\alpha_i}^{[b_i - u_i - v_i, b_i]}(z) \right) & \text{for } i \in N_1^- \cap N_2^+ \\ \max \left( \varphi_{-\alpha_i}^{[-u_i, 0]}(z), \psi_{-\alpha_i}^{[b_i - u_i + v_i, b_i]}(z) \right) & \text{for } i \in N_1^- \cap N_2^-, \end{cases}$$

where

$$\varphi_{\alpha_i}^I(z) = \begin{cases} \alpha_i z & z \in I \\ -\infty & z \notin I \end{cases}$$

$$\psi_{\alpha_i}^I(z) = \begin{cases} \alpha_i(z - b_i) + \beta_i & z \in I \\ -\infty & z \notin I. \end{cases}$$

*Proof.* We only prove the case  $i \in N_1^+ \cap N_2^+$  since the other cases can be proven similarly. Because of (15) we have

$$\begin{aligned} h_i(z) &= \max(\alpha_i x + \beta_i y) \\ &\quad x + b_i y = z \\ &\quad 0 \leq x \leq u_i + v_i y \\ &\quad y \text{ binary.} \end{aligned}$$

If  $y_i = 0$ , then the value the objective value is  $\alpha_i z$  for  $z \in [0, u_i]$ . If  $y_i = 1$ , then the objective value is  $h_i(z) = \alpha_i(z - b_i) + \beta_i$  for  $z \in [b_i, u_i + v_i + b_i]$ . The result follows.  $\square$

Now we determine values for  $\alpha_i$  and  $\beta_i$ , which provide the strongest lifted inequalities.

**Proposition 5.** If  $i \in N_1^+$ , then  $(\alpha_i, \beta_i) = (0, g(b_i))$ . If  $i \in N_1^-$ , then  $(\alpha_i, \beta_i) = (-1, g(b_i))$ .

*Proof.* First assume that  $i \in N_1^+$ . We need to consider the following three cases

1. If  $\alpha_i > 0$ , then  $h_i(z) > 0$  for  $z > 0$ . However we need  $f(z) \geq h(z)$ . Since  $f(z) = 0$ , this case is impossible.
2. If  $\alpha_i = 0$ , then

$$h(z) = \begin{cases} 0 & 0 \leq z < \min\{u_i, b_i\} \\ \beta_i & b_i \leq z \leq u_i \pm v_i + b_i. \end{cases}$$

Since we need  $h(z) \leq g(z)$  and  $h(z)$  as large as possible, it is easy to see that  $\beta_i = g(b_i)$ .

3. If  $\alpha_i < 0$ , then

$$h(z) = \begin{cases} \alpha_i z & 0 \leq z < \min\{u_i, b_i\} \\ \alpha_i(z - b_i) + \beta_i & b_i \leq z \leq u_i \pm v_i + b_i. \end{cases}$$

The strongest restriction on  $\beta_i$  is imposed when  $z = b_i$ . For this  $z$ , we need to impose that  $\beta_i \leq g(b_i)$ . The best possible lifting coefficient for  $y_i$  is therefore obtained by choosing  $\beta_i = g(b_i)$ .

When  $i \in N_1^-$ , note that  $-\alpha_i z \leq h_i(z) \leq f(z) = z$  on  $[-\lambda, 0]$ , and therefore  $\alpha_i \leq -1$ . To make  $h_i(z)$  as large as possible, but still keep it smaller than  $g(z)$  we need  $-\alpha_i(z - b_i) + \beta_i = g(z)$  at  $z = b_i$ , and therefore  $\beta_i = g(b_i)$ . This completes the proof.  $\square$

By using [Proposition 5](#) we obtain the second and the third term in the lifted flow cover inequality (17).

Finally, note that we prove in [Theorem 2](#) that under some conditions (4) is a facet of  $P^+$ . Therefore the best lifting coefficients for the variables in  $N_1^+$  should be 0 in that case. Indeed, if conditions 1-4 hold, it can be verified that  $g(b_i) = 0$ , which implies that  $(\alpha_i, \beta_i) = (0, 0)$  for  $i \in N_1^+$ .

### 3.3.3 Lifting coefficients for variables with zero constant bounds

Let  $i \in (N_2^+ \setminus C) \cap L_l$  and  $u_i = 0$ , which corresponds to the fourth term in the lifted flow cover inequality. This case is more involved since there are less restrictions on the behavior of  $h(z)$  around  $z = 0$ .

1. Assume  $i \in N_2^-$ . This case is impossible because of Assumption 4 about the full-dimensionality of  $P$  in [Section 2.1](#).
2. Assume  $i \in N_1^+ \cap N_2^+$ . In this case

$$h_i(z) = \begin{cases} 0 & z = 0 \\ \alpha_i(z - b_i) + \beta_i & z \in [b_i, b_i + v_i]. \end{cases}$$

If  $b_i \geq d$ , then the support sets of  $g$  and  $h$  intersect only at  $z = 0$  and therefore  $(\alpha_i, \beta_i) = (0, 0)$  is the best lifting pair.

Assume now that  $b_i < d$ . We want to choose  $(\alpha_i, \beta_i)$  in such a way that  $h_i(z) \leq g(z)$  for every  $z$  where both functions are defined. For  $z = 0$  the inequality holds for any  $(\alpha_i, \beta_i)$  and therefore it suffices to consider  $z \in [b_i, b_i + v_i]$ . On this set  $h_i$  is linear while  $g$  is piecewise linear and convex. Convexity follows from the fact that  $g$  is equal to  $f_{LP}$  on this set, and  $f_{LP}$  is the value function of a linear program. The best possible  $(\alpha_i, \beta_i)$  are chosen in such a way that  $h_i(z) = g(z)$  for at least two different  $z$  values. It is clear therefore that  $h_i$  matches one of the linear pieces of  $g$ .

An example of several possible alternatives for the largest  $h_i(z)$  is shown in [Figure 6](#). Be definition  $a_{k_i^1} \leq b_i < a_{k_i^1+1}$  and  $a_{k_i^2} \leq b_i + v_i < a_{k_i^2+1}$ , and therefore the slope and  $y$ -intercept of  $h_i(z)$  are given by

$$(\alpha_i, \beta_i) = (m_j, l_j + m_j b_i) \text{ for } k_i^1 \leq j \leq k_i^2.$$

This corresponds to case 1 in [Appendix A](#).

Note that  $m_p < m_q$  for  $p < q$ , however,  $l_p + m_p b_i > l_q + m_q b_i$ , since these are the values of  $f_p(z) = m_p z + l_p$  and  $f_q(z) = m_q z + l_q$ . Thus  $\alpha_p < \alpha_q$  and  $\beta_p > \beta_q$  for  $p < q$ , and therefore the inequalities, which are constructed by lifting variables with indices  $p$  or  $q$  do not dominate each other.

3. Assume  $i \in N_1^- \cap N_2^+$ . We have

$$h_i(z) = \begin{cases} 0 & z = 0 \\ -\alpha_i(z - b_i) + \beta_i & z \in [b_i - v_i, b_i]. \end{cases}$$

If  $b_i - v_i \geq d$ , then the support set of  $g$  and  $h$  intersect only at  $z = 0$  and therefore  $(\alpha_i, \beta_i) = (0, 0)$  is the best lifting pair; see case 2 in [Appendix A](#).

We assume now that  $b_i - v_i < d$ . For  $b_i - v_i \geq 0$ , the domain of  $h_i(z)$  is a subset of  $\{z : z \geq 0\}$  and this case is identical to the above case by only changing the sign of  $\alpha_i$ . Therefore the slope and  $y$ -intercept of  $h_i(z)$  are given by

$$(\alpha_i, \beta_i) = (-m_j, l_j + m_j b_i) \text{ for } k_i^1 \leq j \leq k_i^2.$$

This is case 3 of [Appendix A](#).

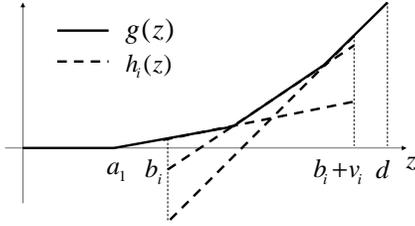


Figure 6: Possible  $h(z)$  for  $i \in N_1^+ \cap N_2^+$

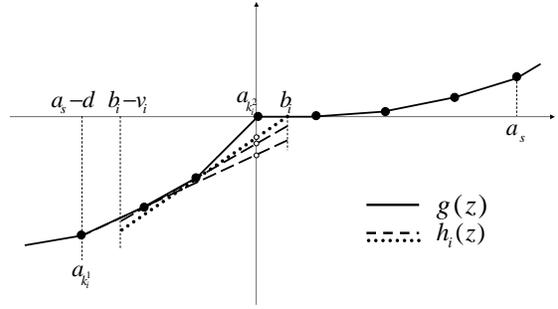


Figure 7: Possible  $h(z)$  for  $i \in N_1^- \cap N_2^+$ ,  $b_i - v_i > a_s - d$  and  $b_i < a_s$

In this case we have  $\alpha_p > \alpha_q$  and  $\beta_p < \beta_q$  for each  $p < q$ , hence again inequalities which are constructed by lifting variables  $p$  or  $q$  do not dominate each other.

The case  $b_i - v_i < 0$  is more complicated. Let  $m_s \leq D/d < m_{s+1}$  for a uniquely defined integer  $s \in \{0, 1, \dots, p\}$ . Consider the following five cases.

- (a)  $a_s - d < b_i - v_i < a_0$  and  $a_1 < b_i < a_s$ : In this case for any  $1 \leq j \leq k_i^2$  we have

$$(\alpha_i, \beta_i) = (-m_j, l_j + m_j b_i) \quad \text{if } m_j(b_i - v_i) + l_j \leq g(b_i - v_i). \quad (19)$$

Let  $j_2$  be the smallest  $j$  such that  $m_j(b_i - v_i) + l_j \leq g(b_i - v_i)$ . Then we have one more choice, which reads

$$(\alpha_i, \beta_i) = \left( -\frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i}, g(a_{j_2}) - \frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i} a_{j_2} + \frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i} b_i \right). \quad (20)$$

We also have another set of possibilities given by

$$(\alpha_i, \beta_i) = (-m_j, l_j - D + m_j(d + b_i)) \quad \text{if } m_j b_i + l_j - D + m_j d \leq g(b_i), \quad (21)$$

for  $k_i^1 \leq j \leq p - 1$  (see dashed lines in [Figure 7](#)), and if  $j_1$  is the largest  $j$  such that  $m_j b_i + l_j - D + m_j d \leq g(b_i)$ , then we have one more choice

$$(\alpha_i, \beta_i) = \left( -\frac{g(b_i) - g(a_{j_1+1} - d)}{b_i - a_{j_1+1} + d}, g(b_i) \right). \quad (22)$$

This choice is depicted by the dotted line in [Figure 7](#). Note that (19) and (20) do not occur in [Figure 7](#) due to the choice of  $b_i$  and  $v_i$ . Note also that all presented coefficients yield non-dominated inequalities. Expressions (19)-(22) correspond to case 4a in [Appendix A](#).

- (b)  $b_i - v_i \geq a_0$  and  $b_i \leq a_1$ : In this case we have just one choice given by

$$(\alpha_i, \beta_i) = \left( -\frac{g(b_i) - g(b_i - v_i)}{v_i}, g(b_i) \right),$$

which is case 4b in [Appendix A](#).

- (c)  $b_i - v_i \leq a_s - d$  and  $b_i < a_s$ : First for  $k_i^1 \leq j \leq s - 1$  we have

$$(\alpha_i, \beta_i) = (-m_j, l_j - t_i D + m_j(t_i d + b_i)). \quad (23)$$

Second, if  $t_i = 1$  we can use (21) and (22). If  $t_i > 1$ , the only possibility is

$$(\alpha_i, \beta_i) = \left( -\frac{D}{d}, g(a_s) - \frac{D}{d} a_s + m_j b_i \right). \quad (24)$$

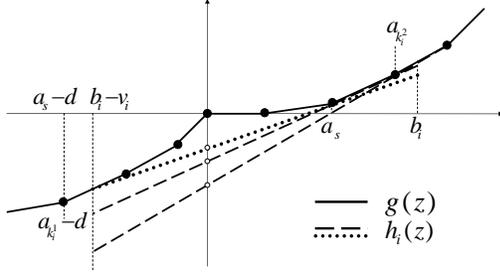


Figure 8: Possible  $h(z)$  for  $i \in N_1^- \cap N_2^+$ ,  $b_i - v_i > a_s - d$  and  $b_i \geq a_s$

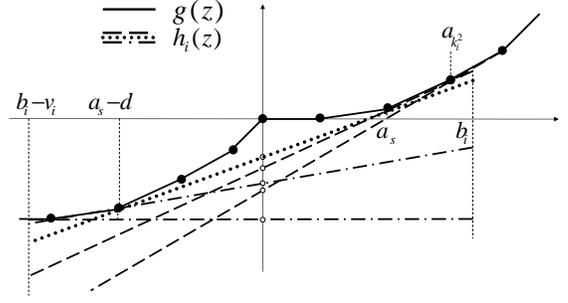


Figure 9: Possible  $h(z)$  for  $i \in N_1^- \cap N_2^+$ ,  $b_i - v_i \leq a_s - d$  and  $b_i \geq a_s$

Note, that in this case  $g(b_i) = 0$  and  $g(b_i - v_i) = b_i - v_i$ . Case 4c in Appendix A corresponds to (21), (22), and (23), (24).

(d)  $b_i - v_i > a_s - d$  and  $b_i \geq a_s$ : For  $s \leq j \leq k_i^2$  we have

$$(\alpha_i, \beta_i) = (-m_j, l_j + m_j b_i). \quad (25)$$

This case is shown by dashed lines in Figure 8. In addition, for  $1 \leq j \leq s - 1$  we obtain

$$(\alpha_i, \beta_i) = (-m_j, l_j + m_j b_i) \quad \text{if } m_j(b_i - v_i) + l_j \leq g(b_i - v_i), \quad (26)$$

which do not occur in Figure 8, and (20), which is represented in Figure 8 by the dotted line. The first set in case 4d, Appendix A, matches (25), the second one corresponds to (20), and the last one to (26).

(e)  $b_i - v_i \leq a_s - d$  and  $b_i \geq a_s$ : We have possibilities (23), (24) and (25); see case 4e in Appendix A, which are represented by dashed-dotted, dotted, and dashed lines, respectively, in Figure 9.

Note that in typical sequence independent lifting, Gu *et al.* (1999), Atamtürk *et al.* (2001), Klabjan and Nemhauser (2002),  $g(z)$  has only two slopes. This is not the case here and therefore we obtain several  $(\alpha_i, \beta_i)$  pairs.

### 3.3.4 Alternative strategies to variable fixing

So far we have considered lifting of variables from  $L_l$ , i.e.  $x_i = y_i = 0$  for  $i \in L_l$ . Similarly we can obtain lifting coefficients for variables from  $L_u$ , which are fixed at the upper bounds. This result is given in Proposition 6. The proof can be found in Shebalov (2004).

**Proposition 6.** If  $x_i = u_i \pm v_i y_i$  and  $y_i = 1$ , then for  $u_i > 0$  the lifting coefficients are given by

$$(\alpha_i, \beta_i) = \begin{cases} (1, \mp v_i - g(\mp v_i - b_i)) & i \in N_1^+ \\ (0, -g(\pm v_i - b_i)) & i \in N_1^- \end{cases}$$

The case  $u_i = 0$  does not produce additional coefficients.  $\square$

This proposition gives the fifth and the sixth terms in the lifted flow cover inequality (17).

Along with fixing variables to upper and lower bounds described earlier we can consider *mixed* variable fixings, where one of the variables from the pair  $(x_i, y_i)$  is fixed to the lower bound and the other one to the upper bound. Lifting coefficients for these cases are obtained by the same procedure. We thus present only the final results.

**Proposition 7.**

1.  $x_i = 0, y_i = 1, i \in N_1^+$ :  $(\alpha_i, \beta_i) = (0, -g(-b_i))$ .
2.  $x_i = 0, y_i = 1, i \in N_1^-$ :  $(\alpha_i, \beta_i) = (-1, -g(-b_i))$ .
3.  $x_i = u_i, y_i = 0, i \in N_1^+$  and  $u_i > 0$ :  $(\alpha_i, \beta_i) = (1, g(b_i \pm v_i))$ .
4.  $x_i = u_i, y_i = 0, i \in N_1^-$  and  $u_i > 0$ :  $(\alpha_i, \beta_i) = (0, g(b_i \pm v_i))$ . □

We did not consider these variable fixings in our lifted flow cover inequality. Their addition is straightforward.

Another possibility are *intermediate* variable fixings, where continuous variable  $x_i$  is fixed in the interior of its range. We now prove that these variable fixings do not give lifting coefficients.

**Proposition 8.** If  $x_i = f_i$  is fixed to a value  $0 < f_i < u_i \pm v_i y_i$ , then the corresponding lifting coefficients  $(\alpha_i, \beta_i)$  do not exist.

*Proof.* Let  $f_i \in (0, u_i)$ , and let  $h_i^0(z)$  be the objective value of  $h_i(z)$  when  $y_i = 0$ , and  $f_i \in (0, u_i \pm v_i)$  and  $h_i^1(z)$  be the objective value of  $h_i(z)$  when  $y_i = 1$ . Note that  $h_i(z) \geq h_i^0(z)$  and  $h_i(z) \geq h_i^1(z)$ . Consider the following possible cases.

1.  $x_i = f_i, y_i = 0, i \in N_1^+$  and  $u_i > 0$ :  $h_i^0(z) = \alpha_i z$  on  $[-f_i, u_i - f_i]$ .
2.  $x_i = f_i, y_i = 1, i \in N_1^+$  and  $u_i \geq 0$ :  $h_i^1(z) = \alpha_i z$  on  $[-f_i, u_i \pm v_i - f_i]$ .
3.  $x_i = f_i, y_i = 0, i \in N_1^-$  and  $u_i > 0$ :  $h_i^0(z) = -\alpha_i z$  on  $[f_i - u_i, f_i]$ .
4.  $x_i = f_i, y_i = 1, i \in N_1^-$  and  $u_i \geq 0$ :  $h_i^1(z) = -\alpha_i z$  on  $[f_i - u_i \mp v_i, f_i]$ .

In all cases  $h_i(z) \geq mz$  on  $[\tilde{a}, \tilde{b}]$  for some  $m \in \mathbb{R}, m \neq 0, \tilde{a} < 0$  and  $\tilde{b} > 0$ . However, we know that  $f(z) = z$  on  $[\tilde{c}, 0]$  and  $f(z) = 0$  on  $[0, \tilde{d}]$  for some  $\tilde{c} < 0$  and  $\tilde{d} > 0$ . To compute the lifting coefficients we need  $h_i(z) \leq f(z)$ . If  $m > 0$ , then  $h_i(z) \geq mz > 0 = f(z)$  on  $(0, \min\{\tilde{b}, \tilde{d}\}]$ , and if  $m \leq 0$ , then  $h_i(z) \geq mz > z = f(z)$  on  $[\max\{\tilde{a}, \tilde{c}\}, 0)$ . Therefore lifting is not possible in any of these cases. □

**Proposition 8** is not surprising since a similar result is obtained by [Richard et al. \(2003a\)](#) when  $v_i = 0$  for all  $i \in N$ .

## 4 Valid inequalities from specially structured covers

The main difficulty in lifting the flow cover inequality (4) is the inability to compute the lifting function. In this section we show that under additional structure imposed on the cover we can explicitly give the lifting function. Let  $C$  be a cover. Assume without loss of generality that the variables in  $C_w^+ \cup C_b^+$  are ordered according to the nonincreasing order of  $b_i \pm v_i$ , i.e.  $b_1 \pm v_1 \geq b_2 \pm v_2 \geq \dots \geq b_p \pm v_p$ . Recall that  $p = |C_w^+| + |C_b^+|$ . If we also assume that  $b_i \geq b_j$  for  $i \leq j$ , we say that  $C$  has *compatible orders*. Note that the order of the variables not in  $C_w^+ \cup C_b^+$  is irrelevant in the definition of compatible orders.

### 4.1 Lifting function

If  $C$  has compatible orders we can derive an explicit form of  $f(z)$ .

**Theorem 7.** If  $C$  has compatible orders, then

$$f(z) = \begin{cases} -\lambda & -\infty < z \leq -\lambda \\ z + k\lambda - \sum_{i=1}^k (b_i \pm v_i) & -\lambda + \sum_{i=1}^k (b_i \pm v_i) < z \leq \min\{\sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i), \lambda\}, \\ & 0 \leq k \leq p-1 \\ k\lambda & \min\{\sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i), \lambda\} < z \leq -\lambda + \sum_{i=1}^{k+1} (b_i \pm v_i), \\ & 0 \leq k \leq p-1 \\ z - \sum_{i \in C_w^+} (b_i + v_i - \lambda) & -\lambda + \sum_{i=1}^p (b_i \pm v_i) \leq z \leq d, \\ -\sum_{i \in C_b^+} (b_i - v_i - \lambda) & \end{cases} \quad \square$$

The proof of [Theorem 7](#) is very technical, and it is presented in [Appendix B](#). In general  $f(z)$  is not superadditive, therefore we again need to find a superadditive function  $g$  such that  $g(z) \leq f(z)$ . We define

$$g(z) = \begin{cases} i\lambda & k(b_1 \pm v_1) \leq z \leq -\lambda + (k+1)(b_1 \pm v_1) & i = 0, \pm 1, \pm 2, \dots \\ z - i(b_1 \pm v_1 - \lambda) & -\lambda + k(b_1 \pm v_1) \leq z \leq k(b_1 \pm v_1) & i = 0, \pm 1, \pm 2, \dots \end{cases}$$

Note that  $f$  might not be continuous. [Gu et al. \(1999\)](#) prove that  $g(z)$  is superadditive and is not dominated with respect to  $f(z)$ . [Figure 10](#) illustrates  $f$  and  $g$  in the compatible orders case.

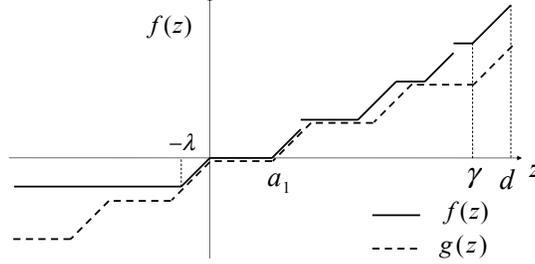


Figure 10:  $f(z)$  and  $g(z)$  for the case of compatible orders

## 4.2 Lifted flow cover inequality from specially structured covers

Lifting coefficients  $(\alpha_i, \beta_i)$  are obtained similarly to the general case presented in [Section 3](#). (See [Shebalov \(2004\)](#) for details.) Here we give only the main conclusion. We define the family of sets  $J_i$  for  $i \in (N_2^+ \cap L_l) \setminus C$  with  $u_i = 0$ , and  $b_i < d$  for  $i \in N_1^+$  and  $b_i - v_i < d$  for  $i \in N_1^-$ , as

$$J_i = \left\{ \left( \frac{g(a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda)) - g(b_i)}{a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda) - b_i}, \right. \right. \\ \left. \left. g(a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda)) - \frac{g(a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda)) - g(b_i)}{a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda) - b_i} (a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda) - b_i) \right) \right\} \\ \cup \left\{ \left( \frac{\lambda}{\lambda + a_1}, -\frac{\lambda a_1}{\lambda + a_1} \right) \right\} \\ \cup \left\{ \left( \frac{g(b_i + v_i) - g(a_1 + \bar{k}_i^2(a_1 + \lambda))}{b_i + v_i - a_1 - \bar{k}_i^2(a_1 + \lambda)}, g(b_i + v_i) - \frac{g(b_i + v_i) - g(a_1 + \bar{k}_i^2(a_1 + \lambda))}{b_i + v_i - a_1 - \bar{k}_i^2(a_1 + \lambda)} (b_i + v_i - a_1 - \bar{k}_i^2(a_1 + \lambda)) \right) \right\},$$

where

$$a_1 + \bar{k}_i^1(a_1 + \lambda) \leq b_i \leq a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda), a_1 + \bar{k}_i^2(a_1 + \lambda) \leq b_i + v_i \leq a_1 + (\bar{k}_i^2 + 1)(a_1 + \lambda) \text{ for } i \in N_1^+ \\ a_1 + \bar{k}_i^1(a_1 + \lambda) \leq b_i - v_i \leq a_1 + (\bar{k}_i^1 + 1)(a_1 + \lambda), a_1 + \bar{k}_i^2(a_1 + \lambda) \leq b_i \leq a_1 + (\bar{k}_i^2 + 1)(a_1 + \lambda) \text{ for } i \in N_1^-.$$

If  $i \in (N_2^+ \cap L_l) \setminus C$  with  $u_i = 0$  and either  $b_i \geq d$  for  $i \in N_1^+$  or  $b_i - v_i \geq d$  for  $i \in N_1^-$ , then  $J_i = \{(0, 0)\}$ .

**Theorem 8.** If  $C$  is a cover with compatible orders, then

$$\begin{aligned}
& \sum_{i \in C} (x_i + b_i y_i) + \sum_{\substack{i \in (N_1^+ \setminus C) \cap L_l \\ u_i > 0}} g(b_i) y_i + \sum_{\substack{i \in N_1^- \cap L_l \\ u_i > 0}} (g(b_i) y_i - x_i) \\
& + \sum_{\substack{i \in ((N_1^+ \cap N_2^+) \setminus C) \cap L_l \\ u_i = 0}} (\alpha_i x_i + \beta_i y_i) + \sum_{\substack{i \in N_1^- \cap N_2^+ \cap L_l \\ u_i = 0}} (-\alpha_i x_i + \beta_i y_i) \\
& + \sum_{i \in (N_1^+ \setminus C) \cap L_u} ((x_i - (u_i \pm v_i)) - (\pm v_i + g(\mp v_i - b_i))(y_i - 1)) - \sum_{i \in N_1^- \cap L_u} g(\pm v_i - b_i)(y_i - 1) \\
& \leq d - \sum_{i \in N_1^+ \cap L_u} (u_i \pm v_i + b_i) + \sum_{i \in N_1^- \cap L_u} (u_i \pm v_i - b_i) \\
& - \sum_{i \in C_w} (v_i + b_i - \lambda)^+(1 - y_i) - \sum_{i \in C_b} (b_i - v_i - \lambda)^+(1 - y_i) + \sum_{i \in C_v} (v_i - b_i - \lambda)^+ y_i \tag{27}
\end{aligned}$$

is valid for  $P$  for any  $(\alpha_i, \beta_i) \in J_i$ .  $\square$

**Example (continued).** For cover  $C = \{1, 2, 3\}$ , we have  $L_l = \{4, 6, 7\}$ ,  $C_w^+ \cup C_b^+ = \{1, 2\}$ ,  $b_i \pm v_i = (6, 2)$  and  $b_i = (4, 3)$ , thus  $C$  has compatible orders. The two functions are

$$f(z) = \begin{cases} -1 & -\infty < z \leq -1 \\ z & -1 \leq z \leq 0 \\ 0 & 0 \leq z \leq 5 \\ z - 5 & 5 \leq z \leq 6 \\ 1 & 6 \leq z \leq 7 \\ z - 6 & 7 \leq z \leq 19 \end{cases} \quad g(z) = \begin{cases} \dots & \\ -1 & -6 < z \leq -1 \\ z & -1 \leq z \leq 0 \\ 0 & 0 \leq z \leq 5 \\ z - 5 & 5 \leq z \leq 6 \\ 1 & 6 \leq z \leq 11 \\ \dots & \end{cases}$$

Lifting variables  $(x_i, y_i)$  for  $i = 4, \dots, 8$  gives the following coefficients.

- $4 \in (N_1^+ \setminus C) \cap L_l$  and  $u_4 = 3 > 0$ , thus  $(\alpha_4, \beta_4) = (0, g(b_4)) = (0, g(6)) = (0, 1)$ .
- $5 \in (N_1^+ \setminus C) \cap L_u$ , hence  $(\alpha_5, \beta_5) = (1, -v_5 - g(\mp v_5 - b_5)) = (1, -2 - g(-5)) = (1, -1)$ .
- $6 \in (N_1^- \cap N_2^+) \setminus C \cap L_l$  and  $u_6 = 0$ , and therefore  $(\alpha_6, \beta_6) = (0, -1)$  or  $(-\frac{1}{6}, \frac{1}{6})$  or  $(-1, 1)$ .
- $7 \in N_1^- \cap L_l$  and  $u_7 = 5 > 0$ , hence  $(\alpha_7, \beta_7) = (-1, g(b_7)) = (-1, g(7)) = (-1, 1)$ .
- $8 \in N_1^- \cap L_u$ , hence  $(\alpha_8, \beta_8) = (0, -g(\pm v_8 - b_8)) = (0, -g(-3)) = (0, 1)$ .

The resulting three inequalities are:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_5 - x_7 - y_1 + 2y_2 + y_4 - y_5 - y_6 + y_7 + y_8 &\leq 19, \\
x_1 + x_2 + x_3 + x_5 - \frac{1}{6}x_6 - x_7 - y_1 + 2y_2 + y_4 - y_5 + \frac{1}{6}y_6 + y_7 + y_8 &\leq 19, \\
x_1 + x_2 + x_3 + x_5 - x_6 - x_7 - y_1 + 2y_2 + y_4 - y_5 + y_6 + y_7 + y_8 &\leq 19. \quad \square
\end{aligned}$$

### 4.3 A note on the single node fixed charge flow covers

Gu *et al.* (1999) consider the set

$$S = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \sum_{i \in N_1^+} x_i - \sum_{i \in N_1^-} x_i \leq d', x_i \leq v_i y_i, i \in N = N_1^+ \cup N_1^-, y \text{ binary} \right\}.$$

They choose  $L_u \subseteq N_1^-$  and project variables with indices from this set to their upper bounds, so  $x_i = v_i$  and  $y_i = 1$  for  $i \in L_u$ . Defining a cover as  $C \subseteq N^+$  with  $\lambda = \sum_{i \in C} v_i - d \geq 0$ , they project variables from  $N \setminus (C \cup L_u)$  to zero and consider the flow cover inequality

$$0 \leq d' + \sum_{i \in L_u} v_i - \sum_{i \in C} x_i - \sum_{i \in C^+} (v_i - \lambda)(1 - y_i), \quad (28)$$

where  $C^+ = \{i \in C : v_i > \lambda\}$ . Lifting (28) they obtain a valid inequality for  $S$  that reads

$$\begin{aligned} & \sum_{i \in C \cup L^+} x_i + \sum_{i \in C^+} (v_i - \lambda)(1 - y_i) - \sum_{i \in L^+} \tilde{\beta}_i y_i \\ & \leq d' + \sum_{i \in N_1^- \cap L_u} v_i - \sum_{i \in N_1^- \cap L_u} g(v_i)(1 - y_i) - \sum_{i \in L^-} g(-v_i)y_i + \sum_{i \in L^{--}} x_i, \end{aligned} \quad (29)$$

where

$$\tilde{\beta}_i = \begin{cases} v_i - k\lambda & kv_1 \leq v_i \leq (k+1)v_1 - \lambda \\ k(v_1 - \lambda) & kv_1 - \lambda \leq v_i \leq kv_1, \end{cases}$$

$L^+$  is an arbitrary subset of  $N^+ \setminus C$ ,  $L^-$  is an arbitrary subset of  $N^- \setminus C^-$ , and  $L^{--} = (N^- \setminus C^-) \setminus L^-$ . We now show that (29) can be obtained from [Theorem 8](#) with  $u_i = b_i = 0$ ,  $N_2^+ = \emptyset$ ,  $N_1^+ \cap L_u = \emptyset$  and particular choice of values for  $\alpha_i$  and  $\beta_i$  for  $i \in (N_1^+ \setminus C) \cup (N_1^- \setminus L_u)$ . Therefore [Theorem 8](#) strictly generalizes the results of [Gu et al. \(1999\)](#). Moreover we show that (29) can be further strengthened by an adequate choice of  $(\alpha, \beta)$ .

For  $i \in N^+ \setminus C$  with the above assumptions we have

$$J_i = \{(0, 0)\} \quad (30)$$

$$\bigcup \left\{ \left( \frac{\lambda}{v_1}, -\frac{\lambda(v_1 - \lambda)}{v_1} \right) \right\} \quad (31)$$

$$\bigcup \left\{ (1, -k(v_1 - \lambda)) : kv_1 - \lambda \leq v_i \leq kv_1 \right\} \quad (32)$$

$$\bigcup \left\{ \left( \frac{\lambda}{v_i - kv_1 + \lambda}, k\lambda - \frac{\lambda}{v_i - kv_1 + \lambda} v_i \right) : kv_1 \leq v_i \leq (k+1)v_1 - \lambda \right\}. \quad (33)$$

Note that (32) and (33) are mutually disjoint. For  $i \in N^- \setminus L_u$  with the above assumptions we have

$$J_i = \{(-1, 0)\} \quad (34)$$

$$\bigcup \left\{ \left( -\frac{\lambda}{v_1}, -\frac{\lambda(v_1 - \lambda)}{v_1} \right) \right\} \quad (35)$$

$$\bigcup \left\{ (0, -(k+1)\lambda) : \lambda + kv_1 \leq v_i \leq (k+1)v_1 \right\} \quad (36)$$

$$\bigcup \left\{ \left( -\frac{v_i - kv_1}{v_i - (k-1)v_1 - \lambda}, k(v_1 - \lambda) - v_i + \frac{v_i - kv_1}{v_i - (k-1)v_1 - \lambda} v_i \right) : kv_1 \leq v_i \leq \lambda + kv_1 \right\}. \quad (37)$$

Similarly, (36) and (37) are mutually disjoint. Let  $(\alpha_i, \beta_i)$  be defined by (30) for  $i \in N_1^+ \setminus (C \cup L^+)$  and by (34) for  $i \in L^{--}$ . Then inequality (27) from [Theorem 8](#) becomes

$$\begin{aligned} & \sum_{i \in C} x_i + \sum_{i \in L^+} \alpha_i x_i + \sum_{i \in C^+} (v_i - \lambda)(1 - y_i) + \sum_{i \in L^+} \beta_i y_i \\ & \leq d' + \sum_{i \in N_1^- \cap L_u} v_i - \sum_{i \in N_1^- \cap L_u} g(v_i)(1 - y_i) - \sum_{i \in L^-} \beta_i y_i + \sum_{i \in L^-} \alpha_i x_i + \sum_{i \in L^{--}} x_i. \end{aligned} \quad (38)$$

If  $(\alpha_i, \beta_i)$  are defined by (32) for all  $i \in L^+$  and by (36) for all  $i \in L^-$ , then (38) is identical to (29). If there exists  $j \in L^+$  such that  $(\alpha_j, \beta_j)$  is defined by (33) or  $j \in L^-$  such that  $(\alpha_j, \beta_j)$  is defined by (37), then (38)

is at least as strong as (29). To prove this let  $(x, y)$  be in the LP relaxation of  $S$  and consider the difference between the lifting term in our case denoted by  $\alpha_j x_j + \beta_j y_j$  and in case considered by Gu *et al.* (1999) and denoted by  $\tilde{\alpha}_j x_j + \tilde{\beta}_j y_j$ . Then for  $j \in L^+$  we have

$$\begin{aligned} (\alpha_j x_j + \beta_j y_j) - (\tilde{\alpha}_j x_j + \tilde{\beta}_j y_j) &= x_j \frac{\lambda}{v_j - kv_1 + \lambda} + \left( k\lambda - \frac{\lambda}{v_j - kv_1 + \lambda} v_j \right) y_j - x_j - (k\lambda - v_j) y_j = \\ x_j \frac{kv_1 - v_j}{v_j - kv_1 + \lambda} + y_j \frac{v_j^2 - kv_1 v_j}{v_j - kv_1 + \lambda} &\geq v_j y_j \frac{kv_1 - v_j}{v_j - kv_1 + \lambda} + y_j \frac{v_j^2 - kv_1 v_j}{v_j - kv_1 + \lambda} = y_j \frac{kv_1 v_j - v_j^2 + v_j^2 - kv_1 v_j}{v_j - kv_1 + \lambda} = 0, \end{aligned}$$

and for  $i \in L^-$

$$\begin{aligned} (\alpha_j x_j + \beta_j y_j) - (\tilde{\alpha}_j x_j + \tilde{\beta}_j y_j) &= -x_j \frac{v_j - kv_1}{v_j - (k-1)v_1 - \lambda} + \left( k(v_1 - \lambda) - v_j + \frac{v_j - kv_1}{v_j - (k-1)v_1 - \lambda} v_j \right) y_j \\ - (k(v_1 - \lambda) - v_j) y_j &\geq \left( -\frac{v_j^2 - kv_1 v_j}{v_j - (k-1)v_1 - \lambda} + \frac{v_j^2 - kv_1 v_j}{v_j - (k-1)v_1 - \lambda} v_j \right) y_j = 0. \end{aligned}$$

Finally note that if we choose (31) or (35), we get additional new inequalities.

## 5 Conclusion

In this work we derive valid inequalities for the convex hull of  $\bar{S}$ . We start with a generalized flow cover inequality and next we lift it by using sequence independent lifting. The main difficulty in this procedure is the fact that the lifting function is NP-hard to evaluate and therefore a lower bound needs to be obtained. If an additional structure on the cover is imposed, then the lifting function can be expressed in a closed form. Unfortunately, it is not superadditive. All proofs of the presented results are long and extremely technical. We believe further generalizations of  $\bar{S}$ , e.g. variable upper bounds involving several binary variables or variable lower bounds, would complicate the proofs even further, most likely to the point, where an analysis along the same lines is no longer possible.

In a sequel paper, Shebalov and Klabjan (2004) discuss sequence dependent lifting for  $\bar{S}$ . Among other variable fixings, we consider fixing all binary variables but one. We are able to obtain the complete polyhedral description of the resulting set. Unfortunately, computing lifting coefficients from resulting inequalities is NP-hard. An interesting line of research is to perform sequence independent lifting starting with these inequalities. This is a challenging task since each binary variable that needs to be lifted is present in two inequalities and therefore the lifting function is multivariate.

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## A Definition of the lifting sets used in **Theorem 6**

1.  $i \in N_1^+ \cap N_2^+$

$$J_i = \begin{cases} \{(m_j, l_j + m_j b_i) : k_i^1 \leq j \leq k_i^2\} & b_i < d \\ \{(0, 0)\} & b_i \geq d \end{cases}$$

2.  $i \in N_1^- \cap N_2^+$  and  $b_i - v_i \geq d$

$$J_i = \{(0, 0)\}$$

3.  $i \in N_1^- \cap N_2^+$  and  $d > b_i - v_i \geq 0$

$$J_i = \{(-m_j, l_j + m_j b_i) : k_i^1 \leq j \leq k_i^2\}$$

4.  $i \in N_1^- \cap N_2^+$  and  $b_i - v_i < 0$  with  $m_s \leq D/d < m_{s+1}$

(a)  $a_s - d < b_i - v_i < a_0$  and  $a_1 < b_i < a_s$

$$J_i = \left\{ (-m_j, l_j + m_j b_i) : 1 \leq j \leq k_i^2, m_j(b_i - v_i) + l_j \leq g(b_i - v_i) \right\}$$

$$\bigcup \left\{ (-m_j, l_j - D + m_j(d + b_i)) : k_i^1 \leq j \leq p-1, m_j(b_i + d) + l_j - D \leq g(b_i) \right\}$$

$$\bigcup \left\{ \left( -\frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i}, g(a_{j_2}) - \frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i}(a_{j_2} - b_i) \right) : j_2 = \min\{j : m_j(b_i - v_i) + l_j \leq g(b_i - v_i)\} \right\}$$

$$\bigcup \left\{ \left( -\frac{g(b_i) - g(a_{j_1+1} - d)}{b_i - a_{j_1+1} + d}, g(b_i) \right) : j_1 = \max\{j : m_j b_i + l_j - D + m_j d \leq g(b_i)\} \right\}$$

(b)  $a_0 \leq b_i - v_i < 0$  and  $0 < b_i \leq a_1$

$$J_i = \left\{ \left( \frac{v_i - b_i}{v_i}, 0 \right) \right\}$$

(c)  $b_i - v_i \leq a_s - d$  and  $b_i < a_s$

$$J_i = \left\{ (-m_j, l_j - t_i D + m_j(t_i d + b_i)) : k_i^1 \leq j \leq s-1 \right\}$$

$$\bigcup \left\{ (-m_j, l_j - D + m_j(d + b_i)) : s \leq j \leq p-1, m_j(b_i + d) + l_j - D \leq g(b_i), t_i = 1 \right\}$$

$$\bigcup \left\{ \left( -\frac{g(b_i) - g(a_{j_1+1} - d)}{b_i - a_{j_1+1} + d}, g(b_i) \right) : j_1 = \max\{j : m_j b_i + l_j - D + m_j d \leq g(b_i)\}, t_i = 1 \right\}$$

$$\bigcup \left\{ \left( -\frac{D}{d}, g(a_s) - \frac{D}{d} a_s + m_j b_i \right) : t_i > 1 \right\}$$

(d)  $b_i - v_i > a_s - d$  and  $b_i \geq a_s$

$$J_i = \left\{ (-m_j, l_j + m_j b_i) : s \leq j \leq k_i^2, m_j(b_i - v_i) + l_j \leq g(b_i - v_i) \right\}$$

$$\bigcup \left\{ \left( -\frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i}, g(a_{j_2}) - \frac{g(a_{j_2}) - g(b_i - v_i)}{a_{j_2} - b_i + v_i}(a_{j_2} - b_i) \right) : j_2 = \min\{j : m_j(b_i - v_i) + l_j \leq g(b_i - v_i)\} \right\}$$

$$\bigcup \left\{ (-m_j, l_j + m_j b_i) : 1 \leq j \leq s-1, m_j(b_i - v_i) + l_j \leq g(b_i - v_i) \right\}$$

(e)  $b_i - v_i \leq a_s - d$  and  $b_i \geq a_s$

$$J_i = \left\{ (-m_j, l_j - t_i D + m_j(t_i d + b_i)) : k_i^1 \leq j \leq s-1 \right\}$$

$$\cup \left\{ (-m_j, l_j + m_j b_i) : s \leq j \leq k_i^2, m_j(b_i - v_i) + l_j \leq g(b_i - v_i) \right\}$$

$$\cup \left\{ \left( -\frac{D}{d}, g(a_s) - \frac{D}{d} a_s + m_j b_i \right) : t_i > 1 \right\}.$$

## B Proof of Theorem 7

*Proof.* Consider first the reduced problem

$$\bar{f}(z) = \max \left( \sum_{i \in C} x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i + \sum_{i \in C_v^+} (2b_i - v_i + \lambda) y_i + \sum_{i \in C \setminus C^+} b_i y_i \right) \quad (39a)$$

$$\sum_{i \in C} x_i + \sum_{i \in C} b_i y_i \leq d - z \quad (39b)$$

$$0 \leq x_i \leq u_i + v_i y_i \quad i \in C_w \quad (39c)$$

$$0 \leq x_i \leq u_i - v_i y_i \quad i \in C_b \cup C_v \quad (39d)$$

$$y \text{ binary.} \quad (39e)$$

At  $z_0 = -\lambda + \sum_{j=1}^k (b_i \pm v_i)$  for  $0 \leq k \leq p$  the solution of the LP relaxation of this problem is integral and has the form

$$(\bar{x}, \bar{y}) = \left( \underbrace{(u_1, \dots, u_k, u_{k+1} \pm v_{k+1}, \dots, u_p \pm v_p, 0, \dots, 0, 1, \dots, 1)}_{C_w^+ \cup C_b^+}, \underbrace{(u_{p+1} \pm v_{p+1}, \dots, u_q \pm v_q, 1, \dots, 1)}_{(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)}, \underbrace{(u_{q+1}, \dots, u_n, 0, \dots, 0)}_{C_v} \right)$$

where  $|C_w^+ \cup C_b^+| = p$ ,  $|C_w \cup C_b| = q$  and  $|C| = n$ . This follows from [Section 3.2](#). Therefore this vector is a solution to the IP problem. The objective function  $\bar{f}(z_0)$  at  $(\bar{x}, \bar{y})$  has the value

$$(p-k)\lambda + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i. \quad (40)$$

If we fix variables from  $(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+) \cup C_v$  to the values they have at  $(\bar{x}, \bar{y})$ , and consider  $z_1 = -\lambda + \sum_{j=1}^k (b_i \pm v_i) + \varepsilon$ , where  $0 < \varepsilon \leq \min\{\lambda, \sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i)\}$ , then problem (39) reduces to

$$\bar{f}(z_1) = \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \bar{g}(z_1),$$

where

$$\bar{g}(z_1) = \max \left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right)$$

$$\sum_{i=1}^p x_i + \sum_{i=1}^p b_i y_i \leq \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon \quad (41)$$

$$0 \leq x_i \leq u_i + v_i y_i \quad i \in C_w^+$$

$$0 \leq x_i \leq u_i - v_i y_i \quad i \in C_b^+$$

$$y \text{ binary.}$$

We now prove that an optimal solution for this problem has to have  $y_i = 0$  for  $1 \leq i \leq k$  and  $y_i = 1$  for  $k+1 \leq i \leq p$ . Let  $(x^1, y^1)$  be any feasible solution and define  $C' = \{i \in C_w^+ \cup C_b^+ : y_i^1 = 1\}$ . The value of the objective function at  $(x^1, y^1)$  is

$$\left( \sum_{i \in C_w^+ \cup C_b^+} x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^1, y^1)} = \sum_{i=1}^p x_i^1 + \sum_{i \in C'} (\lambda \mp v_i).$$

Consider the following three cases.

1. Let  $|C'| = p - k$ .

Since the variables are ordered according to the decreasing order of  $b_i \pm v_i$ , we have

$$\sum_{i \in C'} (b_i \pm v_i) > \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon.$$

If  $\sum_{i=1}^p x_i^1 + \sum_{i \in C'} b_i < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon$ , then there exists a  $j \in C'$  such that  $x_j^1 < u_j \pm v_j$  or  $x_j^1 < u_j$  for some  $j \notin C'$ . Consider  $(x^2, y^2)$  defined as  $y^2 = y^1$  for any  $i$ , and  $x^2 = x^1 + \delta e_j$ , where

$$\delta = \min \left( \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C'} b_i - \sum_{i \in C'} x_i^1, \begin{cases} u_j \pm v_j - x_j^1 & \text{if } j \in C', \\ u_j - x_j^1 & \text{if } j \notin C' \end{cases} \right) > 0. \quad (42)$$

$(x^2, y^2)$  is feasible by the choice of  $\delta$  and gives the objective function value of

$$\sum_{i=1}^p x_i^1 + \sum_{i \in C'} (\lambda \pm v_i) + \delta,$$

which is larger than at  $(x^1, y^1)$ . If  $\delta$  is defined by the second term in (42), then the same update can be repeated for another component of  $(x^2, y^2)$  and eventually  $\delta$  will be defined by the first term in (42). Therefore it suffices to consider only vectors that satisfy (41) at equality. If  $(x^3, y^3)$  is one of them, then

$$\sum_{i=1}^p x_i^3 = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C'} b_i,$$

and

$$\left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^3, y^3)} = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C'} b_i + \sum_{i \in C'} (\lambda \mp v_i). \quad (43)$$

Assume  $C' \neq \{k+1, \dots, p\}$  and let  $l \in C', l \leq k$  and  $m \notin C', m > k$ . Define  $C'' = (C' \setminus \{l\}) \cup \{m\}$ . Then  $|C''| = p - k$  and hence  $\sum_{i \in C''} (b_i \pm v_i) > \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon$ . Define  $(x^4, y^4)$  as  $x^4 = x^3 - x_l^3 e_l - x_m^3 e_m$  and  $y^4 = y^3 - e_l + e_m$ .  $(x^4, y^4)$  is feasible since

$$\begin{aligned} \sum_{i=1}^p x_i^4 + \sum_{i=1}^p b_i y_i^4 &= \sum_{\substack{1 \leq i \leq p \\ i \neq l, m}} x_i^4 + \sum_{\substack{1 \leq i \leq p \\ j \neq l, m}} b_i y_i^4 + x_m^4 + x_l^4 + b_m \\ &= \sum_{1 \leq i \leq p} x_i^3 + \sum_{1 \leq i \leq p} b_i y_i^3 - x_m^3 - x_l^3 - b_l + x_m^4 + x_l^4 + b_m \\ &\leq \sum_{1 \leq i \leq p} x_i^3 + \sum_{1 \leq i \leq p} b_i y_i^3 = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon. \end{aligned} \quad (44)$$

Inequality (44) follows from  $b_m \leq b_l$  by assumption of compatible orders. If  $\sum_{i=1}^p x_i^4 + \sum_{i \in C''} b_i < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon$ , then we can construct  $(x^5, y^5)$  using the procedure described above such that

$$\sum_{i=1}^p x_i^5 + \sum_{i \in C''} b_i = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon. \quad (45)$$

The value of the objective function at  $(x^5, y^5)$  is

$$\left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^5, y^5)} = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C''} b_i + \sum_{i \in C''} (\lambda \mp v_i). \quad (46)$$

Subtracting (43) from (46) yields

$$- \sum_{i \in C''} b_i + \sum_{i \in C''} (\lambda \mp v_i) + \sum_{i \in C'} b_i - \sum_{i \in C'} (\lambda \mp v_i) = (b_l + v_l) - (b_m + v_m) \geq 0,$$

which shows that the value of objective function at  $(x^5, y^5)$  is equal to or larger than at  $(x^3, y^3)$ . Note, that  $(x^5, y^5)$  is obtained from  $(x^3, y^3)$  by swapping the values of two  $y$  coordinates, and then adjusting to equality at (45). Repeating this procedure we end up with  $C' = \{k+1, \dots, p\}$ , which provides the objective function with the value

$$\begin{aligned} \bar{f}(z_1) &= \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i=k+1}^p b_i + \sum_{i=k+1}^p (\lambda \mp v_i) \\ &= (p-k)\lambda + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i - \varepsilon. \end{aligned} \quad (47)$$

If initially  $C' = \{k+1, \dots, p\}$ , then (47) is obtained at  $(x^3, y^3)$  and by the above argument it is an optimal solution.

2. Assume now that  $|C'| = p - k + t$ ,  $0 < t \leq k$ . Repeating the swapping procedure described in the previous case, it is easy to see that an optimum for this case is obtained when (41) is satisfied at equality. This is still possible because

$$\sum_{i \in C'} (b_i \pm v_i) \geq \sum_{i=k-t+1}^p (b_i \pm v_i) \geq \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon.$$

Let  $(x^6, y^6)$  be such a solution. Then  $y_i = 1$  for  $k-t+1 \leq i \leq p$  and  $y_i = 0$  for  $1 \leq i \leq k-t$ . Hence  $C' = \{k-t+1, \dots, p\}$  and  $\sum_{i=1}^p x_i^6 = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i=k-t+1}^p b_i$ . Thus the value of  $\bar{f}(z_1)$  at  $(x^6, y^6)$  is

$$\begin{aligned} & \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^6, y^6)} \\ &= \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i=k-t+1}^p b_i + \sum_{i=k-t+1}^p (\lambda \mp v_i) \\ &= (p-k+t)\lambda + \sum_{i=1}^p u_i - \sum_{i=k-t+1}^k (b_i \pm v_i) + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i - \varepsilon. \end{aligned} \quad (48)$$

Subtracting (48) from (47) we obtain  $\sum_{i=k-t+1}^k (b_i \pm v_i) - t\lambda = \sum_{i=k-t+1}^k (b_i \pm v_i - \lambda) \geq 0$ , which follows from  $C' \subseteq C_w^+ \cup C_b^+$ . So the previous case is better.

3. It remains to consider  $|C'| = p - k - t$ ,  $t > 0$ . If

$$\sum_{i \in C'} (u_i \pm v_i) + \sum_{i \in (C_w^+ \cup C_b^+) \setminus C'} u_i + \sum_{i \in C'} b_i < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon,$$

then consider  $(x^7, y^7)$  such that  $y_i^7 = 1$  if and only if  $i \in C'$ ,  $x_i^7 = u_i$  if  $y_i^1 = 0$  and  $x_i^7 = u_i \pm v_i$  if  $y_i^1 = 1$ . For a given  $C'$ ,  $x_i$ 's are at the upper bounds and therefore  $(x^7, y^7)$  provides the largest objective value, which is

$$\begin{aligned} \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^7, y^7)} = \\ \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i \in C'} (u_i \pm v_i) + \sum_{i \in (C_w^+ \cup C_b^+) \setminus C'} u_i + \sum_{i \in C'} (\lambda \mp v_i). \end{aligned} \quad (49)$$

Subtracting (49) from (47) we obtain

$$(p - k)\lambda + \sum_{i=1}^p u_i - \varepsilon - \sum_{i \in C'} (u_i \pm v_i) - \sum_{i \in (C_w^+ \cup C_b^+) \setminus C'} u_i - \sum_{i \in C'} (\lambda \mp v_i) = t\lambda - \varepsilon \geq 0, \quad (50)$$

since  $\varepsilon \leq \lambda$ . Thus, as before the first case is better.

If

$$\sum_{i \in C'} (u_i \pm v_i) + \sum_{i \in (C_w^+ \cup C_b^+) \setminus C'} u_i + \sum_{i \in C'} b_i \geq \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon,$$

then equality in (41) is still obtainable and we can start the procedure of shifting ones in  $y$  towards the end. Since

$$\sum_{i=k+t+1}^p (u_i \pm v_i) + \sum_{i=1}^{k+t} u_i + \sum_{i=k+t+1}^p b_i = \sum_{i=1}^p u_i + \sum_{i=k+t+1}^p (b_i \pm v_i) < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon,$$

eventually we obtain  $C''$  such that  $|C''| = p - k - t$  and

$$\sum_{i \in C''} (u_i \pm v_i) + \sum_{i \in (C_w^+ \cup C_b^+) \setminus C''} u_i + \sum_{i \in C''} b_i < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon.$$

We have already shown in (50) that a vector with these properties cannot correspond to an optimal solution.

Finally, let  $C'''$  be the set preceding  $C''$  in the sequence obtained in the swapping procedure. Then for  $C'''$  we still have

$$\sum_{i \in C'''} (b_i \pm v_i) \geq \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon. \quad (51)$$

We cannot show that the solution corresponding to set  $C''$  is better than or equal to the one corresponding to  $C'''$  because we are not able to obtain equality after the last swap as we did in the first case at (45). However, we know that the solution corresponding to  $C'''$  is better than the one corresponding to any set preceding  $C'''$  in the constructed sequence. Hence we need to show explicitly that the solution corresponding to  $C'''$  is worse than or equal to the one obtained in the first case. To do this we use the fact that if  $(x^8, y^8)$  is the best solution corresponding to  $C'''$ , than (41) at  $(x^8, y^8)$

is satisfied at equality, and therefore  $\sum_{i=1}^p x_i^8 = \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C'''} b_i$ . Thus the value of  $\bar{f}(z_1)$  at  $(x^8, y^8)$  is

$$\begin{aligned}
& \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \left( \sum_{i=1}^p x_i + \sum_{i \in C_w^+} (\lambda - v_i) y_i + \sum_{i \in C_b^+} (\lambda + v_i) y_i \right) \Big|_{(x^8, y^8)} \\
&= \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon - \sum_{i \in C'''} b_i + \sum_{i \in C'''} (\lambda \mp v_i) \\
&= (p - k - t)\lambda + \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \sum_{i \in C'''} (b_i \pm v_i) + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i - \varepsilon.
\end{aligned} \tag{52}$$

Subtracting (52) from (47) we obtain

$$\begin{aligned}
& (p - k)\lambda + \sum_{i=1}^p u_i - \varepsilon - (p - k - t)\lambda - \sum_{i=1}^p u_i - \sum_{i=k+1}^p (b_i \pm v_i) + \sum_{i \in C'''} (b_i \pm v_i) + \varepsilon \\
&= t\lambda - \sum_{i=k+1}^p (b_i \pm v_i) + \sum_{i \in C'''} (b_i \pm v_i) \geq t\lambda - \sum_{i=k+1}^p (b_i \pm v_i) + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon = t\lambda - \varepsilon \geq 0,
\end{aligned}$$

where the first inequality follows from (51) and the second one from  $\varepsilon \leq \lambda$  and  $t > 0$ . Therefore again the first case is better.

We proved that if  $z = -\lambda + \sum_{j=1}^k (b_j \pm v_j) + \varepsilon$ ,  $0 < \varepsilon \leq \min\{\lambda, \sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i)\}$  and if variables from  $(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+) \cup C_v$  are fixed to the values they have at  $(\bar{x}, \bar{y})$ , then the optimal solution has  $y_i = 0$  for  $1 \leq i \leq p$ ,  $y_i = 1$  for  $k+1 \leq i \leq p$  and  $\sum_{i=1}^p x_i = \sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i) - \varepsilon$ .

The objective value is given by (47). At this point we should consider set  $\{\varepsilon_i : i = 1, \dots, p\}$ , such that  $\sum_{i=1}^p \varepsilon_i = \varepsilon$  and  $\varepsilon_i \leq u_i$  for  $1 \leq i \leq k$ ,  $\varepsilon_i \leq u_i \pm v_i$  for  $k+1 \leq i \leq p$ . However, for ease of exposition we assume that  $u_1 \geq \varepsilon$ , and hence  $\varepsilon_1 = \varepsilon$ . Then the vector

$$(x', y') = \left( \underbrace{(u_1 - \varepsilon, \dots, u_k, u_{k+1} \pm v_{k+1}, \dots, u_p \pm v_p)}_{C_w^+ \cup C_b^+}, \underbrace{(u_{p+1} \pm v_{p+1}, \dots, u_q \pm v_q)}_{(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)}, \underbrace{(u_{q+1}, \dots, u_n, 0, \dots, 0)}_{C_v} \right) \tag{53}$$

satisfies above conditions. Now unfix one variable  $j$  from  $(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)$  and consider the vector  $(x'', y'')$  with  $x_j'' = u_j$  and  $y_j'' = 0$ . We show that it is impossible to have a larger value for the objective function at  $(x'', y'')$  than at  $(x', y')$ . Due to the switch of the value at the  $j$  coordinate,  $\bar{f}(z)$  decreases by  $b_j \pm v_j$ . To compensate this we can use only variables from  $C_w^+ \cup C_b^+$ . We need to choose  $t$  variables, which have  $y_i = 0$  and switch them to 1, because each such switch increases the value of  $\bar{f}(z)$  by  $\lambda$ . Hence  $t$  is determined by  $(t - 1)\lambda \leq b_j \pm v_j < t\lambda$ . From the previous discussion it follows that the best variables to switch have indices  $k - t < i \leq k$ . Without loss of generality assume that  $j = p + 1$ . Thus

$$(x'', y'') = \left( \underbrace{(u_1 - \varepsilon, \dots, u_{k-t}, u_{k-t+1} \pm v_{k-t+1} - \nu, \dots, u_p \pm v_p)}_{C_w^+ \cup C_b^+}, \underbrace{(u_{p+1}, u_{p+2} \pm v_{p+2}, \dots, u_q \pm v_q)}_{(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)}, \underbrace{(u_{q+1}, \dots, u_n, 0, \dots, 0)}_{C_v} \right),$$

where  $\nu$  is the largest number such that  $t\lambda - \nu > b_{p+1} + v_{p+1}$ . However, for  $(x'', y'')$  we have

$$\begin{aligned}
\sum_{i \in C} x_i'' + \sum_{i \in C} y_i'' b_i &= \sum_{i \in C} x_i' + \sum_{i \in C} y_i' b_i + \sum_{i=k-t+1}^k (b_i \pm v_i) - \nu - b_{p+1} - v_{p+1} \\
&\geq \sum_{i \in C} x_i' + \sum_{i \in C} y_i' b_i + t\lambda - \nu - b_{p+1} - v_{p+1} > \sum_{i \in C} x_i' + \sum_{i \in C} y_i' b_i,
\end{aligned}$$

and we know that  $(x', y')$  satisfies (39b) at equality. Hence  $(x'', y'')$  is infeasible. Unfixing more than one variable from  $(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)$  can be considered as a multiple step process, where at each step we unfix one variable. By the above argument setting  $y_i$  to 0 for any variable from  $(C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)$  requires switching  $y_i$  to 1 for some variables from  $C_w^+ \cup C_b^+$ , which yields an infeasible solution. Hence to get an optimal solution we need  $y_i = 1$  for  $i \in (C_w \setminus C_w^+) \cup (C_b \setminus C_b^+)$ . Unfixing variables from  $C_v$  is similar.

So far we showed that  $(x', y')$  given by (53) is an optimal solution for  $z = -\lambda + \sum_{j=1}^k (b_j \pm v_j) + \varepsilon$ ,  $0 < \varepsilon \leq \min\{\lambda, \sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i)\}$ . If  $\sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i) < \lambda$ , then let  $\varepsilon = \sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i)$  and consider  $(x^9, y^9)$  defined by  $x_i^9 = u_i$  for  $1 \leq i \leq k+1$ ,  $x_i^9 = u_i \pm v_i$  for  $k+2 \leq i \leq p$ ,  $y_i = 0$  for  $1 \leq i \leq k+1$  and  $y_i = 1$  for  $k+2 \leq i \leq p$ . Then

$$\begin{aligned} \sum_{i=1}^p x_i^9 + \sum_{i=1}^p b_i y_i^9 &= \sum_{i=1}^{k+1} u_i + \sum_{i=k+2}^p (u_i \pm v_i) + \sum_{i=k+2}^p b_i = \sum_{i=1}^p u_i + \sum_{i=k+2}^p (b_i \pm v_i) \\ &\leq \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \lambda < \sum_{i=1}^p u_i + \sum_{i=k+1}^p (b_i \pm v_i) - \varepsilon. \end{aligned}$$

Hence  $(x^9, y^9)$  is feasible and its objective value is

$$(p-k-1)\lambda + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i. \quad (54)$$

If  $\sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i) \geq \lambda$ , then let  $\varepsilon = \lambda$  and from (47) the optimal value is

$$(p-k)\lambda + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i - \lambda. \quad (55)$$

It is easy to see from (40) that at  $z = -\lambda + \sum_{j=1}^{k+1} (b_j \pm v_j)$  the objective value is

$$(p-k-1)\lambda + \sum_{i=p+1}^q (u_i \pm v_i + b_i) + \sum_{i=q+1}^n u_i + \sum_{i=1}^p u_i. \quad (56)$$

Since (54) equals to (55) and both equal to (56), and  $\bar{f}(z)$  is nondecreasing, we conclude that for  $\sum_{j=1}^k (b_j \pm v_j) + \min\{\sum_{i=1}^k u_i + \sum_{i=k+1}^p (u_i \pm v_i), \lambda\} \leq z \leq -\lambda + \sum_{j=1}^{k+1} (b_j \pm v_j)$  the objective function is constant.  $\square$